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GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION IN THE NONRADIAL CASE

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Abstract. The energy-critical, focusing nonlinear Schrödinger equation in the nonradial case reads as follows:

$$i\partial_t u = -\Delta u - |u|^{\frac{4}{N-2}}u, \quad u(x,0) = u_0 \in H^1(\mathbb{R}^N), \quad N \geq 3.$$

Under a suitable assumption on the maximal strong solution, using a compactness argument and a virial identity, we establish the global well-posedness and scattering in the nonradial case, which gives a positive answer to one open problem proposed by Kenig and Merle [Invent. Math. **166** (2006), 645–675].

Keywords: critical energy, focusing Schrödinger equation, global well-posedness, scattering.

Mathematics Subject Classification: 35Q40, 35Q55.

1. INTRODUCTION AND THE MAIN RESULT

We consider the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^N (N \geq 3)$:

$$\begin{cases} i\partial_t u = -\Delta u \pm |u|^{\frac{4}{N-2}}u & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ u(x,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ denotes the complex-valued wave function, $i = \sqrt{-1}$.

The sign "$-$" corresponds to the focusing problem, while the sign "$+$" corresponds to the defocusing problem. Cazenave-Weissler [6, 7] showed that if $\|\nabla u_0\|_2$ is suitably small, then there exists a unique solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^N))$ of (1.1) satisfying $\|u\|_{L^{\frac{2(N+2)}{N-2}}(\mathbb{R}; L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N))} < \infty$. In the defocusing case, if $u_0 \in H^1(\mathbb{R}^N)$ is radial, Bourgain [1] proved the global well-posedness for (1.1) with $N = 3, 4$, and that for more regular u_0, the solution preserves the smoothness for all time. (Another proof of this last fact is due to Grillakis [13] for $N = 3$.) Bourgain's result is then extended to $N \geq 5$ by Tao [29], still under the assumption that u_0 is radial. Subsequently, Colliander-Keel-Staffilani-Takaoke-Tao [8] obtained the result for general $u_0 \in H^1(\mathbb{R}^3)$. Ryckman-Visan [26] extended this result to $N = 4$ and finally to $N \geq 5$ by Visan [30]. In the focusing case, these results do not hold. In fact, the classical virial identity shows that if $E(u_0) < 0$ and $|x|u_0 \in L^2(\mathbb{R}^N)$, the corresponding solution breaks down in finite time.

Ginibre-Velo [11] considered a general case:

$$\begin{cases} i\partial_t u = -\Delta u - |u|^{q-1}u & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ u(x,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

and established the local well-posedness of the Cauchy problem (1.2) (focusing case) in the energy space $H^1(\mathbb{R}^N)$ with $1 < q < 1 + \frac{4}{N-2}$. Furthermore, they proved the global existence for both small and large initial data in the L^2-subcritical case: $1 < q < 1 + \frac{4}{N}$. In the L^2-supercritical case: $1 + \frac{4}{N} < q < 1 + \frac{4}{N-2}$, Glassey [12], Ogawa-Tsutsumi [24,25] showed that the strong solution of the Cauchy problem (1.2) blows up in finite time for a class of initial data, especially for negative energy initial data. Holmer-Roudenko [15] established sharp conditions on the existence of global solutions of (1.2) with $q = 3$. In the L^2-critical case: $q = 1 + \frac{4}{N}$, Weinstein [31] gave a crucial criterion in terms of L^2-mass initial data. Relevant work on the above topics of (1.2) is referred to [2,3,9,14,16,18,20,23,27] and the references therein.

Using the concentration compactness, which is obtained by Keraani [18], Kenig-Merle [19] considered problem (1.1) in the focusing case for $N = 3, 4, 5$, and discussed global well-posedness and blow-up for the energy-critical problem (1.1) in the radial case. Moreover, they expected their results could be extended to the case of radial data for $N \geq 6$, and believed that it remained an interesting problem to remove the radial symmetry assumption. Subsequently, Killip-Visan [22] considered the focusing problem (1.1) with dimensions $N \geq 5$, and proved that if a maximal-lifespan solution $u : I \times \mathbb{R}^N \to \mathbb{C}$ obeys $\sup_{t\in I} \|\nabla u(t)\|_2 < \|\nabla W\|_2$, then it is global and scatters both forward and backward in time. Here W denotes the ground state, which is a stationary solution of the equation of the focusing problem (1.1). In particular, if a local strong solution has both energy and kinetic energy less than those of the ground state W at some point in time, then the local strong solution is global and scatters in higher dimensions $N \geq 5$. Further results are referred to [10,17].

In the present paper, under a suitable assumption on the local strong solution, we establish the global well-posedness and scattering for the focusing problem (1.1) in the nonradial case, which gives a positive answer to one open problem proposed by Kenig-Merle in [19].

In order to state our main result conveniently, we rewrite the focusing problem (1.1) as follows:

$$\begin{cases} i\partial_t u = -\Delta u - |u|^{\frac{4}{N-2}}u & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ u(x,0) = u_0 \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

Through a standard technical process (see [4]), one can easily check that the solution u of (1.3) defined on the maximal interval $(-T_-(u_0), T_+(u_0))$ obeys conservations of charge and energy:

$$\int_{\mathbb{R}^N} |u(x,t)|^2 dx = \int_{\mathbb{R}^N} |u_0(x)|^2 dx, \quad \forall t \in (-T_-(u_0), T_+(u_0)), \tag{1.4}$$

and

$$E(u(t)) = E(u_0), \quad \forall t \in (-T_-(u_0), T_+(u_0)), \tag{1.5}$$

where

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u(x,t)|^{2^*} dx, \quad 2^* = \frac{2N}{N-2}.$$

Talenti [28] proved that the function

$$W(x) = \frac{(N(N-2))^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$$

satisfies $|\nabla W| \in L^2(\mathbb{R}^N)$ and solves the elliptic equation

$$-\Delta W = |W|^{\frac{4}{N-2}} W \quad \text{in} \quad \mathbb{R}^N.$$

The main result of this paper reads as follows.

