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THE ASYMPTOTIC PROPERTIES OF THE DYNAMIC EQUATION WITH A DELAYED ARGUMENT

Abstract. In this paper, we present some asymptotic results related to the scalar dynamic equation with a delayed argument. Using the time scale calculus we generalize some results known in the differential and difference case to the more general dynamic case.

Keywords: dynamic equation, time scale, asymptotic behavior.

Mathematics Subject Classification: Primary 34K25, 39A10, 39A11; Secondary 39B22.

1. INTRODUCTION

In this paper, we discuss the asymptotic properties of the delay dynamic equation

$$y^{\Delta}(t) = -a(t)y(t) + b(t)y(\tau(t)), \qquad t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real line) and a, b and τ are functions defined on $\mathbb{T}$ such that $a(t) > 0$, $b(t) \neq 0$, $\tau(t) < t$ and $\tau(t)$ is increasing for all $t \in \mathbb{T}$. Throughout this paper, we assume that $\sup \mathbb{T} = \infty$ and $\tau(t) \in \mathbb{T}$ if $t \in \mathbb{T}$ (i.e., the function τ is mapping $\mathbb{T}$ into itself).

To introduce some further requirements on a, b and τ we shortly recall some basic notions of the calculus on time scales (for further explanation, results and details we refer to the books Bohner and Peterson [4, 5]).

We define the *forward jump operator* $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ as well as the *backward jump operator* $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. The function $\mu(t) := \sigma(t) - t$ is of great importance in the theory of time scales and it is called the *graininess function.* Further, for a function $f : \mathbb{T} \to \mathbb{R}$, $f^{\Delta}(t)$ is the *delta-derivative* of f at $t \in \mathbb{T}$ defined by the relation

$$f^{\Delta}(t) := \lim_{\underset{s\to t}{\rightarrow}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

Consequences of this definition and properties of f^Δ can be found in [4]. We particularly emphasize that if $\mathbb{T} = \mathbb{R}$, then $f^\Delta = f'$, while if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta = \Delta f$, where Δ is the usual difference operator. In other words, if $\mathbb{T} = \mathbb{R}$, then equation (1.1) becomes the delay differential equation, while if $\mathbb{T} = \mathbb{Z}$ and $\tau(t) = t - k$, $k \in \mathbb{N}$, then (1.1) becomes the three term difference equation.

Instead of the usual conditions of the continuity of the coefficients a, b and the delayed argument τ we assume that a, b and τ are *rd-continuous*, which means that they are continuous at right-dense points (i.e., points t, where $\sigma(t) = t$) and if the left-hand side limit exists at left-dense points (i.e., points t, where $\rho(t) = t$).We denote the set of all rd-continuous functions on $\mathbb{T}$ by $C_{rd}(\mathbb{T})$. It can be shown (see [4]) that any rd-continuous function f has an antiderivative. Then the Cauchy integral of f is defined by

$$\int_c^d f(t)\Delta t = F(d) - F(c), \qquad \text{where } F^\Delta = f \quad \text{on } \mathbb{T}, \quad c, d \in \mathbb{T}.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive (we write $p \in \mathcal{R}^+(\mathbb{T})$ or shortly $p \in \mathcal{R}^+$) if it is rd-continuous and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$, where μ is the graininess function. It is known that for any $p \in \mathcal{R}^+$ there exists the unique function y satisfying the initial value problem

$$y^\Delta(t) = p(t)y(t), \qquad y(t_0) = 1$$

(where $t_0 \in \mathbb{T}$) and, moreover, this function is positive for all $t \in \mathbb{T}$. We denote this $y(t)$ by $e_p(t, t_0)$ and call it the exponential function. The construction of the explicit form of $e_p(t, t_0)$ can be found in [4]. However, the knowledge of this form is not important for our further investigations and we can omit it.

The origin of equation (1.1) in the differential setting (i.e., for $\mathbb{T} = \mathbb{R}$) goes back to the paper by De Bruijn [7], which started the qualitative (especially asymptotic) investigations of the differential-difference equation

$$y'(t) = -a(t)[y(t) - y(t-1)], \tag{1.2}$$

where a is a positive function satisfying some additional properties. Later, equation (1.2) and its modifications were the subject of further systematic investigations (see, e.g., Arino and Pituk [1], Atkinson and Haddock [2], Diblík [8] or Krisztin [13]).

