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EXISTENCE RESULTS FOR DIRICHLET PROBLEMS WITH DEGENERATED p-LAPLACIAN

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Abstract. In this article, we prove the existence of entropy solutions for the Dirichlet problem

$$(P) \qquad \begin{cases} -\mathrm{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \mathrm{div}(G(x)) & \text{in} \quad \Omega, \\ u(x) = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $f \in L^1(\Omega)$ and $G/\omega \in [L^{p'}(\Omega, \omega)]^N$.

Keywords: degenerate elliptic equations, entropy solutions, weighted Sobolev spaces.

Mathematics Subject Classification: 35J70, 35J60, 35J92.

1. INTRODUCTION

The main purpose of this paper (see Theorem 4.2) is to establish the existence of entropy solutions for the Dirichlet problem

$$(P) \qquad \begin{cases} -\mathrm{div}[\omega(x)|\nabla u|^{p-2}\nabla u = f(x) - \mathrm{div}(G(x)) & \text{in} \quad \Omega, \\ u(x) = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $f \in L^1(\Omega)$, $G/\omega \in [L^{p'}(\Omega, \omega)]^N$, ω is a weight function (i.e., a locally integrable function on $\mathbb{R}^N$ such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$) and $1 < p < \infty$, $p \neq 2$.

The notion of an entropy solution was introduced in [1], where the authors studied the nondegenerate elliptic equation $-\mathrm{div}(a(x, Du)) = f(x)$, with $f \in L^1(\Omega)$. In [3] the author studied the degenerate elliptic equation $Lu = f$, where L is a degenerate elliptic operator in divergence form (i.e., $Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u))$ and $f \in L^1(\Omega)$. Note that, in the proof of our main result, many ideas have been adapted from [1] and [3].

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For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [4–7, 9] and [12]).

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B. Muckenhoupt in the early 1970's (see [9]).

We propose to solve the problem (P) by approximation with variational solutions: we take $f_n \in C_0^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$, $G_n/\omega \in [L^{p'}(\Omega,\omega)]^N$ such that $G_n/\omega \to G/\omega$ in $[L^{p'}(\Omega,\omega)]^N$, we find a solution $u_n \in W_0^{1,p}(\Omega,\omega)$ for the problem with right-hand side f_n and G_n and we will try to pass to the limit as $n \to \infty$.

The paper is organized as follows. In Section 2 we present the definitions and basic results. In Section 3 we prove the existence and uniqueness of solutions when $f/\omega \in L^{p'}(\Omega,\omega)$, $G/\omega \in [L^{p'}(\Omega,\omega)]^N$ and in Section 4 we state and prove our main result about existence of entropy solutions for problem (P) (when $f \in L^1(\Omega)$ and $G/\omega \in [L^{p'}(\Omega,\omega)]^N$).

2. DEFINITIONS AND BASIC RESULTS

By weight we mean a locally integrable function ω on $\mathbb{R}^N$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^N$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^N$.

Definition 2.1. Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, if there is a positive constant $C = C(p,\omega)$ such that, for every ball $B \subset \mathbb{R}^N$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \le C \quad \text{if} \quad p > 1,$$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x\in B} \frac{1}{\omega(x)}\right) \le C \quad \text{if} \quad p = 1,$$

where $|\cdot|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$.

If $1 < q \le p$, then $A_q \subset A_p$ (see [6, 7] or [12] for more information about A_p-weights). As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see [11, Chapter IX, Corollary 4.4]). If $\varphi \in BMO(\mathbb{R}^N)$, then $\omega(x) = \mathrm{e}^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [10]).

Remark 2.2. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \le C\frac{\mu(E)}{\mu(B)}$$

for all measurable subsets E of B (see *15.5 strong doubling property* in [7]). Therefore, if $\mu(E) = 0$, then $|E| = 0$. Thus, if $\{u_n\}$ is a sequence of functions defined in B and $u_n \to u$ μ-a.e., then $u_n \to u$ a.e.

Definition 2.3. Let ω be a weight. We shall denote by $L^p(\Omega,\omega)$ $(1 \le p < \infty)$ the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \bigg(\int\limits_\Omega |f(x)|^p \omega(x)\, dx \bigg)^{1/p} < \infty.$$

We denote $[L^{p'}(\Omega,\omega)]^N = L^{p'}(\Omega,\omega) \times \ldots \times L^{p'}(\Omega,\omega)$.

Remark 2.4. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\mathrm{loc}}(\Omega)$ (see [12, Remark 1.2.4]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.5. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, k a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \bigg(\int\limits_\Omega |u(x)|^p \omega(x)\, dx + \sum_{1 \le |\alpha| \le k} \int\limits_\Omega |D^\alpha u(x)|^p \omega(x)\, dx \bigg)^{1/p}. \tag{2.1}$$

We also define the space $W^{k,p}_0(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)} = \bigg(\sum_{1 \le |\alpha| \le k} \int\limits_\Omega |D^\alpha u(x)|^p \omega(x)\, dx \bigg)^{1/p}.$$

The dual space of $W^{1,p}_0(\Omega,\omega)$ is the space $[W^{1,p}_0(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$,

$$W^{-1,p'}(\Omega,\omega) = \Big\{ T = f - \mathrm{div}(G) : G = (g_1,\ldots,g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega,\omega) \Big\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \le \omega(x) \le C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $\mathrm{W}^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basic result.