Theorem 1.1. *Assume that $u_0 \in H^1(\mathbb{R}^N)$, $N = 3, 4, 5$. Then there exists a unique solution u of* (1.3) *defined on the maximum existence of interval $(-T_-(u_0), T_+(u_0))$ with $u \in C((-T_-(u_0), T_+(u_0)), H^1(\mathbb{R}^N))$, where $0 < T_-(u_0), T_+(u_0) \leq +\infty$.*
Let $E(u_0) < E(W)$, $\|\nabla u_0\|_{L^2(\mathbb{R}^N)} < \|\nabla W\|_{L^2(\mathbb{R}^N)}$. Assume that there exists a non-negative real-valued function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \varphi |u_0|^2 dx > 0 \qquad \textit{and} \qquad \inf_{t\in(0,\,T_+(u_0))} f(t) \geq 0 \ \Big(\textit{resp.} \sup_{t\in(-T_-(u_0),\,0)} f(t) \leq 0\Big), \tag{1.6}$$

where

$$f(t) \triangleq Im \int_{\mathbb{R}^N} \overline{u}(x,t) \nabla\varphi(x) \cdot \nabla u(x,t)\, dx.$$

Then $T_-(u_0) = T_+(u_0) = +\infty$, the solution u belongs to $C(\mathbb{R}^1, H^1(\mathbb{R}^N))$, and there exists $u_{0,+}, u_{0,-} \in H^1(\mathbb{R}^N)$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta} u_{0,+}\|_{H^1(\mathbb{R}^N)} = 0, \qquad \lim_{t\to-\infty} \|u(t) - e^{it\Delta} u_{0,-}\|_{H^1(\mathbb{R}^N)} = 0.$$

Remark 1.2. (i) Let $\varphi_R \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function, which satisfies $\varphi_R(x) \equiv 1$ if $|x| \leq R$; $\varphi_R(x) \equiv 0$ if $|x| \geq 2R$; $|\nabla \varphi_R(x)| \leq \frac{C}{R}$ for any $x \in \mathbb{R}^N$. Then it follows from Lemma 2.2 below that

$$\begin{aligned}
&\sup_{t\in(-T_-(u_0),\,T_+(u_0))} \Big| Im \int\limits_{\mathbb{R}^N} \overline{u}(x,t)\nabla\varphi_R \cdot \nabla u(x,t)\,dx \Big| \leq \\
&\leq \sup_{t\in(-T_-(u_0),\,T_+(u_0))} \frac{C}{R}\|u(t)\|_{L^2(R\leq|x|\leq 2R)}\|\nabla u(t)\|_{L^2(R\leq|x|\leq 2R)} \leq \\
&\leq \frac{C}{R}\|u_0\|_{L^2(\mathbb{R}^N)}\|\nabla u_0\|_{L^2(\mathbb{R}^N)} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty,
\end{aligned}$$

which implies that for any $\epsilon > 0$, there exists a large number $R > 0$ such that

$$\inf_{t\in(0,\,T_+(u_0))} Im \int\limits_{\mathbb{R}^N} \overline{u}(x,t)\nabla\varphi_R \cdot \nabla u(x,t)\,dx \geq -\epsilon.$$

However, this estimate does not work in obtaining (2.26) below because we have to let $t = t_j \longrightarrow +\infty$ in (2.26). That is why we need the additional assumption (1.6) in Theorem 1.1.

(ii) If the initial datum $u_0 \in \dot{H}^1(\mathbb{R}^N)$ $(N = 3, 4, 5)$ is radial. The global existence of the strong solution of (1.3) and the scattering in $\dot{H}^1(\mathbb{R}^N)$ are proved in [19] without assumption (1.6). Here we do not need the radial symmetry assumption on u_0, which is replaced by (1.6). Therefore, our conclusion (i.e., Theorem 1.1) improves the results in [19] in some sense.

(iii) It is well known that if $E(u_0) < 0$, $u_0 \in H^1(\mathbb{R}^N)$ with $|x|u_0 \in L^2(\mathbb{R}^N)$, then the solution u of (1.3) blows up at some finite time. But it does not contradict Theorem 1.1. In fact, under the assumptions in Theorem 1.1, the initial energy $E(u_0) \geq 0$. Indeed, using the assumption $\|\nabla u_0\|_{L^2(\mathbb{R}^N)} < \|\nabla W\|_{L^2(\mathbb{R}^N)}$ and the Sobolev inequality, we get

$$\begin{aligned}
E(u_0) &= \frac{1}{2}\|\nabla u_0\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2^*}\|u_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \geq \\
&\geq \Big(\frac{1}{2} - \frac{N-2}{2N} C_N^{-\frac{N}{N-2}} \|\nabla u_0\|^{\frac{4}{N-2}}_{L^2(\mathbb{R}^N)}\Big)\|\nabla u_0\|^2_{L^2(\mathbb{R}^N)} \geq \\
&\geq \Big(\frac{1}{2} - \frac{N-2}{2N} C_N^{-\frac{N}{N-2}} \|\nabla W\|^{\frac{4}{N-2}}_{L^2(\mathbb{R}^N)}\Big)\|\nabla u_0\|^2_{L^2(\mathbb{R}^N)} = \\
&= \frac{1}{N}\|\nabla u_0\|^2_{L^2(\mathbb{R}^N)},
\end{aligned} \tag{1.7}$$

where $C_N = \|\nabla W\|^{\frac{4}{N}}_{L^2(\mathbb{R}^N)}$ is the best Sobolev constant (see [28] for details).

Throughout this paper, we denote the norm of $H^1(\mathbb{R}^N)$, $\dot{H}^1(\mathbb{R}^N)$ by $\|u\|_{H^1} = \big(\int_{\mathbb{R}^N}(|\nabla u(x)|^2 + |u(x)|^2)dx\big)^{\frac{1}{2}}$, $\|u\|_{\dot{H}^1} = \big(\int_{\mathbb{R}^N}|\nabla u(x)|^2 dx\big)^{\frac{1}{2}}$, respectively, and positive constants (possibly different line to line) by C.

