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Dedicated to the memory of Professor Andrzej Lasota

Michał Kisielewicz

SOME OPTIMAL CONTROL PROBLEMS FOR PARTIAL DIFFERENTIAL INCLUSIONS

Abstract. Partial differential inclusions are considered. In particular, basing on diffusions properties of weak solutions to stochastic differential inclusions, some existence theorems and some properties of solutions to partial differential inclusions are given.

Keywords: partial differential inclusions, diffusion processes, existence theorems.

Mathematics Subject Classification: 93E03, 93C30.

1. INTRODUCTION

The present paper deals with the existence of optimal solutions to some optimal control problem for partial differential inclusions. Partial differential inclusions considered in the paper are investigated by stochastic methods connected with diffusion properties of weak solutions to stochastic differential inclusions considered in the author's paper [9].

In the recent years, some properties of partial differential inclusions have been investigated by G. Bartuzel and A. Fryszkowski (see [1–3]) as applications of some general methods of abstract differential inclusions. Partial differential inclusions considered by G. Bartuzel and A. Fryszkowski have the form $Du \in F(u)$, where D is a partial differential operator and F is a given lower semicontinuous (l.s.c.) multifunction. In the present paper we consider partial differential inclusions of the form $u'_t(t,x) \in (\mathbb{L}_{FG}u)(t,x)+c(t,x)u(t,x)$ and $\psi(t,x) \in (\mathbb{L}_{FG}u)(t,x)+c(t,x)u(t,x)$, where c and ψ are given continuous functions, u denotes an unknown function and $\mathbb{L}_{FG}$ is the set-valued partial differential operator generated by the given l.s.c. set-valued mappings F and G. Partial differential inclusions considered in the paper are investigated together with some initial and boundary conditions and solutions to such initial and boundary valued problems are characterised by weak solutions to stochastic differential inclusions. Such approach leads to natural methods of solving some optimal control problems for systems described by partial differential equations depending on control parameters. These methods use the weak compactness with respect to distributions of the sets of all weak solutions to stochastic differential inclusions (see [7] and [8]).

In what follows we shall denote by $C_0^{1,2}(\mathbb{R}^{n+1})$ and $C_0^2(\mathbb{R}^n)$ the spaces of all functions $\widetilde{h} \in C^{1,2}(\mathbb{R}^{n+1}, \mathbb{R})$ and $h \in C^2(\mathbb{R}^n, \mathbb{R})$, respectively with compact supports. By ∂D and ∂U we denote the boundaries of given domains $D \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^{n+1}$, respectively. By $\mathcal{P}_{\mathbb{F}}$ we shall denote a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions, i.e., such that $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $P(A) = 0$ and $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$ for $t \in [0, T]$. We shall deal with set-valued mappings $F : [0, \infty) \times \mathbb{R}^n \to Cl(\mathbb{R}^n)$ and $G : [0, \infty) \times \mathbb{R}^n \to Cl(\mathbb{R}^{n \times m})$, where $Cl(\mathbb{R}^n)$ and $Cl(\mathbb{R}^{n \times m})$ denote spaces of all nonempty closed subsets of $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$, respectively. Given the set-valued mappings F and G, we shall denote by $\mathcal{C}(F)$ and $\mathcal{C}(G)$ sets of all continuous selectors $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ of F and G, respectively. For a given n-dimensional stochastic process $X_{s,x} = (X_{s,x}(t))_{0 \le t < \infty}$ on $\mathcal{P}_{\mathbb{F}}$ satisfying $X_{s,x}(s) = x$ for $(s, x) \in [0, \infty) \times \mathbb{R}^n$, we shall denote by $Y_{s,x}$, an $(n + 1)$-dimensional stochastic process defined on $\mathcal{P}_{\mathbb{F}}$ by $Y_{s,x} = (Y_{s,x}(t))_{0 \le t < \infty}$, where $Y_{s,x}(t) = (s + t, X_{s,x}(s + t))$ for $0 \le t < \infty$. For the stochastic processes given above, $T > 0$ and a domain $D \subset \mathbb{R}^n$ we can define the first exit times τ_D^s and τ_U^s of $X_{s,x}$ and $Y_{s,x}$ from D and $U = (0, T) \times D$, respectively, namely, $\tau_D^s = \inf\{r > s : X_{s,x}(r) \notin D\}$ and $\tau_U^s = \inf\{t > s : Y_{s,x}(t) \notin U\}$. It can be verified (see [12], p. 226) that $\tau_U^s = \tau_D^s - s$.

In what follows we shall need some continuous selection theorem. We recall it (see [9], Th.3) in the general form. Let (X, ρ), $(Y, |\cdot|)$ and $(Z, ||\cdot||)$ be a Polish and Banach space, respectively. Similarly as above by $Cl(Y)$ we denote the space of all nonempty closed subsets of Y.