Theorem 2.6 (The weighted Sobolev inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p-weight, $1 < p < \infty$. Then there exists positive constants C_Ω and δ such that for all $f \in C_0^\infty(\Omega)$ and $1 \le \eta \le N/(N-1) + \delta$*

$$\|f\|_{L^{\eta p}(\Omega,\omega)} \le C_\Omega \||\nabla f|\|_{L^p(\Omega,\omega)}. \tag{2.2}$$

Proof. See [5, Theorem 1.3]. □

Definition 2.7. We say that $u \in \mathcal{T}^{1,p}_0(\Omega,\omega)$ if $T_k(u) \in W^{1,p}_0(\Omega,\omega)$ for all $k > 0$, where the function $T_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s, & \text{if} \quad |s| \le k, \\ k\,\mathrm{sign}(s), & \text{if} \quad |s| > k. \end{cases}$$

Remark 2.8. (i) Note that for given $h > 0$ and $k > 0$ we have

$$T_h(u - T_k(u)) = \begin{cases} 0 & \text{if} \quad |u| \le k, \\ (|u| - k)\,\mathrm{sign}(u) & \text{if} \quad k < |u| \le k + h, \\ h\mathrm{sign}(u), & \text{if} \quad |u| > k + h. \end{cases}$$

Moreover, if $\alpha \in \mathbb{R}$, $\alpha \neq 0$, we have $T_k(\alpha u) = \alpha T_{k/|\alpha|}(u)$.

(ii) If $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \omega)$, then we have

$$\nabla T_k(u) = \chi_{\{|u|<k\}} \nabla u,$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$.

Definition 2.9. Let $f \in L^1(\Omega)$, $G/\omega \in [L^{p'}(\Omega, \omega)]^N$ and $u \in \mathcal{T}^{1,p}_0(\Omega, \omega)$. We say that u is an entropy solution to problem (P) if

$$\int_\Omega \omega(x)\, |\nabla u|^{p-2} \langle \nabla u, \nabla T_k(u-\varphi)\rangle \, dx = \int_\Omega f\, T_k(u-\varphi)\, dx + \int_\Omega \langle G, \nabla T_k(u-\varphi)\rangle \, dx \quad (2.3)$$

for all $k > 0$ and all $\varphi \in W^{1,p}_0(\Omega, \omega) \cap L^\infty(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$.

We recall that the gradient of u which appears in (2.3) is defined as in Remark 2.8 of [3], that is to say that $\nabla u = \nabla T_k(u)$ on the set where $|u| < k$.

Remark 2.10. Note that if $u_1, u_2 \in W^{1,p}_0(\Omega, \omega)$, then $\varphi = T_k(u_1 + u_2) \in W^{1,p}_0(\Omega, \omega) \cap L^\infty(\Omega)$ and we have

$$\nabla \varphi = \nabla T_k(u_1 + u_2) = \nabla(u_1 + u_2)\, \chi_{\{|u_1+u_2| \le k\}}.$$

Definition 2.11. Let $0 < p < \infty$ and let ω be a weight function. We define the weighted Marcinkiewicz space $\mathcal{M}^p(\Omega, \omega)$ as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that the function

$$\Gamma_k(f) = \mu(\{x \in \Omega \colon |f(x)| > k\}), \quad k > 0,$$

satisfies an estimate of the form $\Gamma_f(k) \le C k^{-p}$, $0 < C < \infty$.

Remark 2.12. If $1 \le q < p$ and $\Omega \subset \mathbb{R}^N$ is a bounded set, we have that

$$L^p(\Omega, \omega) \subset \mathcal{M}^p(\Omega, \omega) \text{ and } \mathcal{M}^p(\Omega, \omega) \subset L^q(\Omega, \omega).$$

(the proof follows the lines of Theorem 2.18.8 in [8]).

Lemma 2.13. *Let $u \in \mathcal{T}^{1,p}_0(\Omega, \omega)$ and $\omega \in A_p$, $1 < p < \infty$, be such that*

$$\frac{1}{k} \int_{\{|u|<k\}} |\nabla u|^p \omega \, dx \le M, \quad (2.4)$$

for every $k > 0$. Then:

(i) $u \in \mathcal{M}^{p_1}(\Omega,\omega)$, *where* $p_1 = \eta(p-1)$ (*where* η *is the constant in Theorem 2.6*). *More precisely, there exists* $C > 0$ *such that* $\Gamma_k(u) \le CM^{\eta}k^{-p_1}$.

(ii) $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$, *where* $p_2 = p\,p_1/(p_1+1)$ *and* $p_1 = \eta(p-1)$. *More precisely, there exists* $C > 0$ *such that* $\Gamma_k(|\nabla u|) \le CM^{(p_1+\eta)/(p_1+1)}k^{-p_2}$.