2. PROOF OF THE MAIN RESULT

Lemma 2.1. *Let $u \in C((-T_-(u_0), T_+(u_0)), H^1(\mathbb{R}^N))$ be a solution of (1.3), and let $\varphi \in C^4([0,\infty))$ with $\varphi(s) \equiv const$ if $s > 0$ is large. Then for any $t \in (-T_-(u_0), T_+(u_0))$*

$$\frac{d}{dt}\int\limits_{\mathbb{R}^N} \varphi(|x|)|u(x,t)|^2 dx = 2Im \int\limits_{\mathbb{R}^N} \nabla\varphi(|x|)\cdot\nabla u(x,t)\overline{u}(x,t)dx$$

and

$$\frac{d^2}{dt^2}\int\limits_{\mathbb{R}^N} \varphi(|x|)|u(x,t)|^2 dx = 4\int\limits_{\mathbb{R}^N} \varphi''(|x|)|\nabla u(x,t)|^2 dx - \frac{4}{N}\int\limits_{\mathbb{R}^N} \Delta\varphi(|x|)|u(x,t)|^{2^*} dx - $$
$$- \int\limits_{\mathbb{R}^N} \Delta^2\varphi(|x|)|u(x,t)|^2 dx.$$

Proof. Since the proof is similar to those of Lemma in [12] and Lemma 7.6.2 in [5], we omit the details here. □

The following variational estimates are Theorem 3.9 and Corollary 3.13 in [19].

Lemma 2.2 ([19]). *Suppose that*

$$\int\limits_{\mathbb{R}^N} |\nabla u_0|^2 dx < \int\limits_{\mathbb{R}^N} |\nabla W|^2 dx \quad and \quad E(u_0) < (1-\delta_0)E(W), \quad where \quad \delta_0 \in (0,1).$$

Let $I \ni 0$ be the maximal interval of existence of the solution $u \in C(I, H^1(\mathbb{R}^N))$ of (1.3). Then there exists $\overline{\delta} = \overline{\delta}(\delta_0, N) > 0$ such that for each $t \in I$

$$\int\limits_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx < (1-\overline{\delta})\int\limits_{\mathbb{R}^N} |\nabla W|^2 dx,$$
$$\overline{\delta}\int\limits_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx < \int\limits_{\mathbb{R}^N} \big(|\nabla u(x,t)|^2 - |u(x,t)|^{2^*}\big)dx,$$
$$E(u(t)) \geq 0.$$

Furthermore, $E(u(t)) \simeq \int\limits_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx \simeq \int\limits_{\mathbb{R}^N} |\nabla u_0|^2 dx$, for all $t \in I$ with comparability constants which depend only on δ_0.

The following rigidity theorem plays a fundamental role in the proof of Theorem 1.1.

Theorem 2.3. *Assume that $u_0 \in H^1(\mathbb{R}^N)$ satisfies*

$$\int\limits_{\mathbb{R}^N} |\nabla u_0|^2 dx < \int\limits_{\mathbb{R}^N} |\nabla W|^2 dx \quad and \quad E(u_0) < E(W).$$

Let u be the solution of (1.3) *with the maximal interval of existence* $(-T_-(u_0), T_+(u_0))$*, and let the assumption* (1.6) *hold. Suppose that there exists* $\lambda(t) > 0$*,* $x(t) \in \mathbb{R}^N$ *with the property that*

$$K = \Big\{ v(x,t) = \frac{1}{\lambda(t)^{\frac{N-2}{2}}} u\Big(\frac{x - x(t)}{\lambda(t)}, t\Big) : \ t \in \big[0, T_+(u_0)\big) \Big\}$$

is such that $\overline{K}$ *is compact in* $\dot{H}^1(\mathbb{R}^N)$*. Then* $T_+(u_0) = +\infty$*,* $u_0 \equiv 0$ *in* $\mathbb{R}^N$*.*

Remark 2.4. If $x(t) \equiv 0$ or $\lambda(t) \geq A_0 > 0$ and $|x(t)| \leq C_0$, Theorem 2.3 is verified in [19] for $u_0 \in \dot{H}^1(\mathbb{R}^N)$.

Proof of Theorem 2.3. Step 1. $T_+(u_0) = +\infty$*.* If $T_+(u_0) < +\infty$, then from Lemma 2.11 in [19], one has

$$\|u\|_{S(0,T_+(u_0))} = +\infty, \quad \text{where} \quad \|u\|_{S(I)} = \|u\|_{L^{\frac{2(N+2)}{N-2}}(I, L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N))}. \tag{2.1}$$

Now we claim that

$$\lambda(t) \longrightarrow +\infty \quad \text{as} \quad t \longrightarrow T_+(u_0). \tag{2.2}$$

Indeed if there exists a sequence $\{t_j\}$, $t_j \longrightarrow T_+(u_0)$ such that $\lambda(t_j) \longrightarrow A < +\infty$ as $j \longrightarrow +\infty$.

Set $v_j(x) = v(x, t_j) = \frac{1}{\lambda(t_j)^{\frac{N-2}{2}}} u\big(\frac{x - x(t_j)}{\lambda(t_j)}, t_j\big)$. It follows from the compactness of $\overline{K}$ in $\dot{H}^1(\mathbb{R}^N)$ that there is a subsequence (still denoted by $\{v_j\}$) and $v_0 \in \dot{H}^1(\mathbb{R}^N)$ such that

$$v_j \longrightarrow v_0 \quad \text{in} \quad \dot{H}^1(\mathbb{R}^N).$$

Then it holds

$$u\Big(y - \frac{x(t_j)}{\lambda(t_j)}, t_j\Big) = \lambda(t_j)^{\frac{N-2}{2}} v_j(\lambda(t_j) y) \longrightarrow A^{\frac{N-2}{2}} v_0(Ay) \quad \text{in} \quad \dot{H}^1(\mathbb{R}^N). \tag{2.3}$$

If $A = 0$, it follows from (2.3) that $u(y - \frac{x(t_j)}{\lambda(t_j)}, t_j) \longrightarrow 0$ in $\dot{H}^1(\mathbb{R}^N)$. So

$$\|\nabla u(t_j)\|_{L^2(\mathbb{R}^N)} \longrightarrow 0 \quad \text{as} \quad t_j \longrightarrow T_+(u_0). \tag{2.4}$$

Using the conservation of energy (1.5), one has

$$E(u_0) = E(u(t_j)) \longrightarrow 0 \quad \text{as} \quad t_j \longrightarrow T_+(u_0). \tag{2.5}$$

In addition, (iii) in Remark 1.2 and the assumption: $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$ yield

$$\|\nabla u_0\|_{L^2}^2 \leq N E(u_0). \tag{2.6}$$

Combining (2.5) and (2.6), we infer $\|\nabla u_0\|_{L^2} = 0$. So $u_0 \equiv 0$ in $\mathbb{R}^N$. Using the conservation of charge (1.4), one has for $t \in [0, T_+(u_0))$

$$\int_{\mathbb{R}^N} |u(t,x)|^2 dx = \int_{\mathbb{R}^N} |u_0(x)|^2 dx = 0,$$

which implies us that $u \equiv 0$ a.e. on $\mathbb{R}^N \times [0, T_+(u_0))$. This is a contradiction with (2.1).

