

Janusz Matkowski

ITERATION GROUPS, COMMUTING FUNCTIONS  
AND SIMULTANEOUS SYSTEMS  
OF LINEAR FUNCTIONAL EQUATIONS

**Abstract.** Let  $(f^t)_{t \in \mathbb{R}}$  be a measurable iteration group on an open interval  $I$ . Under some conditions, we prove that the inequalities  $g \circ f^a \leq f^a \circ g$  and  $g \circ f^b \leq f^b \circ g$  for some  $a, b \in \mathbb{R}$  imply that  $g$  must belong to the iteration group. Some weak conditions under which two iteration groups have to consist of the same elements are given. An extension theorem of a local solution of a simultaneous system of iterative linear functional equations is presented and applied to prove that, under some conditions, if a function  $g$  commutes in a neighbourhood of  $f$  with two suitably chosen elements  $f^a$  and  $f^b$  of an iteration group of  $f$  then, in this neighbourhood,  $g$  coincides with an element of the iteration group. Some weak conditions ensuring equality of iteration groups are considered.

**Keywords:** iteration group, commuting functions, functional equation, functional inequalities.

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## 1. INTRODUCTION

Let  $I \subset \mathbb{R}$  be an open interval. A family of continuous functions  $(f^t)_{t \in \mathbb{R}}$  is said to be an iteration group of a function  $f : I \rightarrow I$  if  $f^1 = f$  and for all  $s, t \in \mathbb{R}$ ,  $f^t : I \rightarrow I$ , and  $f^s \circ f^t = f^{s+t}$ .

An iteration group is called continuous (measurable) if, for every  $x \in I$ , the function  $R \ni t \rightarrow f^t(x)$  is continuous (measurable). The number  $t$  in  $f^t$  is called the iterative index of  $f$ .

According to Zdun's theorem [5], every measurable iteration group  $(f^t)_{t \in \mathbb{R}}$  is continuous and there exists a homeomorphic bijection  $\varphi : I \rightarrow R$ , referred to in the sequel as a generator of the iteration group, such that the following representation formula holds true

$$f^t(x) = \varphi^{-1}(\varphi(x) + t), \quad x \in I, t \in \mathbb{R}. \quad (1.1)$$

It is proved in [3] that if a function  $g : I \rightarrow I$  is continuous at at least one point and commutes with two elements  $f^a$  and  $f^b$  of the iteration group  $(f^t)_{t \in \mathbb{R}}$  and  $\frac{b}{a}$  is irrational, then  $g$  must be an element of the iteration group  $(f^t)_{t \in \mathbb{R}}$  (cf. Theorem 2). In section 3, applying a recent result on a simultaneous system of linear functional inequalities [1] (cf. Lemma 1 in Section 2), we prove that if, additionally,  $a < 0 < b$ , then the commutativity of  $g$  with  $f^a$  and  $f^b$  can be replaced by the inequalities  $g \circ f^a \leq f^a \circ g$  and  $g \circ f^b \leq f^b \circ g$  (Theorem 3). In this section we also give some weak conditions under which two iteration groups have to consist of the same elements (Theorem 4).

In Section 4, applying Theorem 1 (a new result on an extension of a local solution of a simultaneous system of iterative an linear functional equations) we prove Theorem 5, which is a local version of Theorem 2. Here the commutativity is required in a neighbourhood of a fixed point of  $f$ .

The last section begins with a remark that every measurable iteration group  $(f^t)_{t \in \mathbb{R}}$  is uniquely determined by any of its orbit  $\{f^t(x_0) : t \in \mathbb{R}\}$ . Let  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$  be iteration groups. Theorem 6 gives simple conditions on the fixed numbers  $a, b, \alpha, \beta$  under which the inequalities

$$f^\alpha \leq g^\alpha, \quad f^b \leq g^\beta$$

imply the existence of a  $p \in \mathbb{R}$  such that  $g^t = f^{pt}$  for all  $t \in \mathbb{R}$ .

## 2. AUXILIARY RESULTS ON A SYSTEM OF SIMULTANEOUS FUNCTIONAL EQUATIONS AND INEQUALITIES

In the sequel we shall need the following result on an extension of solutions of a simultaneous system of iterative functional equations.