Proof. See Lemma 3.3 and Lemma 3.4 in [3]. □

3. WEAK SOLUTIONS

In this section we prove the existence and uniqueness of weak solutions $u \in W_0^{1,p}(\Omega,\omega)$ to the Dirichlet problem

$$(P1)\qquad \begin{cases} -\mathrm{div}[\omega(x)|\nabla u|^{p-2}\nabla u = f(x) - \mathrm{div}(G(x)) & \text{in} \quad \Omega, \\ u(x) = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \ge 2)$, $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\omega\,[L^{p'}(\Omega,\omega)]^N$.

Definition 3.1. We say that $u \in W_0^{1,p}(\Omega,\omega)$ is a weak solution for problem $(P1)$ if

$$\int_\Omega \omega(x)\,|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,dx = \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G,\nabla\varphi\rangle\,dx, \tag{3.1}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega)$, with $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\omega \in [L^{p'}(\Omega,\omega)]^N$.

Theorem 3.2. *Let* $\omega \in A_p$, $1 < p < \infty$, $f/\omega \in L^{p'}(\Omega,\omega)$ *and* $G/\omega \in [L^{p'}(\Omega,\omega)]^N$. *Then the problem* $(P1)$ *has a unique solution* $u \in W_0^{1,p}(\Omega,\omega)$.

Proof. (I) Existence. By Theorem 2.6, we have that

$$\begin{aligned} \left|\int_\Omega f\,\varphi\,dx\right| &\le \left(\int_\Omega \left|\frac{f}{\omega}\right|^{p'}\omega\,dx\right)^{1/p'}\left(\int_\Omega |\varphi|^p\,\omega\,dx\right)^{1/p} \le \\ &\le C_\Omega\,\|f/\omega\|_{L^{p'}(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)}. \end{aligned} \tag{3.2}$$

Define the functional $J_p : W_0^{1,p}(\Omega,\omega) \to \mathbb{R}$ by

$$J_p(\varphi) = \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\,dx - \int_\Omega f\,\varphi\,dx - \int_\Omega \langle G,\nabla\varphi\rangle\,dx.$$

Using (3.2) and Young's inequality, we have that

$$\begin{aligned} J_p(\varphi) &\geq \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\,dx - (\,C_\Omega\,\|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)}\,)\,\|\nabla\varphi\|_{L^p(\Omega,\omega)} \geq \\ &\geq \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\,dx - \frac{1}{p}\,\|\nabla\varphi\|^p_{L^p(\Omega,\omega)} - \\ &\quad - \frac{1}{p'}[C_\Omega\,\|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)}]^{p'} = \\ &= -\frac{1}{p'}[C_\Omega\,\|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)}\,]^{p'}, \end{aligned}$$

that is, J_p is bounded from below.

Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$J_p(u_n) \;\to \inf_{\varphi\in W^{1,p}_0(\Omega,\omega)} J_p(\varphi).$$

Then for n large enough, we obtain

$$0 \geq J_p(u_n) = \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx - \int_\Omega f\,u_n\,dx - \int_\Omega \langle G, \nabla u_n\rangle\,dx,$$

and we get

$$\begin{aligned} \|\nabla u_n\|^p_{L^p(\Omega,\omega)} &\leq p\left(\int_\Omega f\,u_n\,dx + \int_\Omega \langle G, \nabla u_n\rangle\,dx\right) \leq \\ &\leq p(\|f/\omega\|_{L^{p'}(\Omega,\omega)}\,\|u_n\|_{L^p(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)}\|\nabla u_n\|_{L^p(\Omega,\omega)}) \leq \\ &\leq p(C_\Omega\,\|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)})\|\nabla u_n\|_{L^p(\Omega,\omega)}. \end{aligned}$$

Hence $\|\nabla u_n\|_{L^p(\Omega,\omega)} \leq [p(C_\Omega\,\|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)})]^{1/(p-1)}$. Therefore $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega,\omega)$. Since $W^{1,p}_0(\Omega,\omega)$ is reflexive, there exists $u\in W^{1,p}_0(\Omega,\omega)$ such that $u_n \rightharpoonup$ in $W^{1,p}_0(\Omega,\omega)$. Since $W^{1,p}_0(\Omega,\omega) \ni \varphi \mapsto \int_\Omega f\varphi\,dx + \int_\Omega \langle G, \nabla\varphi\rangle\,dx$, and $\varphi \mapsto \|\nabla\varphi\|_{L^p(\Omega,\omega)}$ are continuous then J_p is continuous. Moreover since $1 < p < \infty$ we have that J_p is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J_p(u) \leq \liminf_n J_p(u_n) = \inf_{\varphi\in W^{1,p}_0(\Omega,\omega)} J_p(\varphi),$$

and thus u is a minimizer of J_p on $W^{1,p}_0(\Omega,\omega)$. For any $\varphi \in W^{1,p}_0(\Omega,\omega)$ the function

$$\lambda \mapsto \frac{1}{p}\int_\Omega |\nabla(u+\lambda\varphi)|^p\,\omega\,dx - \int_\Omega (u+\lambda\,\varphi)\,f\,dx - \int_\Omega \langle G, \nabla(u+\lambda\,\varphi)\rangle\,dx$$

has a minimum at $\lambda = 0$. Hence

$$\frac{d}{d\lambda}\left(J_p(u+\lambda\,\varphi)\right)\Big|_{\lambda=0} = 0, \quad \forall\,\varphi \in W^{1,p}_0(\Omega,\omega).$$