If $\lim_{j\to\infty} \lambda(t_j) = A \in (0, +\infty)$. Let $h(x,t)$ be the solution of (1.3) (which is guaranteed by Remark 2.8 in [19]) on the interval $I_\eta = (T_+(u_0) - \eta, T_+(u_0) + \eta)$, $h(x, T_+(u_0)) = A^{\frac{N-2}{2}} v_0(Ax)$, $\|h\|_{S(I_\eta)} < +\infty$, where $\eta = \eta(\|\nabla v_0\|_{L^2(\mathbb{R}^N)})$.

Let $h_j(x,t)$ be the solution of (1.3) with $h_j(x, T_+(u_0)) = u(x - \frac{x(t_j)}{\lambda(t_j)}, t_j)$. Then the convergence in (2.3) and the continuous dependence on the initial data (see Remark 2.17 in [19]) imply that

$$\|h_j - h\|_{S(I_{\frac{\eta}{2}})} \longrightarrow 0 \quad \text{as} \quad j \longrightarrow +\infty.$$

Then

$$\sup_j \|h_j\|_{S(I_{\frac{\eta}{2}})} < +\infty. \tag{2.7}$$

In addition, the uniqueness theorem on the strong solution of (1.3) (see Definition 2.10 in [19]) yields

$$h_j(x,t) = u\Big(x - \frac{x(t_j)}{\lambda(t_j)}, t + t_j - T_+(u_0)\Big) \quad \text{for every} \quad t \in I_{\frac{\eta}{2}}. \tag{2.8}$$

Combining (2.7) and (2.8), we get

$$+\infty > \sup_j \|h_j\|_{S(I_{\frac{\eta}{2}})} \geq \liminf_{j\longrightarrow\infty} \|u\|_{S(t_j - \frac{\eta}{2}, t_j + \frac{\eta}{2})} \geq \|u\|_{S(T_+(u_0) - \frac{\eta}{2}, T_+(u_0))} = +\infty,$$

which contradicts (2.1).

From the above arguments, we know that (2.2) holds.

Let $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi(x) = \psi(|x|)$, $\psi \equiv 1$ for $|x| \leq 1$ $\psi \equiv 0$ for $|x| \geq 2$ $|\nabla\psi| \leq 2$. Define $\psi_R(x) = \psi(\frac{x}{R})$ and

$$y_R(t) = \int_{\mathbb{R}^N} |u(x,t)|^2 \psi_R(x) dx, \quad \forall t \in [0, T_+(u_0)).$$

Then from Lemma 2.1 and the conservation of charge (1.4), one has

$$\begin{aligned} |y_R'(t)| &\leq 2\Big|Im \int_{\mathbb{R}^N} \overline{u}\nabla u \cdot \nabla\psi_R(x) dx\Big| \leq \\ &\leq \frac{C}{R}\Big(\int_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |u(x,t)|^2 dx\Big)^{\frac{1}{2}} \leq \\ &\leq \frac{C}{R}\Big(\int_{\mathbb{R}^N} |\nabla W(x)|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |u_0(x)|^2 dx\Big)^{\frac{1}{2}}. \end{aligned} \tag{2.9}$$

Note that $u(x,t)=\lambda(t)^{\frac{N-2}{2}}v(\lambda(t)x+x(t),t)$, we deduce for any $R>0$, $\epsilon>0$

$$\begin{aligned}\int\limits_{|x|<R}|u(x,t)|^2dx&=\lambda(t)^{-2}\int\limits_{|y-x(t)|<R\lambda(t)}|v(y,t)|^2dy=\\&=\lambda(t)^{-2}\int\limits_{B(x(t),R\lambda(t))\bigcap B(0,\epsilon R\lambda(t))}|v(y,t)|^2dy+\\&\quad+\lambda(t)^{-2}\int\limits_{B(x(t),R\lambda(t))\setminus B(0,\epsilon R\lambda(t))}|v(y,t)|^2dy.\end{aligned}\tag{2.10}$$

Using Hölder inequality and the compactness property of $\overline{K}$ in $\dot{H}^1(\mathbb{R}^N)$, we conclude from (2.2) that

$$\begin{aligned}\lambda(t)^{-2}\int\limits_{B(x(t),R\lambda(t))\bigcap B(0,\epsilon R\lambda(t))}|v(y,t)|^2dy&\le CR^2\epsilon^2\Big(\int\limits_{|y|\le\epsilon R\lambda(t)}|v(y,t)|^{2^*}dy\Big)^{\frac{2}{2^*}}\le\\&\le CR^2\epsilon^2\int\limits_{\mathbb{R}^N}|\nabla W|^2dx\end{aligned}\tag{2.11}$$

and

$$\lambda(t)^{-2}\int\limits_{B(x(t),R\lambda(t))\setminus B(0,\epsilon R\lambda(t))}|v(y,t)|^2dy\le CR^2\Big(\int\limits_{|y|\ge\epsilon R\lambda(t)}|v(y,t)|^{2^*}dy\Big)^{\frac{2}{2^*}}\longrightarrow 0\quad\text{as}\quad t\longrightarrow T_+(u_0).\tag{2.12}$$

