

**ANALYSIS  
OF INTEGRODIFFERENTIAL CONTROL SYSTEM  
WITH PULSE-WIDTH MODULATED SAMPLER  
ON BANACH SPACES**

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**Abstract.** This paper studies steady-state control and stability for a class of integrodifferential control system with pulse-width modulated sampler on Banach spaces. The existence and stability of the steady-state for the integrodifferential control system with pulse-width modulated sampler are given. An example is given to illustrate the theory.

**Keywords:** integrodifferential system, pulse-width modulated sampler, steady-state control, steady-state stability.

**Mathematics Subject Classification:** 34G20, 93C25, 93C57.

## 1. INTRODUCTION

In design of distributed parameter control systems, one of the important problems is to choose controller and actuator. As the dimension of an industrial controller in actual applications is finite, it restricts us to consider the distributed parameter system with a finite dimensional output. In industrial process control systems on-off actuators have been in engineer's good graces because of the cheap price and the high reliability.

The interest in the pulse-width sampler control systems was aroused as early as 1960s. It was motivated by applications to engineering problems and neural nets modeling. In modern times, the development of neurocomputers promises a rebirth of interest in this field. The theory of pulse-width sampler control systems is treated as a very important branch of engineering and mathematics. Nevertheless, it can supply a technical-minded mathematician with a number of new and interesting problems of mathematical nature. There are some results such as steady-state control, stability analysis, robust control of pulse-width sampler systems [1–9], integral control by variable sampling based on steady-state data and adaptive sampled-data integral control [10–15].

However, to our knowledge, integrodifferential control systems with pulse-width modulated sampler on infinite dimensional spaces have not been investigated extensively. In this paper, we will be concerned with control system governed by a class of integrodifferential equation

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), (Sx)(t)) + Cu(t), \\ z(t) = K_1x(t), \end{cases} \quad (1.1)$$

where state variable  $x(\cdot)$  takes value in a reflexive Banach space  $X$ ,  $A$  is the infinitesimal generator of an exponentially stable  $C_0$ -semigroup  $\{T(t) : t \geq 0\}$  on  $X$ .  $f : [0, \infty) \times X \times X \rightarrow X$  is continuous and bounded on bounded subsets of  $[0, \infty) \times X \times X$ ,  $S$  is a nonlinear integral operator given by

$$(Sx)(t) = \int_0^t k(t, s)g(s, x(s))ds.$$

The function  $g : [0, +\infty) \times X \rightarrow X$  is continuous in  $t$ ,  $k \in C([0, +\infty) \times [0, +\infty), \mathbb{R})$ . Control variable  $u(t)$  is a  $q$  dimensional vector,  $u(t) \in \mathbb{R}^q$ ,  $C : \mathbb{R}^q \rightarrow X$  is a bounded linear operator.  $K_1 : X \rightarrow \mathbb{R}^p$  is a linear operator,  $z(t)$  is a  $p$  dimensional output of the system (1.1).

Suppose that control signal  $u(t)$  is the output of the  $q$  dimensional pulse-width sampler controller.  $v(t)$  is the input of the  $q$  dimensional pulse-width sampler controller, which is the output of some dynamical controller

$$\dot{v}(t) = Jv(t) + K_2z(t), \quad (1.2)$$

where  $J$  is a  $q \times q$  matrix,  $K_2$  is a  $q \times p$  matrix.  $J$  is determined by the dynamic characteristics of the controller,  $K_2$  called to be feedback matrix will be chosen and tuned by the designer. The output signal  $u(t) = (u_1(t), u_2(t), \dots, u_q(t))^T$  and the input signal  $v(t) = (v_1(t), v_2(t), \dots, v_q(t))^T$  of the pulse-width sampler satisfy the following dynamic relation:

$$u_i(t) = \begin{cases} \text{sign } \alpha_{n_i}, & nT_0 \leq t < (n + |\alpha_{n_i}|)T_0, \quad i = 1, 2, \dots, q; \\ 0, & (n + |\alpha_{n_i}|)T_0 \leq t < (n + 1)T_0, \quad n = 0, 1, \dots \end{cases} \quad (1.3)$$

and

$$\alpha_{n_i} = \begin{cases} v_i(nT_0), & |v_i(nT_0)| \leq 1, \quad i = 1, 2, \dots, q; \\ \text{sign } v_i(nT_0), & |v_i(nT_0)| \geq 1, \quad n = 0, 1, \dots, \end{cases} \quad (1.4)$$

where  $\alpha_{n_i}$  are numbers,  $T_0 > 0$  is the sampling period of the pulse-width sampler.

We end this introduction by giving some definitions.

**Definition 1.1.** The closed-loop system (1.1)–(1.4) is called to be a pulse-width sampling integrodifferential control system.

**Definition 1.2.** In the closed-loop system (1.1)–(1.4), the  $q$  dimensional vector  $\vec{\alpha}_n = (\alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_q})^T$  defined by (1.4) is called the duration ratio of the pulse-width sampler in the  $n$ -th sampling period,  $n = 0, 1, \dots$

