Opuscula Math. 33, no. 2 (2013), 283–291
http://dx.doi.org/10.7494/OpMath.2013.33.2.283 Opuscula Mathematica

INEQUALITIES FOR REGULARIZED DETERMINANTS OF OPERATORS WITH THE NAKANO TYPE MODULARS

Michael Gil'

Communicated by P.A. Cojuhari

Abstract. Let $\{p_k\}$ be a nondecreasing sequence of integers, and A be a compact operator in a Hilbert space whose eigenvalues and singular values are $\lambda_k(A)$ and $s_k(A)$ $(k = 1, 2, \ldots)$, respectively. We establish upper and lower bounds for the regularized determinant

$$\prod_{k=1}^{\infty}(1-\lambda_k(A))\exp\Big[\sum_{m=1}^{p_k-1}\frac{\lambda_k^m(A)}{m}\Big], \text{ assuming that } \sum_{j=1}^{\infty}\frac{s_j^{p_j}(A/c)}{p_j}<\infty$$

for a constant $c \in (0, 1)$.

Keywords: Hilbert space, compact operators, regularized determinant, Nakano type modular.

Mathematics Subject Classification: 47B10, 47A55.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let H be a separable Hilbert space. For a compact linear operator A in H, A^* is the adjoint, $\lambda_j(A)$ are the eigenvalues and $s_k(A) = \sqrt{\lambda_k(A^*A)}$ $(k = 1, 2, \ldots)$ are the singular values taken with their multiplicities and ordered in the decreasing way: $|\lambda_j(A)| \geq |\lambda_{j+1}(A)|$ and $s_j(A) \geq s_{j+1}(A)$. Let SN_p $(1 < p < \infty)$ be the Schatten-von Neumann ideal of operators A with the finite norm $N_p(A) := [\mathrm{Trace}(A^*A)^{p/2}]^{1/p}$. We will say that a compact operator in H is of infinite order if it does not belong to any Schatten-von Neumann ideal. Such operators arise in various applications. Many fundamental results on infinite order compact linear operators can be found in the well-known book [9, Section 3.1]. The literature on the determinants of compact operators and their applications is very rich, see the interesting recent papers [2, 3, 10, 15, 16] and references cited therein; about the classical results see [1, 7, 14]. At the same time to the best of our knowledge, bounds for the determinants of infinite order operators are not enough considered in the available literature. The motivation of this paper is to extend some useful results on determinants of Schatten-von Neumann operators to infinite order operators.

Since $s_k(A) \to 0$, there is an integer $\nu \geq 1$ such that

$$\sum_{k=1}^{\nu} s_k(A) \leq \nu \tag{1.1}$$

for a given compact operator A. Everywhere below $\{p_k\}_{k=\nu}^{\infty}$ is a nondecreasing sequence of integers $p_k > 1$ $(k \geq \nu)$. Assume that the condition

$$\sum_{j=\nu}^{\infty} \frac{s_j^{p_j}(A/c)}{p_j} < \infty \tag{1.2}$$

holds for a constant $c \in (0,1)$. Take

$$p_1 = \ldots = p_{\nu-1} = 1. \tag{1.3}$$

If $\nu = 1$, then condition (1.3) is not required. Put $\pi(\nu) := \{p_k\}_{k=1}^{\infty}$ and

$$\gamma_{\pi(\nu)}(A) := \sum_{j=1}^{\infty} \frac{s_j^{p_j}(A)}{p_j}.$$

According to (1.2) $\gamma_{\pi(\nu)}(A/c) < \infty$.

Let Y be an arbitrary vector space over $\mathbb{C}$. A functional $m : Y \to [0, \infty)$ is called modular if it satisfies the properties: a) $m(x) = 0$ iff $x = 0$, b) $m(\alpha x) = m(x)$ for $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, c) $m(\alpha x + \beta y) \leq m(x) + m(y)$ if $\alpha, \beta > 0$ with $\alpha + \beta = 1$ for all $x, y \in Y$, cf. [13] (see also [11, 12]).

Now let Y be a space of number sequences $x = \{x_k\}_{k=1}^{\infty}$, and $m(x) = m(x_1, x_2, \ldots)$ a modular on Y. For example,

$$m(x) = \sum_{k=1}^{\infty} \frac{|x_k|^{p_k}}{p_k}$$

is a modular, cf. [13]. Now, for a compact operator in H put

$$\hat{\gamma}(A) := m(s_1(A), s_2(A), \ldots).$$

Then $\hat{\gamma}(A)$ will be called a modular of A. So $\gamma_{\pi(\nu)}(A)$ is a modular of A.