Combining (2.10), (2.11) and (2.12), we derive for all $R>0$

$$\int\limits_{|x|<R}|u(x,t)|^2dx\longrightarrow 0\quad\text{as}\quad t\longrightarrow T_+(u_0),$$

and so

$$y_R(t)\longrightarrow 0\quad\text{as}\quad t\longrightarrow T_+(u_0).\tag{2.13}$$

From (2.9), (2.13), we obtain for any $t\in[0,T_+(u_0))$ and $R>0$

$$\begin{aligned}y_R(t)&=|y_R(t)-y_R(T_+(u_0))|\le\\&\le\frac{C}{R}(T_+(u_0)-t)\Big(\int\limits_{\mathbb{R}^N}|\nabla W(x)|^2dx\Big)^{\frac12}\Big(\int\limits_{\mathbb{R}^N}|u_0(x)|^2dx\Big)^{\frac12}.\end{aligned}\tag{2.14}$$

Let $R\longrightarrow+\infty$ in (2.14), we get

$$\int\limits_{\mathbb{R}^N}|u(t,x)|^2dx=0\quad\text{for each}\quad t\in[0,T_+(u_0)),$$

and then $u \equiv 0$ a.e. on $\mathbb{R}^N \times [0, T_+(u_0))$, which contradicts (2.1). Therefore, $T_+(u_0) = +\infty$.
Step 2. $u_0 \equiv 0$ in $\mathbb{R}^N$. If $u_0 \not\equiv 0$ in $\mathbb{R}^N$, it holds true that

$$\sup_{t\in[0,+\infty)} |x(t)| < +\infty. \tag{2.15}$$

In fact, assume that there exists an increasing sequence $\{t_j\}$, $t_j \longrightarrow +\infty (= T_+(u_0))$ as $j \longrightarrow +\infty$ such that

$$|x(t_j)| \longrightarrow +\infty \quad \text{as} \quad j \longrightarrow +\infty. \tag{2.16}$$

It follows from the Hardy inequality and the compactness property of $\overline{K}$ in $\dot{H}^1(\mathbb{R}^N)$ that for any $\epsilon > 0$, there exists a large number $M(\epsilon) > 0$ such that for any $M \geq M(\epsilon)$

$$\sup_{t\in[0,+\infty)} \int\limits_{|y|\geq M} \Big(|\nabla v(y,t)|^2 + |v(y,t)|^{2^*} \Big) dy < \epsilon. \tag{2.17}$$

Note that for any $Q > R > 0$ and $t \in [0, +\infty)$

$$\int\limits_{R<|x|<Q} |\nabla u(x,t)|^2 dx = \int\limits_{R\lambda(t)<|y-x(t)|<Q\lambda(t)} |\nabla v(y,t)|^2 dy. \tag{2.18}$$

In the next discussion, we analyze the three possible cases of the limit of the sequence $\{\frac{\lambda(t_j)}{|x(t_j)|}\}$ (select a subsequence if necessary).

(1) If $\lim\limits_{j\to+\infty} \frac{\lambda(t_j)}{|x(t_j)|} = 0$, then for any $Q > 0$

$$\lim_{j\to+\infty} \big(|x(t_j)| - Q\lambda(t_j) \big) = \lim_{j\to+\infty} \Big(|x(t_j)| \Big(1 - \frac{Q\lambda(t_j)}{|x(t_j)|} \Big) \Big) = +\infty > M(\epsilon).$$

From (2.17) and (2.18), one has for any $Q > 0$

$$\begin{aligned} \lim_{j\to+\infty} \int\limits_{|x|<Q} |\nabla u(x,t_j)|^2 dx &\leq \lim_{j\to+\infty} \int\limits_{|y|\geq |x(t_j)|-Q\lambda(t_j)} |\nabla v(y,t_j)|^2 dy \leq \\ &\leq \sup_{t\in[0,+\infty)} \int\limits_{|y|\geq M(\epsilon)} |\nabla v(y,t)|^2 dy \leq \epsilon. \end{aligned} \tag{2.19}$$

Similarly, using the Sobolev inequality, we infer that for any $Q > 0$

$$\lim_{j\to+\infty} \int\limits_{|x|<Q} |u(x,t_j)|^{2^*} dx \leq \epsilon. \tag{2.20}$$

Combination of (2.19), (2.20) yields that (selecting a subsequence if necessary) for any $Q > 0$

$$u(x,t_j) \longrightarrow 0 \quad \text{a.e. on} \quad \{x \in \mathbb{R}^N; \ \ |x| < Q\} \quad \text{as} \quad j \longrightarrow +\infty. \tag{2.21}$$

On the other hand, it follows from the conservation of charge (1.4) and Lemma 2.2 that

$$\sup_j \|u(t_j)\|_{H^1} < \infty.$$

Up to a subsequence if necessary,

$$u(x,t_j) \rightharpoonup \widetilde{u} \quad \text{weakly in} \quad H^1(\mathbb{R}^N) \quad \text{and} \quad L^2(\mathbb{R}^N) \quad \text{as} \quad j \longrightarrow +\infty; \tag{2.22}$$

and

$$u(x,t_j) \longrightarrow \widetilde{u} \quad \text{a.e. on} \quad \mathbb{R}^N \quad \text{as} \quad j \longrightarrow +\infty. \tag{2.23}$$

From (2.21) and (2.23), we infer that

$$\widetilde{u} = 0 \quad \text{a.e. on} \quad \{x \in \mathbb{R}^N : |x| < Q\} \quad \text{as} \quad j \longrightarrow +\infty;$$

and so

$$\widetilde{u} = 0 \quad \text{a.e. on} \quad \mathbb{R}^N \quad \text{due to the arbitrariness of} \quad Q. \tag{2.24}$$

From (2.21)–(2.24), up to a subsequence if necessary, we derive

$$u(x,t_j) \longrightarrow 0 \quad \text{strongly in} \quad L^2_{loc}(\mathbb{R}^N) \quad as \quad j \longrightarrow +\infty. \tag{2.25}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be the given real-valued function in (1.6). Then it follows from assumption (1.6) and Lemma 2.1 that for any $t > 0$

$$\int\limits_{\mathbb{R}^N} \varphi(x)|u(x,t)|^2 dx \geq \int\limits_{\mathbb{R}^N} \varphi(x)|u_0(x)|^2 dx. \tag{2.26}$$

Letting $t = t_j \longrightarrow +\infty$ in (2.26), together with (2.25), we deduce that

$$\int\limits_{\mathbb{R}^N} \varphi(x)|u_0(x)|^2 dx \leq 0,$$

which is a contradiction because of the assumption: $\int_{\mathbb{R}^N} \varphi(x)|u_0(x)|^2 dx > 0$.