Theorem 1 ([9], Th. 3)**.** *Let $\lambda : X \times Y \to Z$ and $u : X \to Z$ be continuous and $H : X \to Cl(Y)$ be l.s.c. such that $u(x) \in \lambda(x, H(x))$ for $x \in X$. Assume $\lambda(x, \cdot)$ is affine and $H(x)$ is a convex subset of Y for every $x \in X$. Then for every $\varepsilon > 0$ there is a continuous function $f_\varepsilon : X \to Y$ such that $f_\varepsilon(x) \in H(x)$ and $||\lambda(x, f_\varepsilon(x)) - u(x)|| \le \varepsilon$ for $x \in X$.*

2. STOCHASTIC DIFFERENTIAL INCLUSIONS AND SET-VALUED PARTIAL DIFFERENTIAL OPERATOR

Given set-valued measurable and bounded mappings $F : [0, \infty) \times \mathbb{R}^n \to Cl(\mathbb{R}^n)$ and $G : [0, \infty) \times \mathbb{R}^n \to Cl(\mathbb{R}^{n \times m})$, by a stochastic differential inclusion $SDI(F, G)$, we mean a relation

$$x_t - x_s \in cl_{L^2} \left(\int_s^t F(\tau, x_\tau) d\tau + \int_s^t G(\tau, x_\tau) dB_\tau \right) \tag{1}$$

which has to be satisfied for every $0 \le s \le t < \infty$ by a system $(\mathcal{P}_{\mathbb{F}}, X, B)$ consisting of a complete filtered probability space $\mathcal{P}_{\mathbb{F}}$, an $\mathbb{L}^2$-continuous $\mathbb{F}$-nonanticipative n-dimensional stochastic process $X = (X(t))_{0 \le t < \infty}$ and m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{0\le t<\infty}$. In a particular case, when F and G take convex values, $SDI(F,G)$ takes the form

$$x_t - x_s \in \int_s^t F(\tau, x_\tau)d\tau + \int_s^t G(\tau, x_\tau)dB_\tau \tag{2}$$

and for its every solution $(\mathcal{P}_{\mathbb{F}}, X, B)$ the process X is continuous. It can be treated as a random variable $X : (\Omega, \mathcal{F}) \to (C, \beta(C))$, where $C = C([0,\infty), \mathbb{R}^n)$ and $\beta(C)$ denotes the Borel σ-algebra on C. We call the above system $(\mathcal{P}_{\mathbb{F}}, X, B)$ a weak solution to $SDI(F,G)$. We call a weak solution $(\mathcal{P}_{\mathbb{F}}, X, B)$ to (2) unique in law if for any other weak solution $(\mathcal{P}'_{\mathbb{F}'}, X', B')$ to $SDI(F,G)$ there is $PX^{-1} = P(X')^{-1}$, where PX^{-1} and $P(X')^{-1}$ denote the distributions of X and X', respectively, defined by $(PX^{-1})(A) = P(X^{-1}(A))$ and $(P(X')^{-1})(A) = P'((X')^{-1}(A))$ for $A \in \beta(C)$. In what follows, we shall consider stochastic differential inclusion (2) together with an initial condition $X_s = x$ a.s., for fixed $(s,x) \in [0,\infty) \times \mathbb{R}^n$. Every weak solution $(\mathcal{P}_{\mathbb{F}}, X, B)$ to (2) will be identified with a pair (X, B) or simply with a stochastic process X defined on $\mathcal{P}_{\mathbb{F}}$. In what follows we shall denote by $\mathcal{X}_{s,x}(F,G)$ the set of all weak solutions to stochastic differential inclusions (2) satisfying an initial condition $x_s = x$ a.s. We shall consider $SDI(F,G)$ of the form (2) with F and G satisfying the following conditions (A).

Conditions (A):

(i) $F : [0,\infty) \times \mathbb{R}^n \to Cl(\mathbb{R}^n)$ and $G : [0,\infty \times \mathbb{R}^n \to Cl(\mathbb{R}^{n\times m})$ are measurable, bounded and take convex values,
(ii) $F(t,\cdot)$ and $G(t,\cdot)$ are continuous for every fixed $t \in [0,\infty)$,
(iii) G is diagonally convex-valued, i.e., for every $(t,x) \in [0,\infty) \times \mathbb{R}^n$, the set $D(G)(t,x) = \{g \cdot g^T : g \in G(t,x)\}$ is convex, where g^T denotes the transposition of g,
(iv) G is such that for every continuous selector σ of a multifunction $D(G)$ defined above, there is a continuous selector g of G such that $g \cdot g^T$ is uniformly positive defined and $\sigma = (g \cdot g^T)$,
(v) F and G are continuous.

Similarly as in [8] and [9] we can prove the following theorems.