We will check that (1.2) implies the condition

$$\sum_{j=1}^{\infty} |\lambda_j(A)|^{p_j} < \infty. \tag{1.4}$$

Then the regularized determinant is defined as

$$\det_{\pi(\nu)}(I - A) := \prod_{j=1}^{\infty} E_{p_j}(\lambda_j(A)),$$

where I is the unit operator,

$$E_p(z) := (1-z)\exp\Big[\sum_{k=1}^{p-1}\frac{z^k}{k}\Big] \quad (p\ge 2) \quad \text{and} \quad E_1(z) := 1-z \quad (z\in\mathbb{C}).$$

Theorem 1.1. *Let conditions* (1.1)–(1.3) *hold. Then*

$$|\det_{\pi(\nu)}(I-A)| \le \exp\Big[\frac{\gamma_{\pi(\nu)}(A/c)}{1-c}\Big].$$

This theorem is proved in the next section. It generalizes the main result from [4]. About the recent results on the Nakano ideals see [6] and references therein.

2. PROOF OF THEOREM 1.1

Lemma 2.1. *Let A be compact and conditions* (1.1), *and* (1.3) *hold. Then for any constant $c\in(0,1)$ we have*

$$\sum_{j=1}^{n}|\lambda_j(A)|^{p_j} \le \frac{1}{1-c}\sum_{j=1}^{n}\frac{s_j^{p_j}(A/c)}{p_j}, \quad p_j\in\pi(\nu), n>\nu.$$

If, in addition, (1.2) *holds, then*

$$\sum_{j=1}^{\infty}|\lambda_j(A)|^{p_j} \le \frac{\gamma_{\pi(\nu)}(A/c)}{1-c}.$$

Proof. Put $\lambda_j(A)=\lambda_j, s_j(A)=s_j$. According to (1.1) and the Weyl inequalities [8] we have

$$\sum_{k=1}^{\nu}|\lambda_k| \le \sum_{k=1}^{\nu}s_k \le \nu.$$

Hence $|\lambda_\nu|\le 1$ and

$$\sum_{k=1}^{n}t_k|\lambda_k| \le \sum_{k=1}^{n}t_k s_k$$

for any nonincreasing sequence t_k. Since $p_k>1$, and $|\lambda_{k+1}|\le|\lambda_k|\le|\lambda_\nu|\le 1$, for $k\ge\nu$, we obtain $|\lambda_{k+1}|^{p_k}\le|\lambda_k|^{p_k-1}$. Take $t_k=1$ for $k<\nu$ and $t_k=|\lambda_k|^{p_k-1}$ for $k\ge\nu$. Then by (1.3),

$$\sum_{k=1}^{n}|\lambda_k|^{p_k} \le \sum_{k=1}^{\nu-1}s_k + \sum_{k=\nu}^{n}|\lambda_k|^{p_k-1}s_k, \quad n\ge\nu. \tag{2.1}$$

By the Young inequality, we arrive at the inequality

$$|\lambda_k|^{p_k-1}s_k \le \frac{c^{q_k}|\lambda_k|^{q_k(p_k-1)}}{q_k} + \frac{(s_k/c)^{p_k}}{p_k}$$

with $1/q_k + 1/p_k = 1$. But $q_k \geq 1, c^{q_k} \leq c$ and $q_k(p_k - 1) = p_k$. So

$$|\lambda_k|^{p_k-1} s_k \leq c|\lambda_k|^{p_k} + \frac{(s_k/c)^{p_k}}{p_k}.$$

Hence, (2.1) implies

$$\sum_{k=1}^{n} |\lambda_k|^{p_k} \leq \sum_{k=1}^{\nu-1} s_k + c\sum_{k=\nu}^{n} |\lambda_k|^{p_k} + \sum_{k=\nu}^{n} \frac{(s_k/c)^{p_k}}{p_k},$$

or according to (1.3),

$$(1-c)\sum_{k=1}^{n} |\lambda_k|^{p_k} \leq \sum_{k=1}^{n} \frac{(s_k/c)^{p_k}}{p_k}.$$

This proves the lemma. □

Lemma 2.2. *For any integer $p \geq 1$ and all $z \in \mathbb{C}$, we have*

$$|E_p(z)| \leq \exp[\eta_p r^p] \leq \exp[r^p], \quad r = |z|,$$

where $\eta_p = \frac{p-1}{p}$ for $p \neq 1, p \neq 3$, and $\eta_1 = \eta_3 = 1$.