**Theorem 1.** *Let  $I \subseteq \mathbb{R}$  be an open interval, let  $\xi \in \mathbb{R} \cup \{-\infty, +\infty\}$  be one of the endpoints of  $I$ , and let  $I_\xi \subset I$  be a nonempty open interval with the endpoint  $\xi$ . Let the functions  $f_1, f_2 : I \rightarrow \mathbb{R}$  be continuous, increasing and such that one of the following conditions is fulfilled:*

1. *if  $\xi$  is finite then*

$$0 < \frac{f_i(x) - \xi}{x - \xi} < 1, \quad x \in I, i = 1, 2;$$

2. *if  $\xi = -\infty$  then*

$$f_i(x) < x, \quad x \in I, i = 1, 2;$$

3. *if  $\xi = +\infty$  then*

$$f_i(x) > x, \quad x \in I, i = 1, 2.$$

*Let the functions  $g_1, g_2, h_1, h_2 : I \rightarrow \mathbb{R}$  be such that  $g_1(x)g_2(x) \neq 0$  for all  $x \in I$ .*

*Suppose that a function  $\gamma : I_\xi \rightarrow \mathbb{R}$  satisfies the simultaneous system of functional equations*

$$\gamma[f_i(x)] = g_i(x)\gamma(x) + h_i(x), \quad x \in I_\xi, i = 1, 2. \quad (2.1)$$

If the functions  $f_1, f_2$  commute, i.e.

$$f_1 \circ f_2 = f_2 \circ f_1, \quad (2.2)$$

and

$$g_1(x)g_2[f_1(x)] = g_2(x)g_1[f_2(x)], \quad x \in I, \quad (2.3)$$

$$g_1[f_2(x)]h_2(x) + h_1[f_2(x)] = g_2[f_1(x)]h_1(x) + h_2[f_1(x)], \quad x \in I, \quad (2.4)$$

then there exists exactly one function  $\Phi : I \rightarrow \mathbb{R}$  such that

$$\Phi|_{I_\xi} = \gamma$$

and

$$\Phi[f_i(x)] = g_i(x)\Phi(x) + h_i(x), \quad x \in I, i = 1, 2.$$

*Proof.* Assume that the functions  $f_1$  and  $f_2$  satisfy condition 1, that is  $\xi$  is finite. Without any loss of generality, we can assume that  $I = (0, \infty)$ ,  $\xi = 0$  and  $I_\xi = (0, \delta)$  for some  $\delta > 0$ . Let  $\gamma : I_\xi \rightarrow \mathbb{R}$  be a solution of system (2.1). For each  $i \in \{1, 2\}$  there exists a unique function  $\Phi_i : (0, \infty) \rightarrow \mathbb{R}$  such that

$$\Phi_i[f_i(x)] = g_i(x)\Phi_i(x) + h_i(x), \quad x \in (0, \infty), i = 1, 2, \quad (2.5)$$

and

$$\Phi_i|_{(0, a)} = \gamma$$

(cf. M. Kuczma [2], p. 246–247). Let

$$c := \sup \{b > 0 : \Phi_1(x) = \Phi_2(x) \text{ for all } x \in (0, b)\}.$$

Obviously  $c \geq \delta$ . To prove the theorem it is enough to show that  $c = +\infty$ . Assume for the contrary that  $c < +\infty$ . Then

$$\Phi_1(x) = \Phi_2(x), \quad x \in (0, c). \quad (2.6)$$

The assumptions on  $f_1$  and  $f_2$  imply that there exists a  $d > c$  such that

$$f_i(x) < c, \quad x \in (0, d), i = 1, 2.$$

Hence, as  $f_i \circ f_j(x) \in (0, c)$  for  $i, j = 1, 2$  and  $x \in (0, d)$ , applying in turn: (2.5) for  $i = 1$ ; (2.6); (2.5) for  $i = 2$ ; (2.2) and (2.6); (2.5) for  $i = 1$ ; (2.6); (2.5) for  $i = 2$ ; and finally (2.3) and (2.4), we get