Theorem 2. *Assume conditions (i)–(iii) of (A) are satisfied. Then for every $(s,x) \in [0,\infty) \times \mathbb{R}^n$ the set $\mathcal{X}_{s,x}(F,G)$ is nonempty and weakly compact with respect to the convergence in distributions.*

Theorem 3. *Assume conditions (i)–(iv) of (A) are satisfied. Then for every $(f,g) \in \mathcal{C}(F) \times \mathcal{C}(G)$ and $(s,x) \in [0,\infty) \times \mathbb{R}^n$ there is $X^{fg}_{s,x} \in \mathcal{X}_{s,x}(F,G)$ such that a process $Y^{fg}_{s,x} = (Y^{fg}_{s,x}(t))_{0\le t<\infty}$ with $Y^{fg}_{s,x}(t) = (s+t, X^{fg}_{s,x}(s+t))$ for $0 \le t < \infty$ is an Itô diffusion such that $Y^{fg}_{s,x}(0) = (s,x)$, a.s.*

Proof. Let $(f,g) \in \mathcal{C}(F) \times \mathcal{C}(G)$ and $(s,x) \in [0,\infty) \times \mathbb{R}^n$ be fixed. By virtue of ([5], Th.IV.6.1) and Strook and Varadhan uniqueness theorem (see [13]) there is a unique in law weak solution $(\mathcal{P}_{\mathbb{F}}, X^{fg}_{s,x}, B)$ to a stochastic differential equation $x_t =$

$x + \int_s^t f(\tau, x_\tau)d\tau + \int_s^t g(\tau, x_\tau)dB_\tau$. Let $a_f = (1, f^T)^T$ and $b_g = (\mathbf{0}, g^1, \ldots, g^n)^T$ with $\mathbf{0}, g^i \in \mathbb{R}^{1\times m}$, where $\mathbf{0} = (0, \ldots, 0)$ and g^i denotes the i -th row of g for $i = 1, \ldots, n$. Similarly as in ([9], p.1044) we can verify that the process $Y_{s,x}^{fg}$ defined above is a unique in law weak solution to stochastic differential equation $y_t = (s, x) + \int_0^t a_f(y_\tau)d\tau + \int_s^t b_g(y_\tau)dB_\tau$. Therefore (see [9]), $Y_{s,x}^{fg}$ is an Itô diffusion on $\mathcal{P}_{\mathbb{F}}$ satisfying $Y_{s,x}^{fg}(0) = (s, x)$. □

Corollary 1. *If conditions (i)–(iv) of (A) are satisfied, then for every* $(f, g) \in \mathcal{C}(F) \times \mathcal{C}(G)$ *there is a nonempty set* $\mathcal{D}_{fg}(R^{n+1})$ *of functions* $\widetilde{h} : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\lim_{t\to 0} \frac{E^{s,x}\widetilde{h}(Y_{s,x}^{fg}(t) - \widetilde{h}(s, x)}{t} \tag{3}$$

exists for every $(s, x) \in [0, \infty) \times \mathbb{R}^n$*, where* $E^{s,x}$ *denotes the mean value operator with respect to a probability law* $Q^{s,x}$ *of* $Y_{s,x}^{fg}$ *so that* $Q^{s,x}[Y_{s,x}^{fg}(t_1) \in E_1, \ldots, Y_{s,x}^{fg}(t_k) \in E_k] = P[Y_{s,x}^{fg}(t_1) \in E_1, \ldots, Y_{s,x}^{fg}(t_k) \in E_k]$ *for* $0 \le t_i < \infty$ *and* $E_i \in \beta(\mathbb{R}^{n+1})$ *with* $1 \le i \le k$*.*

In what follows the above limit will be denoted by $(\mathbb{L}_{fg}^{\mathcal{C}}\widetilde{h})(s, x)$ and called the infinitesimal generator of $Y_{s,x}^{fg}$. Similarly as in ([12], Th.7.3.3), we can verify that for every $\widetilde{h} \in C_0^{1,2}(\mathbb{R}^{n+1})$ there is

$$(\mathbb{L}_{fg}^{\mathcal{C}}\widetilde{h})(s, x) = \widetilde{h}'_t(s, x) + \sum_{i=1}^n f_i(s, x)\widetilde{h}'_{x_i}(s, x) + \frac{1}{2}\sum_{i,j=1}^n (g \cdot g^T)_{ij}(s, x)\widetilde{h}''_{x_i x_j}(s, x) \tag{4}$$

for every $(s, x) \in [0, \infty) \times \mathbb{R}^n$. Hence in particular, for $h \in C_0^2(\mathbb{R}^n)$ we obtain

$$(\mathbb{L}_{fg}^{\mathcal{C}}h)(s, x) = \sum_{i=1}^n f_i(s, x)h'_{x_i}(x) + \frac{1}{2}\sum_{i,j=1}^n (g \cdot g^T)_{ij}(s, x)h''_{x_i x_j}(x) \tag{5}$$