We defined a closed cube  $\Omega$  in  $\mathbb{R}^q$  as

$$\Omega = \{ \vec{\alpha}_n = (\alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_q})^T \in \mathbb{R}^q \mid |\alpha_{n_i}| \leq 1, i = 1, 2, \dots, q \},$$

then we have  $\vec{\alpha}_n \in \Omega$ , for  $n = 0, 1, \dots$

**Definition 1.3.** In the closed-loop system (1.1)–(1.4), if there exists a  $q$  dimensional vector

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_q)^T \in \Omega,$$

and a corresponding periodicity rectangular-wave control signal  $u(t) = u(t, \vec{\alpha})$  defined by

$$u_i(t) = u_i(t, \vec{\alpha}) = \begin{cases} \text{sign } \alpha_i, & nT_0 \leq t < (n + |\alpha_i|)T_0, \quad i = 1, 2, \dots, q; \\ 0, & (n + |\alpha_i|)T_0 \leq t < (n + 1)T_0, \quad n = 0, 1, \dots \end{cases} \quad (1.5)$$

such that the closed-loop system (1.1)–(1.4) has a corresponding  $T_0$ -periodic trajectory  $x(\cdot) = x(\cdot, \vec{\alpha})$ :  $x(t + T_0, \vec{\alpha}) = x(t, \vec{\alpha})$ ,  $t \geq 0$ , then the control signal (1.5) is called to be a steady-state control with respect to the disturbance  $f$ . The  $T_0$ -periodic trajectory  $x(\cdot)$  is called a steady-state corresponding to the steady-state control  $u(\cdot)$  and the constant vector  $\vec{\alpha} \in \Omega$  of defining steady-state control (1.5) is called to be a steady-state duration ratio.

**Definition 1.4.** In the closed-loop system (1.1)–(1.4), if there exist some  $\vec{\alpha} \in \Omega$  such that

$$\lim_{n \rightarrow \infty} \vec{\alpha}_n = \vec{\alpha}, \quad \text{where } \vec{\alpha}_n = (\alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_q})^T, \quad \vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_q)^T.$$

Then system (1.1)–(1.4) is called to be steady-state stable with respect to the disturbance  $f$ .

System (1.1)–(1.4) is called steady-state stabilizability if we can choose a suitable  $T_0 > 0$  and  $K_2$  such that system (1.1)–(1.4) is steady-state stable with respect to the disturbance  $f$ .

## 2. PRELIMINARIES

In order to study system (1.1)–(1.4), we introduce the following assumptions:

[H1]  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $\{T(t) : t \geq 0\}$  on  $X$  with domain  $D(A)$ .

[H2]  $\{T(t) : t \geq 0\}$  is exponentially stable, that is, there exist  $K_0 > 0$  and  $\nu_0 > 0$  such that

$$\|T(t)\| \leq K_0 e^{-\nu_0 t}, \quad t > 0.$$

Here, we renorm the space  $X$  using the semigroup  $T(t)$  and  $e^{\nu_0 t}$ ,  $t \geq 0$ . Define

$$\|x\|_0 = \sup_{s \geq 0} \|e^{\nu_0 s} T(s)x\|. \quad (2.1)$$

It is obvious that

$$\|x\| \leq \|x\|_0 \leq K_0 \|x\|.$$

Thus,  $(X, \|\cdot\|_0)$  is topologically equivalent to  $(X, \|\cdot\|)$ . Clearly for all  $t \geq 0$ ,

$$\|T(t)x\|_0 = \sup_{0 \leq s, t} \|e^{-\nu_0 t} e^{\nu_0(t+s)} T(t+s)x\| \leq \sup_{0 \leq s} \|e^{-\nu_0 t} e^{\nu_0(s)} T(s)x\| \leq e^{-\nu_0 t} \|x\|_0.$$

Let  $\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))$  be the space of linear operators from  $\mathbb{R}^q$  to  $X$  (with norm  $\|\cdot\|_0$ ),  $\mathcal{L}_b((X, \|\cdot\|_0), \mathbb{R}^p)$  be the space of bounded linear operators from  $X$  (with norm  $\|\cdot\|_0$ ) to  $\mathbb{R}^p$ .

[H3] (1)  $f : [0, \infty) \times X \times X \rightarrow X$  is continuous and bounded on bounded subsets of  $[0, \infty) \times X \times X$ , and for any  $x_1, x_2, y_1, y_2 \in X$ , there exists a positive constant  $L_f > 0$  such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\|_0 \leq L_f (\|x_1 - x_2\|_0 + \|y_1 - y_2\|_0).$$

(2)  $f(t, x, y)$  is  $T_0$ -periodic in  $t$ , i.e.,

$$f(t + T_0, x, y) = f(t, x, y), \quad t \geq 0.$$

[H4] Control signal  $u(t)$  is a rectangular wave signal  $u(t, \vec{\alpha})$  with a period  $T_0$  defined by (1.5) for a given  $\vec{\alpha} \in \Omega$ .

[H5] (1)  $g : [0, +\infty) \times X \rightarrow X$  is continuous in  $t$  on  $[0, +\infty)$  and for all  $x_1, x_2 \in X$ , there exists a constant  $L_g > 0$  such that

$$\|g(t, x_1) - g(t, x_2)\|_0 \leq L_g \|x_1 - x_2\|_0.$$

(2)  $k \in C([0, +\infty) \times [0, +\infty), \mathbb{R})$ , are  $T_0$ -periodic in  $t$  and  $s$ , i.e.,

$$k(t + T_0, s + T_0) = k(t, s),$$

and  $g(s, x)$  is  $T_0$ -periodic in  $s$ , i.e.,

$$g(s + T_0, x) = g(s, x), \quad s \geq 0,$$

with

$$\int_0^{T_0} k(t, s) g(s, x) ds = 0, \quad t > T_0 \geq s \geq 0.$$

By assumption [H5], it is not difficult to verify the following result.