(2) If $\lim\limits_{j\to+\infty} \frac{\lambda(t_j)}{|x(t_j)|} \in (0,+\infty)$, there exist $R > 0$ (which is independent of j, ϵ) and $j_1 = j_1(\epsilon) > 0$ such that $R\frac{\lambda(t_j)}{|x(t_j)|} \geq 2$ and $|x(t_j)| \geq M(\epsilon)$ for any $j \geq j_1$. Then from (2.17) and (2.18), one gets for any $j \geq j_1$,

$$\begin{aligned} \int\limits_{|x|>R} |\nabla u(x,t_j)|^2 dx &\leq \int\limits_{|y|\geq (R\frac{\lambda(t_j)}{|x(t_j)|}-1)|x(t_j)|} |\nabla v(y,t_j)|^2 dy \leq \\ &\leq \sup_{t\in[0,+\infty)} \int\limits_{|y|\geq M(\epsilon)} |\nabla v(y,t)|^2 dy \leq \epsilon. \end{aligned} \tag{2.27}$$

If $\lim\limits_{j\to+\infty} \frac{\lambda(t_j)}{|x(t_j)|} = +\infty$, there exists $j_2 = j_2(\epsilon) > 0$ such that $(\frac{\lambda(t_j)}{|x(t_j)|} - 1)|x(t_j)| \geq M(\epsilon)$ for any $j \geq j_2$. Then from (2.17) and (2.18), we derive for any $j \geq j_2$,

$$\begin{aligned} \int\limits_{|x|>1} |\nabla u(x,t_j)|^2 dx \leq & \int\limits_{|y|\geq(\frac{\lambda(t_j)}{|x(t_j)|}-1)|x(t_j)|} |\nabla v(y,t_j)|^2 dy \leq \\ & \leq \sup_{t\in[0,+\infty)} \int\limits_{|y|\geq M(\epsilon)} |\nabla v(y,t)|^2 dy \leq \epsilon. \end{aligned} \tag{2.28}$$

Set $J = \max\{j_1, j_2\}$. From (2.27) and (2.28), we conclude that there exists a positive number R, which is independent of j, ϵ, such that for any $j \geq J$

$$\int\limits_{|x|>R} |\nabla u(x,t_j)|^2 dx \leq \epsilon. \tag{2.29}$$

Using the Sobolev inequality and the Hardy inequality, after a similar argument, we conclude for any $j \geq J$

$$\int\limits_{|x|>R} |u(x,t_j)|^{2^*} dx \leq C(\epsilon), \quad \text{where} \quad C(\epsilon) \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0. \tag{2.30}$$

Here we take the same symbols R, J in (2.29) and (2.30) for the sake of simplicity.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi(x) = \varphi(|x|)$, $\varphi \equiv |x|^2$ for $|x| \leq 1$; $\varphi \equiv 0$ for $|x| \geq 2$. Define $\varphi_R(x) = R^2\varphi(\frac{x}{R})$ and

$$z_R(t) = \int\limits_{\mathbb{R}^N} |u(x,t)|^2 \varphi_R(x) dx, \quad \forall t \in [0,+\infty).$$

It follows from Lemmas 2.1, 2.2 and the Hardy inequality that for any $t \in [0,+\infty)$

$$\begin{aligned} |z_R'(t)| \leq 2\Big| Im \int\limits_{\mathbb{R}^N} \overline{u}\nabla u \cdot \nabla\varphi_R(x) dx\Big| \leq \\ \leq CR^2\Big(\int\limits_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\mathbb{R}^N} \frac{|u(x,t)|^2}{|x|^2} dx\Big)^{\frac{1}{2}} CR^2 \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx. \end{aligned} \tag{2.31}$$

From (2.29), (2.30) and Lemma 2.2, one has for any $j \geq J$

$$8\int\limits_{|x|\leq R} \big(|\nabla u(x,t_j)|^2 - |u(x,t_j)|^{2^*}\big) dx \geq C(\delta_0) \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx, \tag{2.32}$$

where R is independent of j.

From (2.29), (2.30), (2.32) and Lemmas 2.1, 2.2, we obtain for any $j \geq J$

$$\begin{aligned} z_R''(t_j) &= 4\int_{\mathbb{R}^N} \varphi_R''(|x|)|\nabla u(x,t_j)|^2 dx - \frac{4}{N}\int_{\mathbb{R}^N} \Delta\varphi_R(|x|)|u(x,t_j)|^{2^*} dx - \\ &\quad - \int_{\mathbb{R}^N} \Delta^2\varphi_R(|x|)|u(x,t_j)|^2 dx \geq \\ &\geq 8\int_{|x|\leq R} \big(|\nabla u(x,t_j)|^2 - |u(x,t_j)|^{2^*}\big)dx - \\ &\quad - C\int_{|x|>R} \big(|\nabla u(x,t_j)|^2 + |u(x,t_j)|^{2^*}\big)dx - \\ &\quad - C\int_{R\leq|x|\leq 2R} \big(|u(x,t_j)|^{2^*}\big)^{\frac{2}{2^*}} dx \geq C\int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx, \end{aligned} \tag{2.33}$$

where R is given in (2.31), and independent of j.

Combining (2.31), (2.32) and (2.33), we conclude for any $j \geq J$

$$\begin{aligned} CR^2\int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx &\geq |z_R'(2t_j) - z_R'(t_j)| = \\ &= t_j\int_0^1 z_R''(2st_j + (1-s)t_j)ds \geq Ct_j\int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx, \end{aligned}$$

from which we get a contradiction if $j \geq J$ is sufficiently large, because $t_j \longrightarrow +\infty$ as $j \longrightarrow +\infty$, and R is independent of j. Here we have used the fact: replacing t_j by any t with $t \geq t_j$, $j \geq J$, (2.33) still holds. This is not difficult to verify because the sequence $\{t_j\}$ is taken to be increasing on j.