for every $(s, x) \in [0, \infty) \times \mathbb{R}^n$. Similarly as in ([12], Th.7.4.1), we obtain Dynkin's formula

$$E^{s,x}\left[\widetilde{h}(Y_{s,x}^{fg}(\tau)\right] = \widetilde{h}(s, x) + E^{s,x}\left[\int_0^\tau (\mathbb{L}_{fg}^{\mathcal{C}}\widetilde{h})(Y_{s,x}^{fg}(t))dt\right] \tag{6}$$

for $\widetilde{h} \in C_0^{1,2}(\mathbb{R}^{n+1})$, $(s, x) \in [0, \infty) \times \mathbb{R}^n$ and a stopping time τ such that $E^{s,x}[\tau] < \infty$. In particular, for $h \in C_0^2(\mathbb{R}^n)$, one obtains

$$E^{0,x}\left[\widetilde{h}(X_{0,x}^{fg}(\tau)\right] = h(x) + E^{0,x}\left[\int_0^\tau (\mathbb{L}_{fg}^{\mathcal{C}}h)(t, X_{0,x}^{fg}(t))dt\right] \tag{7}$$

for $x \in \mathbb{R}^n$.

Given the above set-valued mappings F and G we can define a set-valued partial differential operator on $C_0^{1,2}(\mathbb{R}^{n+1})$ by setting

$$(\mathbb{L}_{FG}\widetilde{h})(s, x) = \{(\mathbb{L}_{uv}\widetilde{h})(s, x) : u \in F(s, x), v \in G(s, x)\}, \tag{8}$$

where

$$(\mathbb{L}_{uv}\widetilde{h})(s,x) = \sum_{i=1}^{n} u_i \widetilde{h}'_{x_i}(s,x) + \frac{1}{2}\sum_{i,j=1}^{n}(v\cdot v^T)_{ij}\widetilde{h}''_{x_i x_j}(s,x)$$

for $(s,x) \in [0,\infty) \times \mathbb{R}^n$.

From Theorem 1, the next result follows immediately.

Theorem 4. *Assume F and G satisfy conditions (i)–(iv) of (A), let $\mathbb{R}_+ = [0,\infty)$ and $u, v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be continuous and such that $u(t,\cdot) \in C_0^{1,2}(\mathbb{R}^{n+1})$ and $v(t,s,x) \in (\mathbb{L}_{FG}u(t,\cdot))(s,x)$ for $(s,x) \in [0,\infty) \times \mathbb{R}^n$ and $t \in [0,\infty)$. Then for every $\varepsilon > 0$ there is $(f_\varepsilon, g_\varepsilon) \in \mathcal{C}(F) \times \mathcal{C}(G)$ such that $g_\varepsilon \cdot g_\varepsilon^T$ is uniformly positive defined and $|v(t,s,x) - (\mathbb{L}_{f_\varepsilon g_\varepsilon} u(t,\cdot)(s,x)| \le \varepsilon$ for $(s,x) \in [0,\infty) \times \mathbb{R}^n$ and $t \in [0,\infty)$.*

Proof. Let $X = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$, $Y = \mathbb{R}^n \times \mathbb{R}^{n\times m}$, $\lambda((t,s,x),(z,\sigma)) = \sum_{i=1}^n z_i u'_{x_i}(t,s,x) + \frac{1}{2}\sum_{i,j=1}^n \sigma_{ij} u''_{x_i x_j}(t,s,x)$, $\widetilde{F}(t,s,x) = F(\pi(t,s,x))$ and $\widetilde{G}(t,s,x) = G(\pi(t,s,x))$ for $(t,s,x) \in X$, where π denotes the orthogonal projection of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ onto $\mathbb{R} \times \mathbb{R}^n$. From the properties of v there follows $v(t,s,x) \in \lambda((t,s,x), \widetilde{H}(t,s,x))$ for $(t,s,x) \in X$, where $\widetilde{H}(t,s,x) = \widetilde{F}(t,s,x) \times D(\widetilde{G})(t,s,x)$. By virtue of Theorem 1, for every $\varepsilon > 0$ there is a continuous selector $\widetilde{h}_\varepsilon$ of $\widetilde{H}$ such that $|v(t,s,x) - \lambda((t,s,x), \widetilde{h}_\varepsilon(t,s,x))| \le \varepsilon$ for $(t,s,x) \in X$. By the definition of $\widetilde{H}$, there are continuous selectors $\widetilde{f}_\varepsilon$ and $\widetilde{\sigma}_\varepsilon$ of $\widetilde{F}$ and $D(\widetilde{G})$, respectively, such that $\widetilde{h}_\varepsilon = (\widetilde{f}_\varepsilon, \widetilde{\sigma}_\varepsilon)$. From this and (iv) it follows that for every $(s,x) \in [0,\infty) \times \mathbb{R}^n$ there are $f_\varepsilon(s,x) \in F(s,x)$ and $g_\varepsilon(s,x) \in G(s,x)$ such that $\widetilde{f}_\varepsilon(t,s,x) = f_\varepsilon(\pi(t,s,x)) = f_\varepsilon(s,x)$ and $\widetilde{g}_\varepsilon(t,s,x) = g_\varepsilon(\pi(t,s,x)) = g_\varepsilon(s,x)$ and $|v(t,s,x) - (\mathbb{L}_{f_\varepsilon g_\varepsilon} u(t,\cdot)(s,x)| \le \varepsilon$ for $(s,x) \in [0,\infty) \times \mathbb{R}^n$ and $t \in [0,\infty)$. In this way we have defined, on $[0,\infty) \times \mathbb{R}^n$, continuous functions f_ε and g_ε, selectors of F and G, such that $\widetilde{f}_\varepsilon(t,s,x) = f_\varepsilon(s,x)$ and $\widetilde{g}_\varepsilon(t,s,x) = g_\varepsilon(s,x)$. □