**Lemma 2.1.** *Operator  $S$  has the following properties:*

- (1)  $S : C([0, T_0], X) \rightarrow C([0, T_0], X)$ .
- (2) For all  $x_1, x_2 \in C([0, T_0], X)$ , we have

$$\|(Sx_1)(t) - (Sx_2)(t)\|_0 \leq L_g \|k\| T_0 \|(x_1)_t - (x_2)_t\|_{0,B}.$$

- (3) For all  $x \in C([0, T_0], X)$ ,

$$(S(t + T_0))x(t) = (S(t))x(t).$$

**Remark 2.2.** The above results remain true by supposing that the kernel  $k(t, s)$  of the Volterra operator  $S$  is continuous in the domain  $\{(t, s) \in \mathbb{R}_+^2 : 0 \leq s \leq t\}$ .

It will need the following generalized Gronwall's inequality.

**Lemma 2.3.** *Let  $x \in C([0, T_0], X)$  and satisfy the following result:*

$$\|x(t)\|_0 \leq a + b \int_0^t \|x(s)\|_0 ds + c \int_0^t \|x_s\|_{0,B} ds, \quad t \in [0, T_0], \tag{2.2}$$

where  $a, b, c \geq 0$  are constants and

$$\|x_s\|_{0,B} = \sup_{0 \leq \xi \leq s} \left\{ \sup_{\xi \geq 0} \|e^{\nu_0 \xi} T(\xi)x(\xi)\| \right\}.$$

Then

$$\|x(t)\|_0 \leq ae^{(b+c)t}.$$

*Proof.* Using (2.2), we obtain

$$\|x(t)\|_{0,B} \leq a + (b + c) \int_0^t \|x_s\|_{0,B} ds.$$

Define

$$h(t) = a + (b + c) \int_0^t \|x_s\|_{0,B} ds,$$

we have

$$\dot{h}(t) = (b + c)\|x_t\|_{0,B} \leq (b + c)h(t), \quad \text{with } h(0) = a.$$

Thus,

$$h(t) \leq ae^{(b+c)t}.$$

As a result,

$$\|x(t)\|_0 \leq \|x_t\|_{0,B} \leq h(t) \leq ae^{(b+c)t}. \quad \square$$

## 3. STEADY-STATE CONTROL AND STABILITY

**Lemma 3.1.** *Let assumptions [H1]–[H5] hold and suppose that*

$$\nu_0 > L_f + L_f L_g \|k\| T_0.$$

For every  $u(t, \vec{\alpha})$ , system (1.1) has a unique  $T_0$ -periodic mild solution given by

$$x(t, \vec{\alpha}) = T(t)x_0 + \int_0^t T(t-\theta) \left[ f(\theta, x(\theta), \int_0^\theta k(\theta, s)g(s, x(s))ds) + Cu(\theta, \vec{\alpha}) \right] d\theta, \quad (3.1)$$

where  $x(T_0, \vec{\alpha}) = x_0$  and  $x(\cdot, \vec{\alpha})$  is exponentially stable.

*Proof.* We define a map  $H_S(t) : X \rightarrow X$  given by

$$H_S(t)(x_0) = T(t)x_0 + \int_0^t T(t-\theta) (f(\theta, x(\theta), (Sx)(\theta)) + Cu(\theta, \vec{\alpha})) d\theta.$$

For every  $x_1, x_2 \in X$ , it is easy to see that

$$\begin{aligned} & \|H_S(t)(x_1) - H_S(t)(x_2)\|_0 \leq \\ & \leq \|T(t)(x_1 - x_2)\|_0 + L_f \int_0^t \|T(t-\theta)(x_1(\theta) - x_2(\theta))\|_0 d\theta + \\ & + L_f \int_0^t \|T(t-\theta)((Sx_1)(\theta) - (Sx_2)(\theta))\|_0 d\theta \leq \\ & \leq e^{-\nu_0 t} \|x_1 - x_2\|_0 + L_f \int_0^t \|x_1(\theta) - x_2(\theta)\|_0 d\theta + \\ & + L_f L_g \|k\| T_0 \int_0^t \|(x_1)_\theta - (x_2)_\theta\|_{0,B} d\theta. \end{aligned}$$

By Lemma 2.3, we have

$$\|H_S(t)(x_1) - H_S(t)(x_2)\|_0 \leq e^{(L_f + L_f L_g \|k\| T_0 - \nu_0)t} \|x_1 - x_2\|_0, \quad t \in [0, T_0]. \quad (3.2)$$

Thus,

$$\|H_S(T_0)(x_1) - H_S(T_0)(x_2)\|_0 \leq e^{(L_f + L_f L_g \|k\| T_0 - \nu_0)T_0} \|x_1 - x_2\|_0.$$

It comes from the condition

$$\nu_0 > L_f + L_f L_g \|k\| T_0$$

that  $H_S(T_0)$  is a contraction map on Banach space  $(X, \|\cdot\|_0)$ . Thus,  $H_S(T_0)$  has a unique fixed point  $x^* \in X$ :