$$\begin{aligned} \Phi_1(x) &= \frac{1}{g_1(x)} \{ \Phi_1[f_1(x)] - h_1(x) \} = \frac{1}{g_1(x)} \{ \Phi_2[f_1(x)] - h_1(x) \} = \\ &= \frac{1}{g_1(x)} \left\{ \frac{1}{g_2[f_1(x)]} \{ \Phi_2[f_2(f_1(x))] - h_2[f_1(x)] \} - h_1(x) \right\} = \\ &= \frac{1}{g_1(x)} \left\{ \frac{1}{g_2[f_1(x)]} \{ \Phi_1[f_1(f_2(x))] - h_2[f_1(x)] \} - h_1(x) \right\} = \end{aligned}$$

$$\begin{aligned}
&= \frac{g_1[f_2(x)]\Phi_1[f_2(x)] + h_1[f_2(x)] - h_2[f_1(x)] - h_1(x)g_2[f_1(x)]}{g_1(x)g_2[f_1(x)]} = \\
&= \frac{g_1[f_2(x)]\{g_2(x)\Phi_2(x) + h_2(x)\} + h_1[f_2(x)] - h_2[f_1(x)] - h_1(x)g_2[f_1(x)]}{g_1(x)g_2[f_1(x)]} = \\
&= \frac{g_1[f_2(x)]g_2(x)}{g_1(x)g_2[f_1(x)]}\Phi_2(x) + \\
&\quad + \frac{g_1[f_2(x)]h_2(x) + h_1[f_2(x)] - h_2[f_1(x)] - h_1(x)g_2[f_1(x)]}{g_1(x)g_2[f_1(x)]} = \Phi_2(x)
\end{aligned}$$

which contradicts the definition of  $d$ . It completes the proof.  $\square$

A more general result for system of functional equations of higher orders will be presented in [4]. By  $\mathbb{Q}$  we denote the set of all rational numbers. Let us quote the following

**Lemma 1** ([1], Theorem 1). *Let  $a, b, \alpha, \beta \in \mathbb{R}$  be such that*

$$a < 0 < b, \quad \frac{b}{a} \notin \mathbb{Q}, \quad \frac{\alpha}{a} \geq \frac{\beta}{b}.$$

*If a function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$  continuous at at least one point satisfies the simultaneous system of functional inequalities*

$$\gamma(x+a) \leq \gamma(x) + \alpha, \quad \gamma(x+b) \leq \gamma(x) + \beta, \quad x \in \mathbb{R},$$

*then*

$$\gamma(x) = \frac{\alpha}{a}x + \gamma(0), \quad x \in \mathbb{R}.$$

### 3. COMMUTING FUNCTIONS

**Remark 1.** *Functions  $\psi, \varphi : I \rightarrow \mathbb{R}$  are generators of the same iteration group  $(f^t)_{t \in \mathbb{R}}$  iff there exists a constant  $a \in \mathbb{R}$  such that  $\psi = \varphi + a$ .*

In [3] we have proved the following

**Theorem 2.** *Let  $f : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . Suppose that  $(f^t)_{t \in \mathbb{R}}$  is a measurable iteration group of  $f$ . If a function  $g : I \rightarrow I$  is continuous at at least one point and commutes with two functions  $f^a$  and  $f^b$  such that  $\frac{b}{a}$  is irrational, then there exists a  $c \in \mathbb{R}$  such that  $g = f^c$ .*

Assuming that  $a < 0 < b$ , the commutativity assumption may be weakened. Namely, we prove the following

**Theorem 3.** *Let  $f : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . Suppose that  $(f^t)_{t \in \mathbb{R}}$  is a measurable iteration group of  $f$ . If a function  $g : I \rightarrow I$  is continuous at at least one point and*

$$g \circ f^a \leq f^a \circ g, \quad g \circ f^b \leq f^b \circ g,$$

where  $a, b \in \mathbb{R} \setminus \{0\}$  are such that  $\frac{b}{a}$  is irrational and  $a < 0 < b$ , then there exists a  $c \in \mathbb{R}$  such that  $g = f^c$ .