3. INITIAL AND BOUNDARY VALUED PROBLEMS FOR PARTIAL DIFFERENTIAL INCLUSIONS

Let F and G satisfy conditions (i)–(iv) of (A) and $\mathbb{L}_{FG}$ be the set-valued partial differential operator on $C_0^{1,2}(\mathbb{R}^{n+1})$ defined above. By $\mathbb{L}_{FG}^{\mathcal{C}}\widetilde{h}$ we shall denote a family $\mathbb{L}_{FG}^{\mathcal{C}} = \{\mathbb{L}_{fg}^{\mathcal{C}} : (f,g) \in \mathcal{C}(F) \times \mathcal{C}(G)\}$. For every $(f,g) \in \mathcal{C}(F) \times \mathcal{C}(G)\}$ and $\widetilde{h} \in \mathcal{D}_{FG} := \bigcap\{\mathcal{D}_{fg} : (f,g) \in \mathcal{C}(F) \times \mathcal{C}(G)\}$ there is $(\mathbb{L}_{fg}^{\mathcal{C}})\widetilde{h})(t,x) \in (\mathbb{L}_{FG}^{\mathcal{C}}\widetilde{h})(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}^n$. In [10] , the following results were proved.

Theorem 5. *Assume conditions (i)–(iv) of (A) are satisfied, $T > 0$, $\widetilde{h} \in C_0^{1,2}(\mathbb{R}^{n+1})$ and let $c \in C([0,T] \times \mathbb{R}^n, \mathbb{R})$ be bounded. For every $(s,x) \in [0,T) \times \mathbb{R}^n$ and every weak solution $X_{s,x}$ to $SDI(F,G)$ with an initial condition $x_s = x$ a.s., defined on a probability space $(\Omega, \mathcal{F}, P)$, a function*

$$v(t,s,x) = E^{s,x}\left[exp\left(-\int_s^{s+t} c(\tau, X_{s,x}(\tau))d\tau\right)\widetilde{h}(s+t, X_{s,x}(s+t))\right]$$

satisfies

$$\begin{cases} v'_t(t,s,x) \in \left(\mathbb{L}^{\mathcal{C}}_{FG} v(t,\cdot)\right)(s,x) - c(s,x)v(t,s,x) & \textit{for} \quad (s,x) \in [0,T) \times \mathbb{R}^n, \\ & \textit{and} \quad t \in [0, T-s], \\ v(0,s,x) = \widetilde{h}(s,x) & \textit{for} \quad (s,x) \in [0,T) \times \mathbb{R}^n. \end{cases} \tag{9}$$

Theorem 6. *Assume conditions (i), (iii)–(v) of (A) are satisfied, $T > 0$ and let $\widetilde{h} \in C_0^{1,2}(\mathbb{R}^{n+1})$. Suppose $c \in C([0,T] \times \mathbb{R}^n, \mathbb{R})$ and $v \in C^{1,1,2}([0,T] \times [0,T] \times \mathbb{R}^n, \mathbb{R})$ are bounded and such that*

$$\begin{cases} v'_t(t,s,x) - v'_s(t,s,x) \in \left(\mathbb{L}_{FG} v(t,\cdot)\right)(s,x) - c(s,x)v(t,s,x), \\ \textit{for } (s,x) \in [0,T) \times \mathbb{R}^n \quad \textit{and} \quad t \in [0, T-s], \\ v(0,s,x) = \widetilde{h}(s,x) \ for \ (s,x) \in [0,T) \times \mathbb{R}^n. \end{cases} \tag{10}$$

Then for every $(s,x) \in [0,T) \times \mathbb{R}^n$ there exists $\widetilde{X}_{s,x} \in \mathcal{X}_{s,x}(F,G)$ defined on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ such that

$$v(t,s,x) = \widetilde{E}\left[exp\left(-\int_s^{s+t} c(\tau, \widetilde{X}_{s,x}(\tau))d\tau\right)\widetilde{h}(s+t, \widetilde{X}_{s,x}(s+t))\right]$$

for $(s,x) \in [0,T) \times \mathbb{R}^n$ and $t \in [0, T-s]$.