We have

$$\frac{d}{d\lambda}\left(|\nabla(u+\lambda\varphi|^p\,\omega\right) = p\,\{|\nabla(u+\lambda\varphi)|^{p-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\}\,\omega,$$

and we obtain

$$\begin{aligned}
0 &= \frac{d}{d\lambda}\left(J_p(u+\lambda\varphi)\right)\Big|_{\lambda=0} = \\
&= \left[\frac{1}{p}\left(p\int_\Omega |\nabla(u+\lambda\varphi)|^{p-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\omega\,dx\right)-\right.\\
&\quad\left.-\int_\Omega \varphi\,f\,dx-\int_\Omega \langle G,\nabla\varphi\rangle\,dx\right]\Big|_{\lambda=0} = \\
&= \int_\Omega |\nabla u|^{p-2}\,\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx-\int_\Omega f\,\varphi\,dx-\int_\Omega \langle G,\nabla\varphi\rangle\,dx.
\end{aligned}$$

Therefore $\int_\Omega |\nabla u|^{p-2}\,\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx = \int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx$, that is, $u\in W_0^{1,p}(\Omega,\omega)$ is a solution of problem $(P1)$.

(II) Uniqueness. If $u_1,u_2\in W_0^{1,p}(\Omega,\omega)$ are two weak solutions of problem $(P1)$, we have

$$\int_\Omega |\nabla u_i|^{p-2}\,\langle\nabla u_i,\nabla\varphi\rangle\,\omega\,dx = \int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,\quad i=1,2,$$

for all $\varphi\in W_0^{1,p}(\Omega,\omega)$. Hence

$$\int_\Omega \left(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla\varphi\rangle-|\nabla u_2|^{p-2}\langle\nabla u_2,\nabla\varphi\rangle\right)\omega\,dx=0.$$

Taking $\varphi=u_1-u_2$, and using that for every $x,y\in\mathbb{R}^N$ there exist two positive constants α_p and β_p such that

$$\alpha_p\,(|x|+|y|)^{p-2}|x-y|\le\langle\,|x|^{p-2}x-|y|^{p-2}y,x-y\rangle\le\beta_p\,(|x|+|y|)^{p-2}|x-y|,$$

we obtain

$$\begin{aligned}
0 &= \int_\Omega\left(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla u_1-\nabla u_2\rangle-|\nabla u_2|^{p-2}\langle\nabla u_2,\nabla u_1-\nabla u_2\rangle\right)\omega\,dx = \\
&= \int_\Omega \langle|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2,\nabla u_1-\nabla u_2\rangle\,\omega\,dx\ge\\
&\ge\alpha_p\int_\Omega\left(|\nabla u_1|+|\nabla u_2|\right)^{p-2}|\nabla u_1-\nabla u_2|^2\,\omega\,dx.
\end{aligned}$$

Therefore $\nabla u_1=\nabla u_2$ μ-a.e. and since $u_1,u_2\in W_0^{1,p}(\Omega,\omega)$, then $u_1=u_2$ a.e. (by Remark 2.2). $\square$

4. MAIN RESULT

In this section, we prove the main result of this paper. We need the following results.

Lemma 4.1. *Let $\omega \in A_p$, $1 < p < \infty$ and a sequence $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega,\omega)$ satisfies:*

(1) $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega,\omega)$ *and* μ-*a.e. in* Ω.

(2) $\int_\Omega \langle\, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\,,\ \nabla(u_n - u)\rangle\, \omega\, dx \to 0$ *with* $n \to \infty$.

Then $u_n \to u$ *in* $W_0^{1,p}(\Omega,\omega)$.

Proof. The proof of this lemma follows the lines of Lemma 5 in [2]. □

Theorem 4.2. *Let $\omega \in A_p$, $1 < p < \infty$, $f \in L^1(\Omega)$ and $G/\omega \in [L^{p'}(\Omega,\omega)]^N$. There exists an entropy solution u of problem (P). Moreover, $u \in \mathcal{M}^{p_1}(\Omega,\omega)$ and $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$, with $p_1 = \eta\,(p-1)$ and $p_2 = p_1\,p/(p_1+1)$ (where η is the constant in Theorem 2.6).*