*Proof.* According to Zdun's theorem,  $(f^t)_{t \in \mathbb{R}}$  has to be a continuous iteration group (cf. [5]). Consequently, there exists a continuous and strictly monotonic function  $\varphi : I \rightarrow \mathbb{R}$  with  $\varphi(I) = \mathbb{R}$  such that

$$f^t(x) = \varphi^{-1}(\varphi(x) + t), \quad x \in I, t \in \mathbb{R}.$$

Without any loss of generality, we may assume that  $\varphi$  is increasing. By the assumption we hence get

$$g(\varphi^{-1}(\varphi(x) + a)) \leq \varphi^{-1}(\varphi(g(x)) + a), \quad x \in I,$$

and

$$g(\varphi^{-1}(\varphi(x) + b)) \leq \varphi^{-1}(\varphi(g(x)) + b), \quad x \in I.$$

It follows that the function  $\gamma := \varphi \circ g \circ \varphi^{-1}$  satisfies the simultaneous system of functional inequalities

$$\gamma(t + a) \leq \gamma(t) + a, \quad \gamma(t + b) \leq \gamma(t) + b, \quad t \in \mathbb{R}.$$

Since the function  $\gamma$  is continuous at at least one point, there is a  $c \in \mathbb{R}$  such that  $\gamma(t) = t + c$  for all  $t \in \mathbb{R}$  (cf. Theorem 1 in [1]). Thus  $\varphi \circ g \circ \varphi^{-1}(t) = t + c$  for all  $t \in \mathbb{R}$ , whence  $g = \varphi^{-1}(\varphi(x) + c)$  for all  $x \in I$  which, by representation formula (RF), means that  $g = f^c$ .  $\square$

**Theorem 4.** Let  $f, g : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . Suppose that  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$  are measurable iteration groups of  $f$  and  $g$ , respectively. If for some noncommensurable  $\alpha, \beta \in \mathbb{R}$ , each of the functions  $g^\alpha$  and  $g^\beta$  commutes with two functions of the iteration group  $(f^t)_{t \in \mathbb{R}}$  of the noncommensurable iterative indices, then

$$g^t = f^{pt}, \quad t \in \mathbb{R},$$

for some  $p \in \mathbb{R}$ ,  $p \neq 0$ ; in particular,  $\{g^t : t \in \mathbb{R}\} = \{f^t : t \in \mathbb{R}\}$ .

*Proof.* In view of Theorem 1 there are  $a, b \in \mathbb{R}$  such that  $g^\alpha = f^a$  and  $g^\beta = f^b$ . Hence, by induction,  $g^{m\alpha} = f^{ma}$ ,  $g^{n\beta} = f^{nb}$  for all integers  $m, n \in \mathbb{Z}$ , whence,

$$g^{m\alpha+n\beta} = f^{ma+nb}, \quad m, n \in \mathbb{Z}. \quad (3.1)$$

Since, by the representation formula,

$$f^t(x) = \varphi^{-1}(\varphi(x) + t), \quad g^t(x) = \psi^{-1}(\psi(x) + t), \quad x \in I, t \in \mathbb{R},$$

for some homeomorphic functions  $\varphi, \psi : I \rightarrow \mathbb{R}$ , we get

$$\psi^{-1}(\psi(x) + m\alpha + n\beta) = \varphi^{-1}(\varphi(x) + ma + nb), \quad x \in I; m, n \in \mathbb{Z}. \quad (3.2)$$

Take an arbitrary  $t \in \mathbb{R}$ . The density of set  $\{m\alpha + n\beta : m, n \in \mathbb{Z}\}$  in  $\mathbb{R}$  implies that there exist two sequences of integers  $(m_k), (n_k)$  such that

$$t = \lim_{k \rightarrow \infty} (m_k\alpha + n_k\beta). \quad (3.3)$$

The noncommensurability of  $\alpha$  and  $\beta$  implies that  $\alpha\beta \neq 0$  and

$$\lim_{k \rightarrow \infty} |m_k| = \infty = \lim_{k \rightarrow \infty} |n_k|.$$