Theorem 7. *Assume conditions (i), (iii)–(v) of (A) are satisfied, $T > 0$, D is a bounded domain in $\mathbb{R}^n$ and let $\Phi \in C((0,T) \times \partial D, \mathbb{R})$ and $u \in C((0,T) \times D, \mathbb{R})$ be bounded. If $v \in C_0^{1,2}(\mathbb{R}^{n+1})$ is bounded and such that*

$$\begin{cases} u(t,x) - v'_t(t,x) \in (\mathbb{L}_{FG} v)(t,x) \quad \textit{for} \quad (t,x) \in (0,T) \times D, \\ \lim_{D \ni x \to y} v(t,x) = \Phi(t,y) \quad \textit{for} \quad (t,y) \in (0,T] \times \partial D \end{cases}$$

then for every $x \in D$ there exists $\widetilde{X}_{0,x} \in \mathcal{X}_{0,x}(F,G)$ defined on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ such that

$$v(t,x) = \widetilde{E}[\Phi(\widetilde{\tau}^T_D, \widetilde{X}_{0,x}(\widetilde{\tau}^T_D))] - \widetilde{E}\left[\int_0^{\widetilde{\tau}^T_D} u(t, \widetilde{X}_{0,x}(t))dt\right]$$

for $(t,x) \in (0,T) \times D$, where $\widetilde{\tau}^T_D = \inf\{r \in (0,T] : \widetilde{X}_{0,x}(r) \notin D\} \wedge T$.

Theorem 8. *Assume conditions (i), (iii)–(v) of (A) are satisfied, $T > 0$, D is a bounded domain in $\mathbb{R}^n$ and let $\Phi \in C((0,T) \times \partial D, \mathbb{R})$, $c \in C([0,T] \times D, \mathbb{R})$ and $u \in C((0,T) \times D, \mathbb{R})$ be bounded. If $v \in C_0^{1,2}(\mathbb{R}^{n+1})$ is bounded and such that*

$$\begin{cases} u(t,x) - v'_t(t,x) \in (\mathbb{L}_{FG} v)(t,x) - c(t,x)v(t,x) \quad \textit{for} \quad (t,x) \in (0,T) \times D, \\ \lim_{x \to y} v(t,x) = \Phi(t,y) \quad \textit{for} \quad (s,y) \in (0,T] \times \partial D \end{cases} \tag{11}$$

then for every $x \in D$ *there exists* $\widetilde{X}_{0,x} \in \mathcal{X}_{0,x}(F,G)$ *defined on a probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ *such that*

$$v(t,x) = \widetilde{E}\left[\Phi(\widetilde{\tau}_D^T, \widetilde{X}_{0,x}(\widetilde{\tau}_D^T))exp\left(-\int_0^{\widetilde{\tau}_D^T} c(t, \widetilde{X}_{0,x}(t))dt\right)\right] -$$
$$- \widetilde{E}\left\{\int_0^{\widetilde{\tau}_D^T}\left[u(t, \widetilde{X}_{0,x}(t))exp\left(-\int_0^t c(z, \widetilde{X}_{0,x}(z))dz\right)\right]dt\right\}$$

for $(t,x) \in (0,T) \times D$, *where* $\widetilde{\tau}_D^T = \inf\{r \in (0,T] : \widetilde{X}_{0,x}(r) \notin D\} \wedge T$.

Remark 1. *The above results are also true for* $T = \infty$, *because by the boundedness of* D *there is* $\tau_D < \infty$ *a.s., which implies that* $\lim_{T\to\infty} \tau_D^T = \tau_D$ *a.s. The case* $T < \infty$ *is more applicable in practice.*

4. EXISTENCE OF SOLUTION TO OPTIMAL CONTROL PROBLEMS FOR PARTIAL DIFFERENTIAL INCLUSIONS

Given set-valued mappings F, G and functions $\widetilde{h}$, Φ, c and u, by $\Lambda(F,G,\widetilde{h},c)$ we denote the set of all solutions to initial valued problem:

$$\begin{cases} v_t'(t,s,x) - v_s'(t,s,x) \in \left(\mathbb{L}_{FG}^{\mathcal{C}} v(t,\cdot)\right)(s,x) - c(s,x)v(t,s,x), \\ \text{for} \quad (s,x) \in [0,T) \times \mathbb{R}^n \quad \text{and} \quad t \in [0, T-s] \\ v(0,s,x) = \widetilde{h}(s,x) \quad \text{for} \quad (s,x) \in [0,T) \times \mathbb{R}^n, \end{cases} \tag{12}$$

and by and $\Gamma(F,G,\Phi,u,c)$ the set of all solutions to the boundary value problem

$$\begin{cases} u(t,x) \in (\mathbb{L}_{FG}^{\mathcal{C}} v)(t,x) - c(t,x)v(t,x) \quad \text{for} \quad (t,x) \in (0,T) \times D, \\ \lim_{x\to y} v(t,x) = \Phi(t,y) \quad \text{for} \quad (s,y) \in (0,T] \times \partial D. \end{cases} \tag{13}$$