Proof. Considering a sequence $\{f_n\}$, $f_n \in C_0^\infty(\Omega)$, where

$$f_n \to f \text{ in } L^1(\Omega) \ \text{ and } \ \|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)},$$

and a sequence $\{G_n\}$, with $G_n/\omega \in [L^{p'}(\Omega,\omega)]^N$ such that $\frac{G_n}{\omega} \to \frac{G}{\omega}$ in $[L^{p'}(\Omega,\omega)]^N$ and $\|\,|G_n|\,/\omega\|_{L^{p'}(\Omega,\omega)} \le \|\,|G|\,/\omega\|_{L^{p'}(\Omega,\omega)}$. For each n, by Theorem 3.2, there exists a solution $u_n \in W_0^{1,p}(\Omega,\omega)$ of the Dirichlet problem

$$(P_n) \qquad \begin{cases} -\mathrm{div}[\omega(x)|\nabla u_n|^{p-2}\nabla u_n)] = f_n(x) - \mathrm{div}(G_n(x)) & \text{in } \ \Omega, \\ u_n(x) = 0 & \text{in } \ \partial\Omega, \end{cases}$$

that is,

$$\int_\Omega \omega\, |\nabla u_n|^{p-2}\, \langle \nabla u_n, \nabla\varphi\rangle\, dx = \int_\Omega f_n\varphi\, dx + \int_\Omega \langle G, \nabla\varphi\rangle\, dx \tag{4.1}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega)$. For $\varphi = T_k(u_n)$ we obtain in (4.1) that

$$\int_\Omega \omega\, |\nabla u_n|^{p-2}\, \langle\, \nabla u_n), \nabla T_k(u_n)\rangle\, dx = \int_\Omega f_n\, T_k(u_n)\, dx + \int_\Omega \langle G_n, \nabla T_k(u_n)\rangle\, dx. \tag{4.2}$$

We have

$$\left| \int_\Omega f_n\, T_k(u_n)\, dx \right| \le \int_\Omega |f_n||T_k(u_n)|\, dx \le k\|f_n\|_{L^1(\Omega)} \le k\|f\|_{L^1(\Omega)}, \tag{4.3}$$

and since $\nabla T_k(u_n) = \chi_{\{|u_n|<k\}}\nabla u_n$, we obtain

$$\begin{aligned} \int_\Omega \omega\, |\nabla u_n|^{p-2}\, \langle\, \nabla u_n, \nabla T_k(u_n)\rangle\, dx &= \int_\Omega \omega\, |\nabla T_k(u_n)|^{p-2}\, \langle\, \nabla T_k(u_n), \nabla T_k(u_n)\rangle\, dx = \\ &= \int_\Omega |\nabla T_k(u_n)|^p \omega\, dx. \end{aligned} \tag{4.4}$$

We also have, using Young's inequality, that there exists a constant $C_1 > 0$ (depending only on p) such that

$$\begin{aligned}\Big|\int_\Omega \langle G_n, \nabla T_k(u_n)\rangle\, dx\Big| &\le \int_\Omega \Big|\frac{G_n}{\omega}\Big| |\nabla T_k(u_n)|\, \omega\, dx \le \\ &\le \Big(\int_\Omega |G_n/\omega|^{p'}\omega\, dx\Big)^{1/p'} \Big(\int_\Omega |\nabla T_k(u_n)|^p \omega\, dx\Big)^{1/p} \le \\ &\le \frac{1}{2}\int_\Omega |\nabla T_k(u_n)|^p\omega\, dx + C_1\int_\Omega |G_n/\omega|^{p'}\omega\, dx \le \\ &\le \frac{1}{2}\int_\Omega |\nabla T_k(u_n)|^p\omega\, dx + C_1\int_\Omega |G/\omega|^{p'}\omega\, dx.\end{aligned} \tag{4.5}$$

Hence, using (4.3), (4.4) and (4.5), we obtain

$$\int_\Omega |\nabla T_k(u_n)|^p\omega\, dx \le 2\, k\, \|f\|_{L^1(\Omega)} + 2\, C_1\, \|G/\omega\|^{p'}_{L^{p'}(\Omega,\omega)} \le C_2\, k, \tag{4.6}$$

where $C_2 = 2\,\|f\|_{L^1(\Omega)} + 2C_1\,\|G/\omega\|^{p'}_{L^{p'}(\Omega,\omega)}$. By Lemma 2.13, the sequence $\{u_n\}$ is bounded in $\mathcal{M}^{p_1}(\Omega,\omega)$ (with $p_1 = \eta\,(p-1)$), and $\{|\nabla u_n|\}$ is bounded in $\mathcal{M}^{p_2}(\Omega,\omega)$ (with $p_2 = p_1\, p/(p_1+1)$). Moreover, $\{u_n\}$ is a Cauchy sequence in the μ-measure. Consequently, there exists a function u and a subsequence, that we will still denote by $\{u_n\}$, such that

$$u_n \to u \quad \mu - \text{a.e. in } \Omega, \tag{4.7}$$

and $u_n \to u$ a.e. in Ω (by Remark 2.2). Using (4.6) and (4.7), we have

$$\begin{aligned}&T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W^{1,p}_0(\Omega,\omega),\\ &T_k(u_n) \to T_k(u) \text{ strongly in } L^p(\Omega,\omega) \text{ and } \mu - \text{a.e. in } \Omega,\end{aligned} \tag{4.8}$$

for all $k > 0$. Hence $T_k(u) \in W^{1,p}_0(\Omega,\omega)$.