$$H_S(T_0)x^* = x^*. \tag{3.3}$$

Define  $y(t, \vec{\alpha}) = x(t + T_0, \vec{\alpha})$ , for  $t \geq 0$ ,

$$\begin{aligned} y(t, \vec{\alpha}) &= x(t + T_0, \vec{\alpha}) = H_S(t + T_0)(x^*) = \\ &= T(t)(H_S(T_0)x^*) + \\ &\quad + \int_{T_0}^{t+T_0} T(t + T_0 - \theta) \left[ f(\theta, x(\theta), \int_0^\theta k(\theta, s)g(s, x(s))ds) + Cu(\theta, \vec{\alpha}) \right] d\theta = \\ &= T(t)x^* + \\ &\quad + \int_0^t T(t + T_0 - \theta + T_0) \left[ f(\theta + T_0, x(\theta + T_0), \int_0^{\theta+T_0} k(\theta + T_0, s)g(s, x(s))ds) + \right. \\ &\quad \left. + Cu(\theta + T_0, \vec{\alpha}) \right] d(\theta + T_0) = \\ &= T(t)x^* + \\ &\quad + \int_0^t T(t - \theta) \left[ f(\theta, x(\theta), \int_0^{\theta+T_0} k(\theta + T_0, s)g(s, x(s))ds) + Cu(\theta + T_0, \vec{\alpha}) \right] d(\theta + T_0) = \\ &= T(t)x^* + \\ &\quad + \int_0^t T(t + T_0 - \theta - T_0) \left[ f(\theta + T_0, x(\theta + T_0), \int_0^{\theta+T_0} k(\theta + T_0, s)g(s, x(s))ds) + \right. \\ &\quad \left. + Cu(\theta + T_0, \vec{\alpha}) \right] d(\theta + T_0) = \\ &= T(t)x^* + \\ &\quad + \int_0^t T(t - \theta) \left[ f(\theta, x(\theta + T_0), \int_0^\theta k(\theta + T_0, s + T_0)g(s + T_0, x(s + T_0))d(s + T_0)) + \right. \\ &\quad \left. + Cu(\theta + T_0, \vec{\alpha}) \right] d(\theta + T_0) = \\ &= T(t)x^* + \\ &\quad + \int_0^t T(t - \theta) \left[ f(\theta, y(\theta), \int_0^\theta k(\theta, s)g(s, y(s))ds) + Cu(\theta, \vec{\alpha}) \right] d\theta = \\ &= H_S(t)(x^*), \end{aligned}$$

which implies that  $y$  is also a solution. Then, the uniqueness implies that  $x(\cdot, \vec{\alpha}) = x(\cdot, x^*)$  is just the  $T_0$ -periodic mild solution of system (1.1). Note that (3.1) and condition (3.2), we know  $x(\cdot)$  is exponentially stable.  $\square$

By Lemma 3.1, we have the following results.

**Theorem 3.2.** *Under the assumptions of Lemma 3.1, if the sampler periodic  $T_0$  has the following properties:*

$$i\omega_n \in \rho(J), \quad \omega_n = \frac{2n\pi}{T_0}, \quad n = 0, \pm 1, \pm 2, \dots, \quad (3.4)$$

where  $\rho(J)$  is the resolvent set of the matrix  $J$ ,  $i$  satisfies  $i^2 = -1$ , then the following open-loop control system

$$\begin{cases} \dot{x}(t, \vec{\alpha}) = Ax(t, \vec{\alpha}) + f(t, x(t), (Sx)(t)) + Cu(t, \vec{\alpha}), \\ z(t) = K_1x(t), \\ \dot{v}(t, \vec{\alpha}) = Jv(t, \vec{\alpha}) + K_2z(t, \vec{\alpha}), \end{cases} \quad (3.5)$$

has a unique  $T_0$ -periodic mild solution  $v(t, \vec{\alpha})$  given by

$$v(t, \vec{\alpha}) = e^{Jt} \left[ (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2z(s, \vec{\alpha}) ds \right] + \int_0^t e^{J(t-s)} K_2z(s, \vec{\alpha}) ds.$$

*Proof.* By (3.4), we know that  $e^{i\omega_n T_0} = e^{i2n\pi} = 1$ , that is  $1 \in \rho(e^{JT_0})$ . Thus  $(I - e^{JT_0})^{-1}$  exists and is bounded. It is not difficult to see that

$$v(t, \vec{\alpha}) = e^{Jt}v_0 + \int_0^t e^{J(t-s)} K_2z(s, \vec{\alpha}) ds, \quad (3.6)$$

where  $v_0 = v(0, \vec{\alpha})$ . Consider

$$y = (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2z(s, \vec{\alpha}) ds,$$

which is the unique solution of the following equation

$$y = e^{Jt}y + \int_0^t e^{J(t-s)} K_2z(s, \vec{\alpha}) ds.$$

Let

$$v_0 = y = (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2z(s, \vec{\alpha}) ds,$$

it comes from Lemma 3.1 that

$$z(t + T_0, \vec{\alpha}) = z(t, \vec{\alpha}), t \geq 0.$$

It is easy to verify that

$$v(t, \vec{\alpha}) = e^{Jt} \left[ (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 z(s, \vec{\alpha}) ds \right] + \int_0^t e^{J(t-s)} K_2 z(s, \vec{\alpha}) ds$$

is just the  $T_0$ -periodic mild solution  $v(t, \vec{\alpha})$  of the open-loop control system (3.5).

In order to discuss existence of steady-state for system (1.1), we define a map  $G: \Omega \in \mathbb{R}^q \rightarrow \mathbb{R}^q$  given by

$$G(\vec{\alpha}) = (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 K_1 x(s, \vec{\alpha}) ds, \vec{\alpha} \in \Omega,$$

where  $x(\cdot, \vec{\alpha})$  is the  $T_0$ -periodic mild solution of system (1.1) corresponding to  $\vec{\alpha} \in \Omega$ . □

**Lemma 3.3.** *Under the assumptions of Theorem 3.2, if*

$$1 - e^{(L_f + L_f L_g \|k\| T_0) T_0} \frac{L_f (1 + L_g \|k\| T_0)}{1 - e^{-\nu_0 T_0}} T_0 > 0, \tag{3.7}$$

there exists a constant  $\bar{M} > 0$  such that

$$\|G(\vec{\alpha}) - G(\vec{\alpha}')\| \leq \bar{M} \|K_2\| \|\vec{\alpha} - \vec{\alpha}'\|, \quad \vec{\alpha}, \vec{\alpha}' \in \Omega.$$