Let us recall that by a solution to initial value problem (12) we mean a function $v \in C^1([0,T]\times[0,T]\times\mathbb{R}^n, \mathbb{R})$ such that $v(t,\cdot) \in \mathcal{D}_{FG}$ for every $t \in [0,T]$ and conditions (12) are satisfied. Similarly by a solution to boundary valued problem (13) we mean a function $v \in C^1([0,T] \times \mathbb{R}^n)$ such that $v \in \mathcal{D}_{FG}$ and conditins (13) are satisfied. In what follows, we shall denote the sets $\Lambda(F,G,\widetilde{h},c) \cap C^{1,1,2}([0,T] \times [0,T] \times \mathbb{R}^n, \mathbb{R})$ and $\Gamma(F,G,\Phi,u,c) \cap C^{1,2}(\mathbb{R}^{n+1})$ by $\Lambda^{\mathcal{C}}(F,G,\widetilde{h},c)$ and $\Gamma^{\mathcal{C}}(F,G,\Phi,u,c)$, respectively. It is easy to see that $\Lambda^{\mathcal{C}}(F,G,\widetilde{h},c)$ and $\Gamma^{\mathcal{C}}(F,G,\Phi,u,c)$ are solutions sets to (10) and (11), respectively. Indeed, by the definitions of $\mathbb{L}_{FG}^{\mathcal{C}}$ and $\mathbb{L}_{FG}$ there is $(\mathbb{L}_{FG}^{\mathcal{C}}\widetilde{h})(t,x) = \widetilde{h}_t'(t,x) + (\mathbb{L}_{FG}\widetilde{h})(t,x)$ for $\widetilde{h} \in C^{1,2}(\mathbb{R}^{n+1})$ and $(t,x) \in [0,T] \times \mathbb{R}^n$.

Let $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a given measurable and uniformly integrably bounded function and let $\mathcal{H}$ and $\mathcal{Z}$ be mappings defined on $\Lambda(F,G,\widetilde{h},c)$ and $\Gamma(F,G,\Phi,u,c)$ for fixed $(s,x) \in [0,T) \times \mathbb{R}^n$ by settings:

$$\mathcal{H}(v)(s,x) = \int_0^T H(t, v(t,s,x))dt \quad \text{for} \quad v \in \Lambda(F,G,\widetilde{h},c)$$

and

$$\mathcal{Z}(v)(x) = \int_0^T H(t, v(t,x))dt \quad \text{for} \quad v \in \Gamma(F, G, \Phi, u, c).$$

Theorem 9. *Assume conditions (i), (iii)–(v) of (A) are satisfied and let $c \in C([0,T] \times \mathbb{R}^n, \mathbb{R})$ be bounded. Let $\widetilde{h} \in C^{1,2}(\mathbb{R}^{n+1})$ and assume $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable, uniformly integrably bounded and such that $H(t,\cdot)$ is continuous. If F and G are such that for the $\widetilde{h}$ and c given above the set $\Lambda^{\mathcal{C}}(F, G, \widetilde{h}, c)$ is nonempty, then there is $\widetilde{v} \in \Lambda(F, G, \widetilde{h}, c)$ such that $\mathcal{H}(\widetilde{v})(s,x) = \inf\{\mathcal{H}(v)(s,x) : v \in \Lambda^{\mathcal{C}}(F, G, \widetilde{h}, c)\}$ for every $(s,x) \in [0,T] \times \mathbb{R}^n$.*

Proof. Let $(s,x) \in [0,T] \times \mathbb{R}^n$ be fixed. The set $\{\mathcal{H}(v)(s,x) : v \in \Lambda^{\mathcal{C}}(F, G, \widetilde{h}, c)\}$ is nonempty and bounded, because there is a $k \in L([0,T], \mathbb{R}_+)$ such that $|\mathcal{H}(v)(s,x)| \le \int_0^T k(t)dt$ for every $v \in \Lambda^{\mathcal{C}}(F, G, \widetilde{h}, c)$. Therefore, there is a sequence $(v^n)_{n=1}^\infty$ of $\Lambda^{\mathcal{C}}(F, G, \widetilde{h}, c)$ such that $\alpha := \inf\{\mathcal{H}(v)(s,x) : v \in \Lambda^{\mathcal{C}}(F, G, \widetilde{h}, c) = \lim_{n\to\infty} \mathcal{H}(v^n)(s,x)$. By virtue of Theorem 6 for every $n = 1, 2, \ldots$ and fixed $(s,x) \in [0,T] \times \mathbb{R}^n$ there is $X_{s,x}^n \in \mathcal{X}_{s,x}(F,G)$ such that