Furthermore, by the weak lower semicontinuity of the norm $W^{1,p}_0(\Omega,\omega)$, we have that (4.6) still holds for u, that is,

$$\int_\Omega |\nabla T_k(u)|^p\omega\, dx \le k\, C_2.$$

Applying Lemma 2.13, we deduce that $u \in \mathcal{M}^{p_1}(\Omega,\omega)$ (with $p_1 = \eta\,(p-1)$) and $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$ (with $p_2 = p_1\, p/(p_1+1)$).

We need to show that $T_k(u_n) \to T_k(u)$ strongly in $W^{1,p}_0(\Omega,\omega)$ for all $k > 0$.

Let $h > k$ and applying (4.1) with function $\varphi_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$, we get

$$\int_\Omega \omega\, |\nabla u_n|^{p-2}\langle \nabla u_n, \nabla\varphi_n\rangle\, dx = \int_\Omega f_n\, \varphi_n\, dx + \int_\Omega \langle G, \nabla\varphi_n\rangle\, dx. \tag{4.9}$$

If we set $M = 4k + h$, we have $\nabla\varphi_n = 0$ for $|u_n| > M$. We can write

$$\int_\Omega \omega\,|\nabla T_M(u_n)|^{p-2}\,\langle \nabla T_M(u_n), \nabla\varphi_n\rangle\,dx = \int_\Omega f_n\varphi_n\,dx + \int_\Omega \langle G, \nabla\varphi_n\rangle\,dx. \tag{4.10}$$

In the left-hand side of (4.10), we have

$$\begin{aligned}
&\int_\Omega \omega\,|\nabla T_M(u_n)|^{p-2}\,\langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\,dx = \\
&= \int_{\{|u_n|\le k\}} \omega\,|\nabla T_M(u_n)|^{p-2}\,\langle \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\,dx + \\
&\quad + \int_{\{|u_n|>k\}} \omega\,|\nabla T_M(u_n)|^{p-2}\,\langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle dx
\end{aligned} \tag{4.11}$$

(a) If $|u_n| \le k$.
Since $h > k$, if $|u_n| \le k < h$, then $T_h(u_n) = T_k(u_n) = u_n$. Hence, $u_n - T_h(u_n) + T_k(u_n) - T_k(u) = u_n - T_k(u)$. We also have that $|u_n - u| \le 2k$. Then, since $\nabla T_M(u_n) = \nabla T_k(u_n)$ (because $|u_n| \le k < M$),

$$\begin{aligned}
&\int_{\{|u_n|\le k\}} \omega\,|\nabla T_M(u_n)|^{p-2}\langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\,dx = \\
&= \int_{\{|u_n|\le k\}} \omega|\nabla T_k(u_n)|^{p-2}\langle \nabla T_k(u_n), \nabla(T_k(u_n) - T_k(u))\,dx = \\
&= \int_\Omega \omega\,|\nabla T_k(u_n)|^{p-2}\langle \nabla T_k(u_n), \nabla(T_k(u_n) - T_k(u))\rangle\,dx.
\end{aligned}$$

(b) If $|u_n| > k$.
Since $u_n, T_k(u_n)$ and $T_k(u)$ are in $W_0^{1,p}(\Omega,\omega)$, if $|u_n - T_h(u_n) + T_k(u_n) - T_k(u)| \le 2k$, we obtain

$$\begin{aligned}
\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) &= \nabla(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = \\
&= \nabla u_n - \nabla T_h(u_n) + \nabla T_k(u_n) - \nabla T_k(u) = \\
&= \nabla u_n - \nabla T_h(u_n) - \nabla T_k(u)
\end{aligned}$$

(because $\nabla T_k(u_n) = 0$ if $|u_n| > k$). There are two possible cases:

(i) If $k < |u_n| < h$, we have $\nabla T_h(u_n) = \nabla u_n$. Then

$$\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = -\nabla T_k(u).$$

(ii) If $h < |u_n| \le M$, we have $\nabla T_h(u_n) = 0$. Then

$$\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = \nabla u_n - \nabla T_k(u) = \nabla T_M(u_n) - \nabla T_k(u).$$

In both cases we obtain

$$\begin{aligned}&|\nabla T_M(u_n)|^{p-2}\langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle \geq \\ &\geq -|\nabla T_M(u_n)|^{p-2}\langle \nabla T_M(u_n), \nabla T_k(u)\rangle \geq \\ &\geq -|\nabla T_M(u_n)|^{p-2}\,|\nabla T_M(u_n)|\,|\nabla T_k(u)|.\end{aligned}$$

Therefore, we obtain in (4.11)

$$\begin{aligned}&\int_\Omega \omega\,|\nabla T_M(u_n)|^{p-2}\langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\,dx = \\ &= \int_{\{|u_n|\leq k\}} \omega\,|\nabla T_M(u_n)|^{p-2}\langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\,dx + \\ &\quad + \int_{\{|u_n|>k\}} \omega\,|\nabla T_M(u_n)|^{p-2}\langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\,dx \geq \\ &\geq \int_\Omega \omega|\nabla T_M(u_n)|^{p-2}\langle \nabla T_k(u_n), \nabla(T_k(u_n) - T_k(u))\,dx - \\ &\quad - \int_{\{|u_n|>k\}} \omega|\nabla T_M(u_n)|^{p-2}|\nabla T_M(u_n)|\,|\nabla T_k(u)|\,dx.\end{aligned}$$