Whence (2.15) holds. Now we claim that there exists a positive number C_0 (which is independent of t) such that

$$\lambda(t) \geq C_0 \quad \text{for any} \quad t \in [0,+\infty). \tag{2.34}$$

We present a proof by contradiction. Assume that there is a sequence $\{t_m\}$, $t_m \longrightarrow +\infty$ as $m \longrightarrow +\infty$ such that

$$\lambda(t_m) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty.$$

Observe that $u(x,t) = \lambda(t)^{\frac{N-2}{2}} v(\lambda(t)x + x(t), t)$. From the conservation of charge (1.4), one has

$$\int_{\mathbb{R}^N} |v(x,t_m)|^2 dx = \lambda(t_m)^2\int_{\mathbb{R}^N} |u(x,t_m)|^2 dx = \lambda(t_m)^2\int_{\mathbb{R}^N} |u_0(x)|^2 dx,$$

which implies that

$$v(x,t_m) \longrightarrow 0 \quad \text{a.e. on} \quad \mathbb{R}^N \quad \text{as} \quad m \longrightarrow \infty.$$

Whence from the compactness property of the set $\overline{K}$ in $\dot{H}^1(\mathbb{R}^N)$, we can find a subsequence of $\{v(x,t_m)\}$ (still denoted by $\{v(x,t_m)\}$) such that

$$v(x,t_m) \longrightarrow 0 \quad \text{in} \quad \dot{H}^1(\mathbb{R}^N) \quad \text{as} \quad m \longrightarrow \infty. \tag{2.35}$$

However, one gets from Lemma 2.2

$$\int\limits_{\mathbb{R}^N} |\nabla v(x,t_m)|^2 dx = \int\limits_{\mathbb{R}^N} |\nabla u(x,t_m)|^2 dx \simeq \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx > 0. \tag{2.36}$$

This contradicts (2.35) by passing the limit $m \longrightarrow \infty$ in (2.36). Therefore (2.34) holds.

From (2.15) and (2.34), we conclude that for any $t \in [0, +T_+(u_0))$ and $R > 0$

$$\begin{aligned}\int\limits_{|x|>R} |\nabla u(x,t)|^2 dx &= \int\limits_{|y-x(t)|>R\lambda(t)} |\nabla v(y,t)|^2 dy \le \\ &\le \int\limits_{|y|>R\lambda(t)-|x(t)|} |\nabla v(y,t)|^2 dy \le \int\limits_{|y|>CR-C} |\nabla v(y,t)|^2 dy.\end{aligned}$$

Whence it follows from (2.34) that for $\epsilon > 0$, there exists a large number $R(\epsilon) > 0$ such that for any $t \in [0, +\infty)$

$$\int\limits_{|x|>R(\epsilon)} \big(|\nabla u(x,t)|^2 + |u(x,t)|^{2^*}\big) dx < \epsilon. \tag{2.37}$$

In addition, Lemma 2.2 implies that

$$8 \int\limits_{\mathbb{R}^N} \big(|\nabla u(x,t)|^2 - |u(x,t)|^{2^*}\big) dx \ge \widetilde{C}_{\delta_0} \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx, \tag{2.38}$$

It follows from (2.37) and (2.38) that there exists a sufficiently large number $M_0 > 0$ such that for all $t \in [0, +\infty)$

$$8 \int\limits_{|x|\le M_0} \big(|\nabla u(x,t)|^2 - |u(x,t)|^{2^*}\big) dx \ge C \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx, \tag{2.39}$$

where we take $\epsilon = \epsilon_0 \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx$ in (2.37) with $\epsilon_0 > 0$ suitably small.

Let $z_R(t)$ be defined as in the above. From Lemma 2.1, one has for any $t \in [0, +\infty)$

$$|z_R'(t) - z_R'(0)| \le CR^2 \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx. \tag{2.40}$$

From (2.40) and Lemmas 2.1, 2.2, we obtain for every $t \in [0, +\infty)$

$$\begin{aligned} z''_{M_0}(t) = 4 \int\limits_{\mathbb{R}^N} \varphi''_{M_0}(|x|)|\nabla u(x,t)|^2 dx - \frac{4}{N} \int\limits_{\mathbb{R}^N} \Delta\varphi_{M_0}(|x|)|u(x,t)|^{2^*} dx - \\ - \int\limits_{\mathbb{R}^N} \Delta^2 \varphi_{M_0}(|x|)|u(x,t)|^2 dx \geq \\ \geq 8 \int\limits_{|x|\leq M_0} \big(|\nabla u(x,t)|^2 - |u(x,t)|^{2^*}\big) dx - \\ - C \int\limits_{|x|>M_0} \big(|\nabla u(x,t)|^2 + |u(x,t)|^{2^*}\big) dx - \\ - C \int\limits_{M_0\leq|x|\leq 2M_0} \big(|u(x,t)|^{2^*}\big)^{\frac{2}{2^*}} dx \geq C \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx. \end{aligned} \tag{2.41}$$

Combining (2.40) and (2.41), we obtain for every $t \in [0, +\infty)$

$$CM_0^2 \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx \geq |z'_{M_0}(t) - z'_{M_0}(0)| = \int\limits_0^t z''_{M_0}(s) ds \geq Ct \int\limits_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx,$$

from which we get a contradiction if $t > 0$ is large enough unless $\int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx = 0$.

From the above argument of *Steps 1, 2*, we complete the proof of Theorem 2.3. $\square$

Proof of Theorem 1.1. We first introduce notation (see [19]): $(SC)(u_0)$ holds if for the particular function u_0 with $\int_{\mathbb{R}^N} |\nabla u_0|^2 dx < \int_{\mathbb{R}^N} |\nabla W|^2 dx$ and $E(u_0) < E(W)$. Let u be the corresponding strong solution of problem (1.3) with maximal interval of existence I, then $I = (-\infty, +\infty)$ and $\|u\|_{S((-\infty,+\infty))} < \infty$, where $\|\cdot\|_{S(I)} = \|\cdot\|_{L^{\frac{2(N+2)}{N-2}}(I, L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N))}$.