$$v^n(t,s,x) = E^{s,x}\left[exp\left(-\int_s^{s+t} c(\tau, X_{s,x}^n(\tau))d\tau\right)\widetilde{h}(s+t, X_{s,x}^n(s+t))\right]$$

for $t \in [0, T-s]$. By the weak compactness of $\mathcal{X}_{s,x}(F,G)$ and ([5],Th.I.2.1) there are a subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=1}^\infty$, a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, stochastic processes $\widetilde{X}^{n_k}$ and $\widetilde{X}$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ such that $P(X_{s,x}^{n_k})^{-1} = P(\widetilde{X}^{n_k})^{-1}$ for $k = 1, 2, \ldots$ and $\sup_{0\le t\le T} |\widetilde{X}^{n_k}(t) - \widetilde{X}(t)| \to 0$, $\widetilde{P}$ - a.s. Hence, in particular there follows

$$\begin{aligned} v^{n_k}(t,s,x) &= E^{s,x}\left[exp\left(-\int_s^{s+t} c(\tau, X_{s,x}^{n_k}(\tau))d\tau\right)\widetilde{h}(s+t, X_{s,x}^{n_k}(s+t))\right] = \\ &= \widetilde{E}\left[exp\left(-\int_s^{s+t} c(\tau, \widetilde{X}^{n_k}(\tau))d\tau\right)\widetilde{h}(s+t, \widetilde{X}^{n_k}(s+t))\right]. \end{aligned}$$

By the properties of processes $\widetilde{X}^{n_k}$, $\widetilde{X}$ and functions c and $\widetilde{h}$, thence there follows that

$$\lim_{k\to\infty} v^{n_k}(t,s,x) = \widetilde{E}\left[exp\left(-\int_s^{s+t} c(\tau, \widetilde{X}(\tau))d\tau\right)\widetilde{h}(s+t, \widetilde{X}(s+t))\right].$$

Let

$$\widetilde{v}(t,s,x) = \widetilde{E}\left[exp\left(-\int_s^{s+t} c(\tau, \widetilde{X}(\tau))d\tau\right)\widetilde{h}(s+t, \widetilde{X}(s+t))\right].$$

By virtue of Theorem 5, $\widetilde{v} \in \Lambda(F, G, \widetilde{h}, c)$. Hence, by the properties of the function H we get $\alpha = \lim_{k\to\infty} \mathcal{H}(v^{n_k})(s,x) = \mathcal{H}(\widetilde{v})(s,x)$ for $(s,x) \in [0,T] \times \mathbb{R}^n$. □

In a similar way, we can also prove the following theorem.

Theorem 10. *Assume conditions (i), (iii)–(v) of (A) are satisfied and let D be a bounded domain in $\mathbb{R}^n$. Let $c \in C([0,T] \times \mathbb{R}^n, \mathbb{R})$, $u \in C((0,T) \times D)$ and $\Phi \in C((0,T) \times \partial D, \mathbb{R})$ be bounded. Assume $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable, uniformly integrably bounded and such that $H(t,\cdot)$ is continuous. If F and G are such that for the Φ, u and c given above the set $\Gamma^{\mathcal{C}}(F,G,\Phi,u,c)$ is nonempty, then for every $x \in \mathbb{R}^n$ there is $\widetilde{X}_{0,x} \in \mathcal{X}_{0,x}(F,G)$ such that $\mathcal{Z}(\widetilde{v})(x) = \inf\{\mathcal{Z}(v)(x) : v \in \Gamma^{\mathcal{C}}(F,G,\Phi,u,c)\}$ for $x \in \mathbb{R}^n$, where*

$$\widetilde{v}(t,x) = E^{0,x}\left[\Phi(\tau_D, \widetilde{X}_{0,x}(\tau_D))exp\left(-\int_0^{\tau_D} c(t,\widetilde{X}_{0,x}(t))dt\right)\right] - \\ - E^{0,x}\left\{\int_0^{\tau_D}\left[u(t,\widetilde{X}_{0,x}(t))exp\left(-\int_0^t c(z,\widetilde{X}_{0,x}(z))dz\right)\right]dt\right\}$$

for $(t,x) \in (0,T) \times D$, where $\tau_D = \inf\{r \in (0,T] : X_{0,x}(r) \notin D\}$.

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Michał Kisielewicz
M.Kisielewicz@wmie.uz.zgora.pl

University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
ul. Podgórna 50, 65–246 Zielona Góra, Poland

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