Hence, in (4.10) we obtain

$$\begin{aligned}&\int_\Omega \omega\langle |\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u), \nabla(T_k(u_n) - T_k(u))\rangle\,dx \leq \\ &\leq \int_{\{|u_n|>k\}} \omega\,|\nabla T_M(u_n)|\,|\nabla T_k(u)|\,dx + \\ &\quad + \int_\Omega f_n T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\,dx + \\ &\quad + \int_\Omega \langle G_n, \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\,dx - \\ &\quad - \int_\Omega \omega\,|\nabla T_k(u)|^{p-2}\langle \nabla T_k(u), \nabla(T_k(u_n) - T_k(u))\rangle\,dx.\end{aligned} \tag{4.12}$$

Considering the test function $\psi_n = T_{2k}(u_n - T_h(u_n))$ in (4.1), we have

$$\int_\Omega \omega\,|\nabla u_n|^{p-2}\,\langle \nabla u_n, \nabla\psi_n\rangle\,dx = \int_\Omega f_n\,\psi_n\,dx + \int_\Omega \langle G_n, \nabla\psi_n\rangle\,dx,$$

and using that

$$\left|\int_\Omega f_n\psi_n\,dx\right| \le \int_\Omega |f_n||\psi_n|\,dx \le (2k+1)\|f_n\|_{L^1(\Omega)} \le (2k+1)\|f\|_{L^1(\Omega)},$$

and

$$\begin{aligned}\int_\Omega \omega\,|\nabla u_n|^{p-2}\langle\nabla u_n,\nabla\psi_n\rangle\,dx &= \int_\Omega \omega\,|\nabla\psi_n|^{p-2}\langle\nabla\psi_n,\nabla\psi_n\rangle\,dx = \\ &= \int_\Omega |\nabla\psi_n|^p\omega\,dx = \int_\Omega |\nabla T_{2k}(u_n - T_h(u_n))|^p\omega\,dx,\end{aligned}$$

we obtain

$$\int_\Omega |\nabla T_{2k}(u_n - T_h(u_n))|^p\omega\,dx \le (2k+1)\,C_2.$$

Now using that $T_{2k}(u_n - T_h(u_n)) \rightharpoonup T_{2k}(u - T_h(u))$ weakly in $W_0^{1,p}(\Omega,\omega)$ (by (4.8) and Remark 2.8 (i)), we have

$$\int_\Omega |\nabla T_{2k}(u - T_h(u))|^p\omega\,dx \le (2k+1)\,C_2. \tag{4.13}$$

We have (by Remark 2.8 (i) and (ii) and (4.13))

$$\begin{aligned}\int_\Omega |G||\nabla T_{2k}(u-T_h(u))|\,dx &= \int_{\{h<|u|<2k+h\}} |G||\nabla u|dx \le \\ &\le \left(\int_{\{|u|\ge h\}} |G/\omega|^{p'}\omega dx\right)^{1/p'}\left(\int_{\{h<|u|<2k+h\}} |\nabla u|^p\omega dx\right)^{1/p} = \\ &= \left(\int_{\{|u|\ge h\}} |G/\omega|^{p'}\omega dx\right)^{1/p'}\left(\int_\Omega |\nabla T_{2k}(u-T_h(u))|^p\omega dx\right)^{1/p} \le \\ &\le C_3\left(\int_{\{|u|\ge h\}} |G/\omega|^{p'}\omega dx\right)^{1/p'},\end{aligned}$$

where C_3 depends on k but not on h. Therefore, we have

$$\lim_{h\to\infty}\int_\Omega \langle G,\nabla T_{2k}(u - T_h(u))\rangle\,dx = 0.$$

We also have (by Theorem 2.6 and (4.13))

$$\int_\Omega |T_{2k}(u - T_h(u))|^p\omega\,dx \le C_\Omega\int_\Omega |\nabla T_{2k}(u - T_h(u))|^p\omega\,dx \le C_\Omega\,C_2\,(2k+1).$$

Moreover, by Lebesgue's theorem, we obtain

$$\lim_{h\to\infty}\int_\Omega f\,T_{2k}(u-T_h(u))\,dx=0.$$

We can fix a positive real number h_ε sufficiently large to have

$$\int_\Omega f\,T_{2k}(u-T_{h_\varepsilon})\,dx+\int_\Omega \langle G,\nabla T_{2k}(u-T_{h_\varepsilon}(u))\rangle\,dx\le\varepsilon. \tag{4.14}$$