Note that if $\|\nabla u_0\|_{L^2(\mathbb{R}^N)} \leq \delta$, $(SC)(u_0)$ holds. Whence there exists a number E_C with $\delta \leq E_C \leq E(W)$ such that if u_0 is as in $(SC)(u_0)$ and $E(u_0) < E_C$, $(SC)(u_0)$ holds and E_C is optimal with this property.

From Remark 2.8 in [19] and the uniqueness theory on strong solutions of (1.3) (see Definition 2.10 in [19]), we know that problem (1.3) admits a unique maximal strong solution $u \in ((-T_-(u_0), T_+(u_0)), H^1(\mathbb{R}^N))$. If $T_+(u_0) < \infty$ then by Lemma 2.11 in [19], $\|u\|_{S(I_+)} = +\infty$, where $I_+ = [0, T_+(u_0)]$. By the definition of E_C, we infer that $E(u_0) \geq E_C$. If $E(u_0) = E_C$, then by Proposition 4.2 in [19], there exists $x(t) \in \mathbb{R}^N$ and $\lambda(t) \in \mathbb{R}^+$ such that

$$K = \Big\{ v(x,t) = \frac{1}{\lambda(t)^{\frac{N-2}{2}}} u\Big(\frac{x - x(t)}{\lambda(t)}, t\Big) : \ t \in I_+ \Big\}$$

has the property that $\overline{K}$ is compact in $\dot{H}^1(\mathbb{R}^N)$. Therefore it follows from Theorem 2.3 that $T_+(u_0) = +\infty$, $u_0 \equiv 0$ in $\mathbb{R}^N$, which is a contradiction (we may always assume $u_0 \not\equiv 0$ in $\mathbb{R}^N$. Otherwise, the uniqueness theory on strong solutions of (1.3) in Definition 2.10 in [19] implies that problem (1.3) has only a trivial (global) solution).

If $E(u_0) > E_C$. Note that $E(su_0) \longrightarrow 0$ as $s \longrightarrow 0$, there exists $s_0 \in (0,1)$ such that $E(s_0u_0) = E_C$. Repeating the proof in the case $E(u_0) = E_C$, we also infer $u_0 \equiv 0$ in $\mathbb{R}^N$, which is a contradiction. Similarly, a contradiction appears if $T_-(u_0) < \infty$.

From the above arguments, we conclude that (SC) holds. That is, $T_-(u_0) = T_-(u_0) = +\infty$ and $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$, $u \in L^{\frac{2(N+2)}{N-2}}(\mathbb{R}, L^{\frac{2N(N+2)}{N^2+4}})$. Moreover from Remark 2.8 in [19] and following the proof of Theorem 2.5 in [19], $\nabla u \in L^{\frac{2(N+2)}{N-2}}(\mathbb{R}, L^{\frac{2N(N+2)}{N^2+4}})$.

Note that

$$u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}|u(s)|^{\frac{4}{N-2}}u(s)ds.$$

Set $\mathcal{F}(t) = e^{it\Delta}$. Then the solution u can be rewritten as

$$u(t) = \mathcal{F}(t)u_0 + i\int_0^t \mathcal{F}(t-s)|u(s)|^{\frac{4}{N-2}}u(s)ds.$$

Let $v(t) = \mathcal{F}(-t)u(t)$. It follows from the Strichartz estimates (see [4, 21]) that for any $0 < \tau < t$

$$\begin{aligned}
&\|v(t) - v(\tau)\|_{H^1} = \\
&= \|\mathcal{F}(t)(v(t) - v(\tau))\|_{H^1} = \|i\int_\tau^t \mathcal{F}(t-s)|u(s)|^{\frac{4}{N-2}}u(s)ds\|_{H^1} \leq \\
&\leq C\Big(\||u|^{\frac{4}{N-2}}u\|_{L^2((\tau,t),L^{\frac{2N}{N+2}}(\mathbb{R}^N))} + \|\nabla\big(|u|^{\frac{4}{N-2}}u\big)\|_{L^2((\tau,t),L^{\frac{2N}{N+2}}(\mathbb{R}^N))}\Big) \leq \\
&\leq C\|u\|_{S((\tau,t))}^{\frac{4}{N-2}}\Big(\|u\|_{W((\tau,t))} + \|\nabla u\|_{W((\tau,t))}\Big),
\end{aligned}$$

where $\|u\|_{S(I)} = \|u\|_{L^{\frac{2(N+2)}{N-2}}(I,L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N))}$, $\|u\|_{W(I)} = \|u\|_{L^{\frac{2(N+2)}{N-2}}(I,L^{\frac{2N(N+2)}{N^2+4}}(\mathbb{R}^N))}$, and the Sobolev inequality is used: $\|u\|_{S(I)} \leq C\|u\|_{W(I)}$, $\forall I \subseteq \mathbb{R}$.

Whence $\|v(t) - v(\tau)\|_{H^1} \longrightarrow 0$ as $\tau, t \longrightarrow +\infty$. Therefore, there exists $u_+ \in H^1(\mathbb{R}^N)$ such that $v(t) \longrightarrow u_+$ in $H^1(\mathbb{R}^N)$ as $t \longrightarrow +\infty$. So

$$\begin{aligned}
&\|u(t) - e^{it\Delta}u_+\|_{H^1(\mathbb{R}^N)} = \\
&= \|\mathcal{F}(t)(v(t) - u_+)\|_{H^1(\mathbb{R}^N)} = \|v(t) - u_+\|_{H^1(\mathbb{R}^N)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty.
\end{aligned}$$

Similarly there exists $u_- \in H^1(\mathbb{R}^N)$ such that

$$\|u(t) - e^{it\Delta}u_-\|_{H^1(\mathbb{R}^N)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow -\infty.$$

Here it is not difficult to verify that

$$u_+ = u_0 + i\int_0^{+\infty} e^{-is\Delta}|u(s)|^{\frac{4}{N-2}}u(s)ds, \quad u_- = u_0 - i\int_{-\infty}^0 e^{-is\Delta}|u(s)|^{\frac{4}{N-2}}u(s)ds. \quad \square$$

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