This lemma is proved in [5] but according to the referee's suggestion, for the sake of completeness, we give the proof of Lemma 2.2 in Section 4 below.

Corollary 2.3. *Let condition* (1.4) *hold. Then*

$$|\det_\pi\ (I - A)| \leq \exp\Big[\sum_{j=1}^{\infty} |\lambda_j(A)|^{p_j}\Big].$$

Indeed, in view of Lemma 2.2,

$$|\det_\pi\ (I - A)| \leq \prod_{j=1}^{\infty} \exp[|\lambda_j(A)|^{p_j}].$$

Hence the required result follows.

The assertion of Theorem 1.1 follows from the previous corollary and Lemma 2.1.

3. LOWER BOUNDS FOR DETERMINANTS

Again, for brevity put $\lambda_j(A) = \lambda_j$. We begin with the following lemma, in which $\pi = \{p_k\}$ is a nondecreasing sequence of positive integers.

Lemma 3.1. *Let A be a compact operator satisfying the conditions*

$$\sup_k |\lambda_k| < 1 \tag{3.1}$$

and

$$\sum_{k=1}^{\infty} \frac{|\lambda_k|^{p_k}}{p_k} < \infty. \tag{3.2}$$

Then

$$|\det_\pi(I - A)| \geq \exp\left[- \frac{1}{\tilde{\phi}_1(A)} \sum_{k=1}^{\infty} \frac{|\lambda_k|^{p_k}}{p_k} \right],$$

where $\tilde{\phi}_1(A) := \inf_{j=1,2,\dots;s\in[0,1]} |1 - s\lambda_j|$.

Proof. Put $w_j(z) := E_{p_j}(z\lambda_j)$. Clearly,

$$w_k'(z) = \Big[- \lambda_k + (1 - z\lambda_k) \sum_{m=0}^{p_k-2} z^m \lambda_k^{m+1} \Big] \exp\Big[\sum_{s=1}^{p_k-1} \frac{z^s \lambda_k^s}{s} \Big].$$

But

$$-\lambda_j + (1 - z\lambda_j) \sum_{m=0}^{p_j-2} z^m \lambda_j^{m+1} = -z^{p_j-1} \lambda_j^{p_j},$$

since

$$\sum_{m=0}^{p_j-2} z^m \lambda_j^m = \frac{1 - (z\lambda_j)^{p_j-1}}{1 - z\lambda_j}.$$

So

$$w_j'(z) = -z^{p_j-1} \lambda_j^{p_j} \exp\Big[\sum_{m=1}^{p_j-1} \frac{z^m \lambda_j^m}{m} \Big] = h_j(z) w_j(z),$$

where

$$h_j(z) := -\frac{z^{p_j-1} \lambda_j^{p_j}}{1 - z\lambda_j}.$$

Therefore,

$$E_{p_j}(\lambda_j) = w_j(1) = \exp\left[\int_0^1 h_j(s) ds \right].$$

But

$$\left| \int_0^1 h_j(s)\, ds \right| \leq \int_0^1 \frac{s^{p_j-1}\, ds}{|1 - s\lambda_j|} \leq \frac{1}{p_j \tilde{\phi}_1(A)}.$$

Hence,

$$|E_{p_j}(\lambda_j)| \geq \exp\Big[- \frac{|\lambda_j|^{p_j}}{p_j \tilde{\phi}_1(A)} \Big].$$

This proves the required result. $\square$

Now assume that, instead of (3.1), the condition (1.1) holds. Again take $p_1 = \ldots = p_{\nu-1} = 1$. With the notation

$$\psi_{\nu-1}(A) = \Big(\min_{k=1,\ldots,\nu-1} |1-\lambda_k| \Big)^{\nu-1},$$

we have $|E_1(\lambda_j)| = |1-\lambda_j| \geq (\psi_{\nu-1}(A))^{1/(\nu-1)}$ $(j \leq \nu - 1)$ and

$$|\det_{\pi(\nu)}(I-A)| = \prod_{j=1}^{\infty} |E_{p_j}(\lambda_j)| \geq \psi_{\nu-1}(A) \prod_{j=\nu}^{\infty} |E_{p_j}(\lambda_j)|.$$

By Lemma 3.1,

$$\prod_{j=\nu}^{\infty} |E_{p_j}(\lambda_j)| \geq \exp\left[- \frac{1}{\tilde{\phi}_\nu(A)} \sum_{k=\nu}^{\infty} \frac{|\lambda_k|^{p_k}}{p_k} \right],$$

where $\tilde{\phi}_\nu(A) := \inf_{j=\nu,\nu+1,\ldots;s\in[0,1]} |1-s\lambda_j|$.