Considering $h=h_\varepsilon$ in (4.12) (and $M=M_\varepsilon=4k+h_\varepsilon$), by (4.6), we have

$$\int_\Omega \big||\nabla T_M(u_n)|^{p-2}\,\nabla T_M(u_n)\big|^{p'}\omega\,dx=\int_\Omega |\nabla T_M(u_n)|^{(p-2)\,p'}|\nabla T_M(u_n)|^{p'}\omega\,dx=$$
$$=\int_\Omega |\nabla T_M(u_n)|^{p}\,\omega\,dx\le\, M\,C_2,$$

that is, $|\nabla T_M(u_n)|^{p-2}\,\nabla T_M(u_n)|$ is bounded in $L^{p'}(\Omega,\omega)$. Moreover,

$$\chi_{\{|u_n|>k\}}|\nabla T_k(u)|\to 0$$

in $L^p(\Omega,\omega)$ as $n\to\infty$. Therefore,

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}} |\nabla T_M(u_n)|^{p-2}\,\nabla T_M(u_n)||\nabla T_k(u)|\,\omega\,dx=0. \tag{4.15}$$

Futhermore, we have that $T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\rightharpoonup T_{2k}(u-T_h(u))$, weakly in $W_0^{1,p}(\Omega,\omega)$, as $n\to\infty$.

Hence, by (4.8), (4.14) and (4.15), passing to the limit in (4.12), we have

$$\lim_{n\to\infty}\int_\Omega \langle\,|\nabla T_k(u_n)|^{p-2}\,\nabla T_k(u_n)-|\nabla T_k(u)|^{p-2}\,\nabla T_k(u),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega\,dx\le$$
$$\le\int_\Omega f\,T_{2k}(u-T_{h_\varepsilon})\,dx+\int_\Omega \langle G,\nabla T_{2k}(u-T_{h_\varepsilon}(u))\rangle\,dx\le\varepsilon$$

for all $\varepsilon>0$, that is,

$$\lim_{n\to\infty}\int_\Omega \langle\,|\nabla T_k(u_n)|^{p-2}\,\nabla T_k(u_n)-|\nabla T_k(u)|^{p-2}\,\nabla T_k(u),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega\,dx=0.$$

Applying Lemma 4.1 we get

$$T_k(u_n)\to T_k(u) \tag{4.16}$$

strongly in $W_0^{1,p}(\Omega,\omega)$ for every $k>0$.

This convergence implies that for every fixed $k > 0$

$$|\nabla T_k(u_n)|^{p-2}\,\nabla T_k(u_n)) \to |\nabla T_k(u)|^{p-2}\,\nabla T_k(u)) \tag{4.17}$$

in $(L^{p'}(\Omega,\omega))^N = L^{p'}(\Omega,\omega)\times\ldots\times L^{p'}(\Omega,\omega)$.

Finally, we need to show that u is an entropy solution to the Dirichlet problem (P). Let us take $\psi_n = T_k(u_n - \varphi)$ as test function in (4.1), with $\varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$. We obtain

$$\int_\Omega \omega\,|\nabla u_n|^{p-2}\langle \nabla u_n, \nabla\psi_n\rangle\,dx = \int_\Omega f_n\psi_n\,dx + \int_\Omega \langle G_n, \nabla\psi_n\rangle\,dx. \tag{4.18}$$

If $M = k + \|\varphi\|_{L^\infty(\Omega)}$ and $n > M$, we have

$$\int_\Omega \omega|\nabla u_n|^{p-2}\langle \nabla u_n, \nabla T_k(u_n-\varphi)\rangle dx = \int_\Omega \omega|\nabla T_M(u_n)|^{p-2}\langle \nabla T_M(u_n), \nabla T_k(u_n-\varphi)\rangle dx.$$

Hence, in (4.18) we obtain

$$\begin{aligned}&\int_\Omega \omega\,|\nabla T_M(u_n)|^{p-2}\langle \nabla T_M(u_n), \nabla T_k(u_n-\varphi)\rangle\,dx = \\ &= \int_\Omega f_n\,T_k(u_n-\varphi)\,dx + \int_\Omega \langle G, \nabla T_k(u-\varphi)\rangle\,dx.\end{aligned} \tag{4.19}$$

Therefore, by (4.8) and (4.17), passing to the limit as $n\to\infty$ in (4.19), we obtain

$$\int_\Omega \omega\,|\nabla u|^{p-2}\langle \nabla u, \nabla T_k(u-\varphi)\rangle\,dx = \int_\Omega f\,T_k(u-\varphi)\,dx$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ and for each $k > 0$.

Therefore u is an entropy solution of problem (P). □

Example 4.3. Let $\Omega = \{(x,y)\in\mathbb{R}^2 \;:\; x^2+y^2<1\}$, $\omega(x,y) = (x^2+y^2)^{-1/6}$ ($\omega\in A_3$, $p=3$), $f(x,y) = \frac{\sin(xy)}{(x^2+y^2)^{1/3}}$ ($f \in L^1(\Omega)$), $G(x,y) = ((x^2+y^2)\sin(x\,y),$ $(x^2+y^2)^{-1/3}\cos(x\,y))$. By Theorem 4.2, the problem

$$(P)\qquad \begin{cases} -\mathrm{div}[(x^2+y^2)^{-1/6}|\nabla u|\nabla u] = \frac{\sin(xy)}{(x^2+y^2)^{1/3}} - \mathrm{div}(G(x,y)) & \text{in } \Omega,\\ u(x,y) = 0 & \text{in } \partial\Omega\end{cases}$$

has an entropy solution.

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