*Proof.* Let  $x_1(t, \vec{\alpha})$  and  $x_2(t, \vec{\alpha}')$  be the  $T_0$ -periodic mild solution of system (1.1) corresponding to  $\vec{\alpha}$  and  $\vec{\alpha}' \in \Omega$  respectively, then

$$\begin{aligned} x_1(0) - x_2(0) &= x_1(T_0) - x_2(T_0) = \\ &= T(T_0)(x_1(0) - x_2(0)) + \\ &\quad + \int_0^{T_0} T(T_0 - \theta) (f(\theta, x_1(\theta), (Sx_1)(\theta)) - f(\theta, x_2(\theta), (Sx_2)(\theta))) d\theta + \\ &\quad + \int_0^{T_0} T(T_0 - \theta) C(u(\theta, \vec{\alpha}) - u(\theta, \vec{\alpha}')) d\theta. \end{aligned}$$

Thus,

$$\begin{aligned}
& \|x_1(0) - x_2(0)\|_0 \leq \\
& \leq \|[I - T(T_0)]^{-1}\| L_f \left[ \int_0^{T_0} \|x_1(\theta) - x_2(\theta)\|_0 d\theta + \int_0^{T_0} \|(Sx_1)(\theta) - (Sx_2)(\theta)\|_0 d\theta \right] + \\
& \quad + \|[I - T(T_0)]^{-1}\| \|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} \int_0^{T_0} \|u(\theta, \vec{\alpha}) - u(\theta, \vec{\alpha})\|_{\mathbb{R}^q} d\theta \leq \\
& \leq \frac{1}{1 - e^{-\nu_0 T_0}} L_f \left[ \int_0^{T_0} \|x_1(\theta) - x_2(\theta)\|_0 d\theta + L_g \|k\| T_0 \int_0^{T_0} \|(x_1)_\theta - (x_2)_\theta\|_{0,B} d\theta \right] + \\
& \quad + \frac{1}{1 - e^{-\nu_0 T_0}} \|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} \int_0^{T_0} \|u(\theta, \vec{\alpha}) - u(\theta, \vec{\alpha})\|_{\mathbb{R}^q} d\theta.
\end{aligned}$$

For  $0 \leq t \leq T_0$ , we obtain

$$\begin{aligned}
& \|x_1(t, \vec{\alpha}) - x_2(t, \vec{\alpha})\|_0 \leq \\
& \leq \|x_1(0) - x_2(0)\|_0 + \\
& \quad + L_f \int_0^t \|x_1(\theta) - x_2(\theta)\|_0 d\theta + L_f L_g \|k\| T_0 \int_0^t \|(x_1)_\theta - (x_2)_\theta\|_{0,B} d\theta + \\
& \quad + \|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} \int_0^t \|u(\theta, \vec{\alpha}) - u(\theta, \vec{\alpha})\|_{\mathbb{R}^q} d\theta \leq \\
& \leq \frac{1}{1 - e^{-\nu_0 T_0}} L_f \left[ \int_0^{T_0} \|x_1(\theta) - x_2(\theta)\|_0 d\theta + L_g \|k\| T_0 \int_0^{T_0} \|(x_1)_\theta - (x_2)_\theta\|_{0,B} d\theta \right] + \\
& \quad + \|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} \left( \frac{1}{1 - e^{-\nu_0 T_0}} + 1 \right) \int_0^{T_0} \|u(\theta, \vec{\alpha}) - u(\theta, \vec{\alpha})\|_{\mathbb{R}^q} d\theta + \\
& \quad + L_f \int_0^t \|x_1(\theta) - x_2(\theta)\|_0 d\theta + L_f L_g \|k\| T_0 \int_0^t \|(x_1)_\theta - (x_2)_\theta\|_{0,B} d\theta.
\end{aligned}$$

By Lemma 2.3 again, we have

$$\begin{aligned} & \|x_1(t, \vec{\alpha}) - x_2(t, \vec{\alpha})\|_0 \leq \\ & \leq e^{(L_f + L_f L_g \|k\| T_0) T_0} \left\{ \frac{1}{1 - e^{-\nu_0 T_0}} L_f \left[ \int_0^{T_0} \|x_1(\theta) - x_2(\theta)\|_0 d\theta + \right. \right. \\ & \quad \left. \left. + L_g \|k\| T_0 \int_0^{T_0} \|(x_1)_\theta - (x_2)_\theta\|_{0,B} d\theta \right] + \right. \\ & \quad \left. + \|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} \left( \frac{1}{1 - e^{-\nu_0 T_0}} + 1 \right) \int_0^{T_0} \|u(\theta, \vec{\alpha}) - u(\theta, \vec{\alpha})\|_{\mathbb{R}^q} d\theta \right\}. \end{aligned}$$

Integrating from 0 to  $T_0$ , we obtain

$$\begin{aligned} \int_0^{T_0} \|x_1(t, \vec{\alpha}) - x_2(t, \vec{\alpha})\|_0 dt & \leq \frac{M_2}{M_1} \int_0^{T_0} \|u(\theta, \vec{\alpha}) - u(\theta, \vec{\alpha})\|_{\mathbb{R}^q} d\theta \leq \\ & \leq \frac{M_2}{M_1} \int_0^{T_0} \sum_{l=1}^q |u_l(\theta, \alpha_l) - u_l(\theta, \bar{\alpha}_l)| d\theta, \end{aligned}$$

where

$$\begin{aligned} M_1 & = 1 - e^{(L_f + L_f L_g \|k\| T_0) T_0} \frac{L_f (1 + L_g \|k\| T_0)}{1 - e^{-\nu_0 T_0}} > 0, \\ M_2 & = e^{(L_f + L_f L_g \|k\| T_0) T_0} \|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} \left( \frac{1}{1 - e^{-\nu_0 T_0}} + 1 \right). \end{aligned}$$