Therefore, from (3.3),

$$\lim_{k \rightarrow \infty} \left( \frac{m_k}{n_k} \alpha + \beta \right) = \lim_{k \rightarrow \infty} \frac{m_k\alpha + n_k\beta}{n_k} = \lim_{k \rightarrow \infty} \frac{t}{n_k} = 0,$$

which implies that

$$\lim_{k \rightarrow \infty} \frac{m_k}{n_k} = -\frac{\beta}{\alpha}.$$

From (3.2), for an arbitrarily fixed  $x \in I$ ,

$$m_k a + n_k b = \varphi [\psi^{-1}(\psi(x) + m_k\alpha + n_k\beta)] - \varphi(x), \quad k \in \mathbb{N},$$

whence, by the continuity of  $\varphi, \varphi^{-1}, \psi, \psi^{-1}$ ,

$$\lim_{k \rightarrow \infty} (m_k a + n_k b) = \varphi [\psi^{-1}(\psi(x) + t)] - \varphi(x).$$

Repeating now the above argument we infer that

$$\lim_{k \rightarrow \infty} \frac{m_k}{n_k} = -\frac{b}{a},$$

and, consequently  $\frac{b}{a} = \frac{\beta}{\alpha}$ , whence  $\frac{a}{\alpha} = \frac{b}{\beta}$ . Setting

$$p := \frac{a}{\alpha} = \frac{b}{\beta}$$

we get

$$a = p\alpha, \quad b = p\beta,$$

which implies that  $a$  and  $b$  are not commensurable and, by (3.1),

$$g^{m\alpha+n\beta} = f^{p(m\alpha+n\beta)}, \quad m, n \in \mathbb{Z},$$

whence, in particular,

$$g^{m_k\alpha+n_k\beta} = f^{p(m_k\alpha+n_k\beta)}, \quad k \in \mathbb{N}.$$

Letting here  $k \rightarrow \infty$ , we obtain

$$g^t = f^{pt},$$

which completes the proof.  $\square$

Applying Theorem 2 and Theorem 3, we obtain

**Theorem 5.** *Let  $f, g : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . Suppose that  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$  are measurable iteration groups of  $f$  and  $g$ , respectively. If for some noncommensurable  $\alpha, \beta \in \mathbb{R}$  there are two pairs  $(a, b)$  and  $(c, d)$  of noncommensurable real numbers such that*

$$a < 0 < b, \quad g^\alpha \circ f^a \leq f^a \circ g^\alpha, \quad g^\alpha \circ f^b \leq f^b \circ g^\alpha,$$

and

$$c < 0 < d, \quad g^\beta \circ f^c \leq f^c \circ g^\beta, \quad g^\beta \circ f^d \leq f^d \circ g^\beta,$$

then

$$g^t = f^{pt}, \quad t \in \mathbb{R},$$

for some  $p \in \mathbb{R}$ ,  $p \neq 0$ ; in particular,  $\{g^t : t \in \mathbb{R}\} = \{f^t : t \in \mathbb{R}\}$ .

**Remark 2.** *Note that the function  $f^{pt}$  in the previous result can be treated as the  $p$ -th iterate of  $f^t$ . Indeed, let  $\varphi : I \rightarrow \mathbb{R}$  be a generator of the iteration group of  $f$ . If  $n \in \mathbb{N}$ , then, by induction,*

$$(f^t)^{\frac{m}{n}}(x) = \varphi^{-1}\left(\varphi(x) + \frac{m}{n}t\right) = f^{\frac{m}{n}t}, \quad n \in \mathbb{N}, m \in \mathbb{Z}, t \in \mathbb{R}, x \in I,$$

which means that

$$(f^t)^p(x) = \varphi^{-1}(\varphi(x) + pt) = f^{pt}, \quad p \in \mathbb{Q}, t \in \mathbb{R}, x \in I.$$

The continuity of the iteration group and the functions  $\varphi$  and  $\varphi^{-1}$  implies that this relation can be uniquely extended onto all  $p \in \mathbb{R}$ .

**Corollary 1** ([3]). *Let  $f, g : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . If the measurable iteration groups  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$  of  $f$  and  $g$ , respectively, are commuting in pairs, i.e.,*

$$f^t \circ g^t = g^t \circ f^t, \quad t \in \mathbb{R},$$

then, for every  $t \in \mathbb{R}$ ,  $g^t = f^{tp}$  for some  $p \in \mathbb{R}$ ,  $p \neq 0$ .