We thus have proved the following result.

Lemma 3.2. *Let A be a compact operator, such that conditions* (1.3) *and* (3.2) *are fulfilled. Then*

$$|\det_{\pi(\nu)}(I-A)| \geq \psi_{\nu-1}(A) \exp\left[- \frac{1}{\tilde{\phi}_\nu(A)} \sum_{k=\nu}^{\infty} \frac{|\lambda_k|^{p_k}}{p_k} \right].$$

Lemma 2.1 and the previous one imply our next result.

Theorem 3.3. *Let conditions* (1.1)–(1.3) *be fulfilled. Then*

$$|\det_\pi(I-A)| \geq \psi_{\nu-1}(A) \exp\left[- \frac{\gamma_{\pi(\nu)}(A/c)}{(1-c)\tilde{\phi}_\nu(A)} \right].$$

4. PROOF OF LEMMA 2.2

Put $b_p = \sum_{k=2}^{p} \frac{1}{k}$. We begin with the following result.

Lemma 4.1. *For any integer $p \geq 2$ and all $z \in \mathbb{C}$, we have the inequality*

$$|E_p(z)| \leq 1 + \frac{e^{b_p}}{p-1} \left(\exp\left[\frac{p-1}{p} r^p \right] - 1 \right), \quad r = |z|.$$

Proof. Clearly,

$$E_p'(z) = \left[-1 + (1-z) \sum_{m=0}^{p-2} z^m \right] \exp\left[\sum_{m=1}^{p-1} \frac{z^m}{m} \right].$$

But $-1+(1-z)\sum_{m=0}^{p-2} z^m = -z^{p-1}$. So

$$E_p'(z) = -z^{p-1}\exp\Big[\sum_{m=1}^{p-1}\frac{z^m}{m}\Big].$$

With $z = re^{it}$ and a fixed t we obtain

$$\frac{d|E_p(z)|}{dr} \le |E_p'(z)| \le r^{p-1}\exp\Big[\sum_{m=1}^{p-1}\frac{r^m}{m}\Big]. \tag{4.1}$$

Let us check that

$$\sum_{m=1}^{p-1}\frac{r^m}{m} \le \frac{p-1}{p}r^p + b_p. \tag{4.2}$$

To this end note that by the classical Young inequality we have $x \le x^s/s + (s-1)/s$ $(x>0, s>1)$. Hence taking $s = p/m$, we get,

$$\sum_{m=1}^{p-1}\frac{r^m}{m} \le \sum_{m=1}^{p-1}\Big(\frac{r^p}{p} + \frac{p-m}{pm}\Big).$$

But

$$\sum_{m=1}^{p-1}\frac{p-m}{pm} = \sum_{m=1}^{p-1}\frac{1}{m} - \frac{p-1}{p} = \sum_{m=1}^{p-1}\frac{1}{m} - 1 + \frac{1}{p} = b_p.$$

Hence (4.2) follows. Let us point to another proof of (4.2). Put

$$h(r) = \sum_{m=1}^{p-1}\frac{r^m}{m} - \frac{(p-1)r^p}{p}.$$

Since

$$h'(r) = \sum_{m=1}^{p-1} r^{m-1} - (p-1)r^p,$$

we have $h'(1) = 0$. Since the maximum of $h(r)$ is unique, and

$$h(1) = \sum_{m=1}^{p-1}\frac{1}{m} - \frac{p-1}{p} = b_p$$

and

$$\sum_{m=1}^{p-1}\frac{r^m}{m} = h(r) + \frac{(p-1)r^p}{p} \le b_p + \frac{(p-1)r^p}{p},$$

we obtain (4.2). So by (4.1), $\frac{d|E_p(z)|}{dr} \le r^{p-1}\exp\,[\frac{p-1}{p}r^p + b_p]$. Since $E_p(0) = 1$, this inequality implies

$$|E_p(z)| \le 1 + e^{b_p}\int_0^r s^{p-1}\exp\Big[\frac{p-1}{p}s^p\Big]ds =$$

$$= 1 + \frac{e^{b_p}}{p} \int_0^{r^p} \exp\left[\frac{p-1}{p} t\right] dt = 1 + \frac{e^{b_p}}{p-1} \left(\exp\left[\frac{p-1}{p} r^p\right] - 1 \right),$$

as claimed. □

Lemma 4.2. *For any integer $p \geq 2$ and all $z \in \mathbb{C}$, the inequality*

$$|E_p(z)| \leq C_p \exp\left[\frac{p-1}{p} r^p\right], \quad r = |z|,$$

is true, where $C_p = 1$ for $p \neq 3$ and $C_3 = e^{5/6}\frac{1}{2} \geq 1$.