For  $\alpha_l \bar{\alpha}_l > 0$ , without loss of any generality, we suppose that  $0 < \alpha_l < \bar{\alpha}_l$ , then we have

$$\int_0^{T_0} |u_l(\theta, \alpha_l) - u_l(\theta, \bar{\alpha}_l)| d\theta \leq \int_{\alpha_l T_0}^{\bar{\alpha}_l T_0} |u_l(\theta, \alpha_l) - u_l(\theta, \bar{\alpha}_l)| d\theta \leq T_0 |\alpha_l - \bar{\alpha}_l|.$$

For  $\alpha_l \bar{\alpha}_l < 0$ , for example,  $\alpha_l < 0 < \bar{\alpha}_l$ ,  $|\bar{\alpha}_l| > \alpha_l$ , we have

$$\int_0^{T_0} |u_l(\theta, \alpha_l) - u_l(\theta, \bar{\alpha}_l)| d\theta \leq \int_{\alpha_l T_0}^{|\bar{\alpha}_l| T_0} |u_l(\theta, \alpha_l) - u_l(\theta, \bar{\alpha}_l)| d\theta \leq 2T_0 |\alpha_l - \bar{\alpha}_l|.$$

By elementally computation,

$$\begin{aligned}
& \|G(\vec{\alpha}) - G(\vec{\alpha}')\| \leq \\
& \leq \|(I - e^{JT_0})^{-1}\| \|e^{JT_0}\| \|K_2\| \|K_1\|_{\mathcal{L}_b((X, \|\cdot\|_0), \mathbb{R}^p)} \int_0^{T_0} \|x_1(s, \vec{\alpha}) - x_2(s, \vec{\alpha}')\|_0 ds \leq \\
& \leq \|(I - e^{JT_0})^{-1}\| \|e^{JT_0}\| \|K_2\| \|K_1\|_{\mathcal{L}_b((X, \|\cdot\|_0), \mathbb{R}^p)} \frac{M_2}{M_1} \int_0^{T_0} \|u(\theta, \vec{\alpha}) - u(\theta, \vec{\alpha}')\|_{\mathbb{R}^q} d\theta \leq \\
& \leq 2\|(I - e^{JT_0})^{-1}\| \|e^{JT_0}\| \|K_2\| \|K_1\|_{\mathcal{L}_b((X, \|\cdot\|_0), \mathbb{R}^p)} \frac{M_2}{M_1} T_0 \|\vec{\alpha} - \vec{\alpha}'\|.
\end{aligned}$$

By virtue of (3.7),

$$\bar{M} = 2\|(I - e^{JT_0})^{-1}\| \|e^{JT_0}\| \|K_1\|_{\mathcal{L}_b((X, \|\cdot\|_0), \mathbb{R}^p)} \frac{M_2}{M_1} T_0 > 0,$$

then

$$\|G(\vec{\alpha}) - G(\vec{\alpha}')\| \leq \bar{M} \|\vec{\alpha} - \vec{\alpha}'\|, \quad \vec{\alpha}, \vec{\alpha}' \in \Omega. \quad \square$$

By Lemma 3.3, we have the following result.

**Theorem 3.4.** *Under the assumptions Lemma 3.3, we can choose a suitable  $\|K_2\| > 0$  such that system (1.1)–(1.4) has a unique steady-state for any given  $f \in X$  and the fixed point of  $G$  is just the steady-state duration ratio.*

*Proof.* Let  $x(t, \vec{\alpha})$  be the  $T_0$ -periodic mild solution of system (1.1) corresponding to  $\vec{\alpha} \in \Omega$ , then

$$x(0) = x(T_0) = T(T_0)x(0) + \int_0^{T_0} T(T_0 - \theta)(f(\theta, x(\theta), (Sx)(\theta)) + Cu(\theta, \vec{\alpha}))d\theta,$$

that is,

$$x(0) = [I - T(T_0)]^{-1} \int_0^{T_0} T(T_0 - \theta)(f(\theta, x(\theta), (Sx)(\theta)) + Cu(\theta, \vec{\alpha}))d\theta.$$

It is obvious that

$$\begin{aligned}
\|x(0)\|_0 & \leq \frac{1}{1 - e^{-\nu_0 T_0}} \frac{1}{\nu_0} (1 - e^{-\nu_0 T_0})(MT_0 + q\|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} T_0) = \\
& = \frac{1}{\nu_0} (MT_0 + q\|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} T_0) \equiv M_3.
\end{aligned}$$

It comes from

$$\begin{aligned}
 G(\vec{\alpha}) &= \\
 &= (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 K_1 T(t) x(0) ds + \\
 &\quad + (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 K_1 \left( \int_0^t T(t-\theta) (f(\theta, x(\theta), (Sx)(\theta)) + Cu(\theta, \vec{\alpha})) d\theta \right) ds
 \end{aligned}$$

that

$$\|G(\vec{\alpha})\| \leq \overline{M}_4 \|K_2\|,$$

where

$$\overline{M}_4 = \|(I - e^{JT_0})^{-1}\| \|e^{JT_0}\| \|K_1\|_{\mathcal{L}_b(X, \mathbb{R}^p)} T_0 (M_3 + T_0 \|C\|_{\mathcal{L}_b(\mathbb{R}^q, (X, \|\cdot\|_0))} q + MT_0).$$

It is not difficult to see that  $G : \Omega \rightarrow \Omega$  is a contraction map when

$$0 < \|K_2\| < \frac{1}{\max(\overline{M}, \overline{M}_4)}.$$

By Banach fixed point theorem,  $G$  has a unique fixed point  $\vec{\alpha}^* \in \Omega$ . Obviously, the  $T_0$ -periodic mild solution of system (1.1) corresponding to  $\vec{\alpha}^*$  is just the unique steady-state.  $\square$

We end this section by discussing the steady-state stability. The following notations and assumptions are needed.