#### 4. LOCALLY COMMUTING FUNCTIONS

In this section we prove the following local counterpart of Theorem 2.

**Theorem 6.** *Let  $f : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . Suppose that  $(f^t)_{t \in \mathbb{R}}$  is a measurable iteration group of  $f$ . Let  $\xi$  be one of the endpoints of  $I$  and  $I_\xi \subset I$  be an arbitrary nonempty open interval with the endpoint  $\xi$ . If a function  $g : I_\xi \rightarrow I$  is continuous at at least one point and there are  $a, b \in \mathbb{R}$ ,  $a \neq 0$ ,  $\frac{b}{a}$  irrational and such that*

$$f^a(I_\xi) \subset I_\xi, \quad f^b(I_\xi) \subset I_\xi \tag{4.1}$$

and

$$g \circ f^a = f^a \circ g, \quad g \circ f^b = f^b \circ g, \quad (4.2)$$

then  $g = f^c|_{I_\xi}$  for some  $c \in \mathbb{R}$ .

*Proof.* We shall only consider the case of  $\xi$  finite, as in the opposite case the argument is analogous. Without any loss of generality, we may assume that  $\xi = 0$  is the left-end point of  $I$  and  $I_\xi = (0, \delta)$  for some  $\delta > 0$ . Since

$$f^t(x) = \varphi^{-1}(\varphi(x) + t), \quad x \in I, t \in \mathbb{R},$$

for a homeomorphic function  $\varphi : I \rightarrow \mathbb{R}$ , then

$$f^a(x) = \varphi^{-1}(\varphi(x) + a), \quad f^b(x) = \varphi^{-1}(\varphi(x) + b), \quad x \in I.$$

Assume that, for instance,  $\varphi$  is increasing. Then, by (4.1), both  $a$  and  $b$  must be negative, and by (4.2),

$$g[\varphi^{-1}(\varphi(x) + a)] = \varphi^{-1}(\varphi(g(x)) + a), \quad x \in (0, \delta),$$

$$g[\varphi^{-1}(\varphi(x) + b)] = \varphi^{-1}(\varphi(g(x)) + b), \quad x \in (0, \delta).$$

Since  $\varphi((0, \delta)) = (-\infty, \varphi(\delta))$ , the function  $\gamma := \varphi \circ g \circ \varphi^{-1}$  is defined on  $(-\infty, \varphi(\delta))$  and satisfies the simultaneous system of functional equations

$$\gamma(t + a) = \gamma(t) + a, \quad \gamma(t + b) = \gamma(t) + b, \quad t < \varphi(\delta).$$

Assuming  $I = \mathbb{R}$ ,  $\xi = -\infty$ ,  $I_\xi = (-\infty, \varphi(\delta))$ ,  $f_1(t) = t + a$ ,  $f_2(t) = t + b$ ,  $g_1 = g_2 = 1$  it is easy to verify that the conditions of Theorem 1 are fulfilled. Therefore there exists a unique function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\Phi|_{(-\infty, \varphi(\delta))} = \gamma$  and

$$\Phi(t + a) = \Phi(t) + a, \quad \Phi(t + b) = \Phi(t) + b, \quad t \in \mathbb{R}.$$

Since the function  $\Phi$  is continuous at at least one point, applying Theorem 1 of [3], we conclude that, for some  $c \in \mathbb{R}$ ,

$$\Phi(t) = t + c, \quad t \in \mathbb{R}.$$

It follows that  $\gamma(t) := \varphi \circ g \circ \varphi^{-1}(t) = t + c$  for all  $t < \varphi(\delta)$ , whence, by representation formula (RF)

$$g(x) = \varphi^{-1}(\varphi(x) + c) = f^c(x), \quad x \in (0, \delta).$$

This completes the proof. □

## 5. EQUALITY OF ITERATION GROUPS

We begin this section with the following

**Proposition 1.** *Let  $f, g : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . Suppose that  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$  are measurable iteration groups of  $f$  and  $g$ , respectively. If for some  $x_0 \in I$ ,*