Proof. First note that

$$|(1-z)e^z|^2 = (1 - 2Re\ z + |z|^2)e^{2Re\ z} \leq e^{-2Re\ z + |z|^2} e^{2Re\ z} = e^{|z|^2}, \quad z \in \mathbb{C},$$

and thus $|E_2(z)| \leq e^{\frac{1}{2}|z|^2}$. Furthermore, if $e^{b_p} \leq p-1$, then the required result follows from the previous lemma. We have $e^{b_4} = e^{13/12} \leq 3$ and $e^{b_5} = e^{77/60} \leq 4$. Clearly,

$$b_p = \frac{1}{2} + \sum_{k=3}^{p} \frac{1}{k} \leq \frac{1}{2} + \int_2^p \frac{dt}{t} = \frac{1}{2} + \ln\left(\frac{p}{2}\right).$$

Therefore, for $p \geq 6$,

$$e^{b_p} \frac{1}{p-1} \leq e^{1/2} \frac{p}{2(p-1)} \leq e^{1/2} \frac{3}{5} \leq 1.$$

This proves the lemma. □

Proof of Lemma 2.2. Clearly $|E_1(z)| = |1-z| \leq e^{|z|}$. Consider the function $f_3(r) = r + \frac{r^2}{2} - r^3$. Its maximum is attained at

$$r_0 = 1/6 + \sqrt{1/36 + 1/3} \approx 0.7676.$$

So $f_3(r_0) = r_0 + \frac{r_0^2}{2} - r_0^3 \leq 0.69 \leq \ln\ 2$. Consequently, $r + \frac{r^2}{2} = f_3(r) + r^3 \leq \ln\ 2 + r^3$. Thus by (4.1),

$$\frac{d|E_3(z)|}{dr} \leq r^2 \exp\left[r + \frac{r^2}{2}\right] \leq 2r^2 \exp\,[r^3].$$

Hence, $|E_3(z)| \leq \exp\,[r^3]$. Now the previous lemma yields the required result. □

REFERENCES

[1] N. Dunford, J.T. Schwartz, *Linear Operators, Part II, Spectral Theory*, Interscience Publishers, New York, London, 1963.

[2] T. Ehrhardt, *Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel*, Comm. Math. Phys. **262** (2006) 2, 317–341.

[3] J. Fujii, M. Nakamura, Y. Seo, *Operator valued determinant and Hadamard product*, Sci. Math. Jpn. **60** (2004) 1, 1–8.

[4] M.I. Gil', *Upper and lower bounds for regularized determinants*, JIPAM. J. Inequal. Pure Appl. Math. **9** (2008) 1, Article 2, 6 pp.

[5] M.I. Gil', *Ideals of compact operators with the Orlicz norms*, Ann. Mat. Pura Appl. Online First, 11 October 2011.

[6] M.I. Gil', *Ideals of compact operators with Nakano type norms in a Hilbert space*, Oper. Matrices **6** (2012) 3, 493–501.

[7] I.C. Gohberg, S. Goldberg, N. Krupnik, *Traces and Determinants of Linear Operators*, Birkhäuser Verlag, Basel, 2000.

[8] I.C. Gohberg, M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Trans. Mathem. Monographs, vol. 18, Amer. Math. Soc., Providence, R.I., 1969.

[9] I.C. Gohberg, M.G. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, Trans. Mathem. Monographs, vol. 24, Amer. Math. Soc., R.I. 1970.

[10] A.Yu. Karlovich, *Higher order asymptotics of Toeplitz determinants with symbols in weighted Wiener algebras*, J. Math. Anal. Appl. **320** (2006), 944–963.

[11] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, Berlin, 1977.

[12] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, Berlin, 1983.

[13] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen Co. Ltd., Tokyo, 1950.

[14] B. Simon, *Trace Ideals and their Applications*, 2nd ed., AMS, 2005.

[15] H. Rugh, *Intermittency and regularized Fredholm determinants*, Invent. Math. **135** (1999) 1, 1–24.

[16] O. Tsuchiya, *Determinant formula for the six-vertex model with reflecting end*, J. Math. Phys. **39** (1998) 11, 5946–5951.

Michael Gil'
gilmi@bezeqint.net

Ben Gurion University of the Negev
Department of Mathematics
P.O. Box 653, Beer-Sheva 84105, Israel

Received: November 12, 2012.
Revised: December 11, 2012.
Accepted: December 12, 2012.