Denote two  $q \times q$  matrices

$$\begin{aligned}
 g(t, \eta) &= \int_s^t e^{J(t-s)} K_2 K_1 T(s - \eta) ds, \\
 g(t) &= g(t, 0) = \int_0^t e^{J(t-s)} K_2 K_1 T(s) ds.
 \end{aligned}$$

[H6]  $g(t)$  is  $q \times q$  matrices function continuously on  $[0, \infty)$  and satisfies

$$\|g(t)\| \leq M_g e^{-\nu_g t}, t \geq 0, \text{ where } M_g > 0 \text{ and } \nu_g > 0.$$

[H7] (1) There exists an inverse matrix  $[I - \int_0^\infty g(t) dt]^{-1}$ ,  $I$  is identity matrix, and

$$\left[ I - \int_0^\infty g(t) dt \right]^{-1} J^{-1} K_2 K_1 A^{-1} f \in \text{Int } \Omega, \quad f \in X.$$

(2) There exists some constant  $\delta > 0$  such that

$$\left\| J^{-1}K_2K_1A^{-1}f + \left( \int_0^\infty g(t)dt \right) \vec{\alpha} - v \right\| \geq \delta, v \in \mathbb{R}^q - \Omega,$$

where  $\vec{\alpha} = \text{Proj}_\Omega(v)$  is the projection of  $v$  on the closed convex set  $\Omega$ .

[H8] There exist  $M_J > 0, \nu_J > 0$  such that  $\|e^{Jt}\| \leq M_J e^{-\nu_J t}, t > 0$ .

**Theorem 3.5.** *System (1.1)–(1.4) is steady-state stable under the assumptions [H1], [H2], [H4], [H6], [H7] and [H8], that is, there exists a  $\vec{\alpha} \in \Omega$  given by*

$$\vec{\alpha} = [I - \int_0^\infty g(t)dt]^{-1} J^{-1}K_2K_1A^{-1}f,$$

such that

$$\lim_{n \rightarrow \infty} \vec{\alpha}_n = \vec{\alpha},$$

where

$$\vec{\alpha}_n = v(nT_0) = e^{JnT_0}v_0 + \int_0^{nT_0} e^{J(nT_0-\theta)} K_2K_1x(\theta)d\theta.$$

*Proof.* The output  $v(t)$  of dynamical controller (1.2) is

$$v(t) = v_e(t) + v_{K_1}(t), \quad (3.8)$$

where

$$v_e(t) = e^{Jt}v_0 + \int_0^t e^{J(t-\tau)} K_2K_1[T(\tau)(x_0) + A^{-1}f] - A^{-1}f d\tau,$$

$$v_{K_1}(t) = \int_0^t g(t, \eta)u(\eta)d\eta.$$

From [H6], [H8], it is easy to see that

$$\lim_{t \rightarrow \infty} v_e(t) = J^{-1}K_2K_1A^{-1}f.$$

From (3.8), after some calculation one can arrive at

$$\sum_{n=0}^{\infty} [v_e(nT_0) + \sum_{k=1}^{\infty} T_0g(kT_0 - \vec{\beta}_n T_0)\vec{\alpha}_n - v(nT_0)] = 0, \quad (3.9)$$

where  $\vec{\beta}_n = (\beta_{n_1}, \dots, \beta_{n_q})^T, 0 \leq \beta_{n_j} \leq |\alpha_{n_j}| \leq 1, j = 1, 2, \dots, q$ .

By Lemma 3.3 of [4], there exists some  $q \times q$  matrix  $E(T_0, n)$  such that

$$\sum_{k=1}^{\infty} T_0 g(kT_0 - \vec{\beta}_n T_0) = \int_0^{\infty} g(t) dt + E(T_0, n),$$

and

$$\lim_{T_0 \rightarrow 0} \|E(T_0, n)\| = 0 \text{ uniformly for } n = 0, 1, 2, \dots \tag{3.10}$$

From (3.9), we have

$$J^{-1}K_2K_1A^{-1}f + \lim_{n \rightarrow \infty} \left\{ \left[ \int_0^{\infty} g(t) dt + E(T_0, n) \right] \vec{\alpha}_n - v(nT_0) \right\}. \tag{3.11}$$

Let  $\delta > 0$  be the positive number in [H7]. It is not difficult to prove that there exists some  $T^* > 0$  and  $N^*(T^*, \delta) > 0$  such that

$$v(nT_0) = \vec{\alpha}_n, \quad 0 < T_0 < T^*, \quad n \geq N^*(T^*, \delta). \tag{3.12}$$

Combined (3.11) and (3.12), we obtain

$$\lim_{n \rightarrow \infty} \left[ I - \int_0^{\infty} g(t) dt - E(T_0, n) \right] \vec{\alpha}_n = J^{-1}K_2K_1A^{-1}f, \quad 0 < T_0 < T^*.$$

Note that by (3.10) and [H7], one can complete the rest of the proof. □

#### 4. EXAMPLE

Consider a heat conduction temperature control system with an one dimensional control and an one dimensional output

$$\begin{cases} \frac{\partial}{\partial t} x(t, y) = \frac{\partial^2 x(t, y)}{\partial^2 y} + bu(t) + \\ \quad + L_f x(t, y) + L_f \int_0^t \psi(s) \sin(t-s) L_g x(s, y) ds, \quad y \in (0, 2\pi), \quad t > 0, \\ x(t, 0) = x(t, 2\pi) = 0, \quad t \geq 0, \\ z(t) = \int_0^{2\pi} k_1 x(t, y) dy, \quad t \geq 0, \end{cases} \tag{4.1}$$

and the output  $v(t)$  satisfies

$$\dot{v} = \frac{1}{m} v(t) + k_2 z(t), \tag{4.2}$$

where  $m, b, b_k, k_1$  and  $k_2$  are positive constants.