$$f^t(x_0) = g^t(x_0) \quad \text{for all } t \in \mathbb{R},$$

then

$$f^t = g^t \quad \text{for all } t \in \mathbb{R}.$$

*Proof.* Let  $\varphi, \psi : I \rightarrow \mathbb{R}$  be the generators of the iteration groups  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$ , respectively. By the assumption,

$$\varphi^{-1}(\varphi(x_0) + t) = \psi^{-1}(\psi(x_0) + t), \quad t \in \mathbb{R},$$

whence, after simple calculations, we obtain

$$\psi(x) = \varphi(x) + \psi(x_0) - \varphi(x_0),$$

and Remark 1 completes the proof.  $\square$

**Remark 3.** *According to the above proposition, every measurable iteration group is uniquely determined by any of its orbits  $\{f^t(x_0) : t \in \mathbb{R}\}$ .*

**Theorem 7.** *Let  $f, g : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . Suppose that  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$  are measurable iteration groups of  $f$  and  $g$ , respectively. If*

$$f^a \leq g^\alpha, \quad f^b \leq g^\beta,$$

for some  $a, b, \alpha, \beta \in \mathbb{R}$  such that

$$a < 0 < b, \quad \frac{b}{a} \notin \mathbb{Q}, \quad \frac{\alpha}{a} \geq \frac{\beta}{b},$$

then

$$g^t = f^{pt}, \quad t \in \mathbb{R},$$

for some  $p \in \mathbb{R}$ ,  $p \neq 0$ ; in particular,  $\{g^t : t \in \mathbb{R}\} = \{f^t : t \in \mathbb{R}\}$ .

*Proof.* Let  $\varphi, \psi : I \rightarrow \mathbb{R}$  be the generators of the iteration groups  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$ , respectively. By the assumption,

$$\varphi^{-1}(\varphi(x) + a) \leq \psi^{-1}(\psi(x) + \alpha), \quad x \in I,$$

and

$$\varphi^{-1}(\varphi(x) + b) \leq \psi^{-1}(\psi(x) + \beta), \quad x \in I.$$

Setting  $\gamma := \psi \circ \varphi^{-1}$  we hence get

$$\gamma(t + a) \leq \gamma(t) + \alpha, \quad \gamma(t + b) \leq \gamma(t) + \beta, \quad t \in \mathbb{R}.$$

Applying Lemma 1 we obtain  $\gamma(t) = \frac{1}{p}t + q$  for all  $t \in \mathbb{R}$ , where  $p := \frac{a}{\alpha}$  and  $q = \gamma(0)$ , that is  $\psi \circ \varphi^{-1}(t) = \frac{1}{p}t + q$  ( $t \in \mathbb{R}$ ) whence

$$\psi(x) = p\varphi(x) + q, \quad t \in \mathbb{R}.$$

Now, by simple calculations, we obtain

$$g^t(x) = \psi^{-1}(\psi(x) + t) = \varphi^{-1}(\varphi(x) + pt) = f^{pt}(x)$$

for all  $x \in I$  and  $t \in \mathbb{R}$ . □

If  $\alpha = a$  and  $\beta = b$  in the above theorem, then  $p = 1$  and we obtain the following

**Corollary 2.** *Let  $f, g : I \rightarrow I$  be strictly increasing, onto and without fixed points in an open interval  $I$ . Suppose that  $(f^t)_{t \in \mathbb{R}}$  and  $(g^t)_{t \in \mathbb{R}}$  are measurable iteration group of  $f$  and  $g$ , respectively. If*

$$f^a \leq g^a, \quad f^b \leq g^b,$$

for some  $a, b \in \mathbb{R}$  such that

$$a < 0 < b, \quad \frac{b}{a} \notin \mathbb{Q},$$

then

$$f^t = g^t, \quad t \in \mathbb{R}.$$

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Janusz Matkowski  
j.matkowski@wmie.uz.im.zgora.pl

University of Zielona Góra  
Institute of Mathematics, Computer Science and Econometry,  
ul. Podgórna 50, 65-246 Zielona Góra, Poland

Silesian University  
Institute of Mathematics  
40-007 Katowice, Poland

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