Let  $X = L^2(0, 2\pi)$ . Define

$$(Ax)(y) = x''(y), \text{ for arbitrary } x \in D(A),$$

$$D(A) = \{x \in L^2(0, 2\pi) \mid x, x'' \in L^2(0, 2\pi), x(0) = x(2\pi) = 0\}.$$

Then  $A$  can generate an exponentially stable  $C_0$ -semigroup  $\{T(t), t \geq 0\}$  in  $X = L^2(0, 2\pi)$ , has the form

$$(T(t)x)(y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle x, \phi_n \rangle \phi_n(y),$$

where

$$\lambda_n = \left(\frac{n}{2}\right)^2, \quad \phi_n(y) = \sqrt{\frac{1}{\pi}} \sin \frac{ny}{2}, \quad \langle x, \phi_n \rangle = \int_0^{2\pi} x(y) \phi_n(y) dy.$$

Obviously,

$$\|T(t)\| \leq e^{-\frac{1}{4}t}, \quad t \geq 0.$$

Define  $x(\cdot)(y) = x(\cdot, y)$ ,  $C(\cdot)u(\cdot)(y) = bu(\cdot, y)$ ,  $k(t, s) = \psi(s) \sin(t - s)$ ,  $g(t, x(s)) = L_g x(s)$ ,

$$f(\cdot, x(\cdot), (Sx)(\cdot))(y) = L_f x(\cdot)(y) + L_f \int_0^{\cdot} k(\cdot, s) g(\cdot, x(s)) ds(y),$$

where

$$\psi(\cdot + 2\pi) = \psi(\cdot) \in L^1_{loc}([0, +\infty); X), \quad \int_0^{2\pi} \psi(s) \sin(t - s) x(t) ds = 0$$

and

$$K_1 x(\cdot)(y) = K_1 x(\cdot, y) = \int_0^{2\pi} k_1 x(\cdot, y) dy.$$

Thus problem (4.1) can be rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), (Sx)(t)) + Cu(t), \\ x(0) = x(2\pi) = 0, \\ z(t) = K_1 x, \end{cases}$$

and the output  $v(t)$  satisfies

$$\frac{dv(t)}{dt} = Jv(t) + K_2 z(t),$$

where  $J = \frac{1}{m}$ ,  $K_2 = k_2$ .

(1) For the condition of the steady-state control, we choose  $L_f$  and  $\psi$  such that

$$L_f < \frac{1}{4} \quad \text{and} \quad 1 - e^{2\pi L_f} \frac{2\pi L_f}{1 - e^{-\frac{\pi}{2}}} 2\pi > 0, \quad \|\psi\| = \min \left\{ o\left(\frac{1}{2L_f L_g \pi}\right), o\left(\frac{1}{2L_g \pi}\right) \right\}.$$

Then all the assumptions given in Theorem 3.4, our results can be used to system (4.1), (4.2), (1.3), (1.4).

(2) For the condition of the steady-state stability, take into account that

$$\int_0^\infty g(t)dt = mk_2 \sum_{i=1}^\infty \frac{4}{i^2} \langle b, \phi_i \rangle \langle k_1, \phi_i \rangle = mk_2 \sum_{i=1}^\infty \frac{4}{i^2} \left[ \frac{4k_1 b}{i^2} \frac{1}{\pi} (1 - \cos i\pi y)^2 \right],$$

where  $\cos i\pi y \neq 1$ , and

$$J^{-1}K_2K_1A^{-1}f = mk_2 \sum_{i=1}^\infty \frac{4}{i^2} \left[ \frac{2k_1}{i} \sqrt{\frac{1}{\pi}} (1 - \cos i\pi y) |\langle f, \phi_i \rangle| \right],$$

where  $\langle f, \phi_i \rangle \neq 0$ .

We can choose  $m$  and  $k_2$  such that

$$mk_2 \sum_{i=1}^\infty \frac{4}{i^2} \left[ \frac{2k_1}{i} \sqrt{\frac{1}{\pi}} (1 - \cos i\pi y) (|\langle f, \phi_i \rangle| + b) \right] < 1.$$

Then

$$mk_2 \sum_{i=1}^\infty \frac{4}{i^2} (|\langle f, \phi_i \rangle \langle k_1, \phi_i \rangle| + |\langle b, \phi_i \rangle \langle k_1, \phi_i \rangle|) < 1.$$

Obviously,

$$\left| mk_2 \sum_{i=1}^\infty \frac{4}{i^2} \langle f, \phi_i \rangle \langle k_1, \phi_i \rangle \right| \leq \left| 1 - mk_2 \sum_{i=1}^\infty \frac{4}{i^2} \langle b, \phi_i \rangle \langle k_1, \phi_i \rangle \right| = \left| 1 - \int_0^\infty g(t)dt \right|,$$

$$mk_2 \sum_{i=1}^\infty \frac{4}{i^2} [(\langle f, \phi_i \rangle \langle k_1, \phi_i \rangle + \langle b, \phi_i \rangle \langle k_1, \phi_i \rangle \text{sign} v) - v] \geq \delta, \quad \text{where } |v| > 1.$$

Thus, all the assumptions given in Theorem 3.5, our results can be used to system (4.1), (4.2), (1.3), (1.4).

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