Similarly, asymptotic properties of linear delay differential equations with a negative coefficient at $y(t)$ were also derived for other types of delays. Linear delay differential equations with rescaling (i.e. such that $\tau(t) = \lambda t$, $t \geq 0$, $0 < \lambda < 1$) have been studied, e.g., by Kato and McLeod [12], Iserles [10], Liu [14] and others. For related asymptotic results on linear differential equations with a general delayed argument, we refer to papers by Heard [11], Makay and Terjéki [15] or [6].

The qualitative investigation of equation (1.1) in the difference setting (i.e. for $\mathbb{T} = \mathbb{Z}$) is less developed than in the continuous case. Some relevant results can be found in Győri and Pituk [9] or Péics [17], where the additional assumption $a(t) < 1$

has been used. In Section 3, we mention the connection between this assumption and the positive regressivity of the function $-a$.

In this paper, we wish to generalize some known asymptotic estimates of solutions of delay differential or difference equations to the case of delay dynamic equation (1.1). To the best of our knowledge, this is the first paper dealing with the asymptotic properties of delay dynamic equations on time scales. For some recent oscillation results on delay dynamic equations on time scales, we refer to papers by Bohner [3], Mathsen, Wang and Wu [16], and Zhang and Deng [18]. The common idea of these papers is very close to ours: they unify and extend some oscillation criteria known in the differential and/or difference setting to the corresponding delay dynamic equations on time scales.

2. ASYMPTOTIC BOUNDS OF SOLUTIONS

In this section, we consider delay dynamic equation (1.1) and the auxiliary functional inequality

$$-a(t)\varphi(t) + |b(t)|\varphi(\tau(t)) \leq 0, \qquad t \in \mathbb{T}. \tag{2.1}$$

The main result of this paper yields the estimate of all solutions of (1.1) in terms of a solution of (2.1).

Theorem 1. *Consider equation* (1.1)*, where* $-a \in \mathcal{R}^+$*,* $b, \tau \in C_{rd}(\mathbb{T})$*,* $a(t) > 0$*,* $b(t) \neq 0$ *for all* $t \in \mathbb{T}$*,* $\tau : \mathbb{T} \to \mathbb{T}$ *is increasing on* $\mathbb{T}$*, satisfying* $\tau(t) < t$ *for all* $t \in \mathbb{T}$ *and* $\underset{t\to\infty}{\to} \lim \tau(t) = \infty$*. Further, assume that there exists a positive and delta-differentiable function* $\varphi : \mathbb{T} \to \mathbb{R}$ *fulfilling* (2.1) *such that* φ^Δ *is nonnegative on* $\mathbb{T}$*. Then*

$$y(t) = O(\varphi(t)) \quad \text{as } t \to \infty$$

for any solution y *of* (1.1)*.*

Proof. Let y be a solution of (1.1) defined on $[t_0, \infty) \cap \mathbb{T}$, $t_0 \in \mathbb{T}$. We introduce the substitution $z(t) = \frac{y(t)}{\varphi(t)}$, where φ has all the properties listed above. Our aim is to show that $z(t)$ is bounded as $t \to \infty$.

Using the basic properties of the delta-derivative, especially the product rule

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t)\,,$$

we can rewrite equation (1.1) as

$$z^\Delta(t)\varphi(t) = -z(\sigma(t))\varphi^\Delta(t) - a(t)z(t)\varphi(t) + b(t)z(\tau(t))\varphi(\tau(t))\,. \tag{2.2}$$

Further, multiplying (2.2) by $1/e_{-a}(\sigma(t), t_0)$ and applying the quotient rule

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}$$

we can check that relation (2.2) is equivalent to

$$\Big[\frac{z(t)\varphi(t)}{\mathrm{e}_{-a}(t,t_0)}\Big]^{\Delta}=\frac{b(t)z(\tau(t))\varphi(\tau(t))}{\mathrm{e}_{-a}(\sigma(t),t_0)}, \tag{2.3}$$

where, in the accordance with our previous notation $\mathrm{e}_{-a}(t,t_0)$ is the (unique) positive exponential function solving the initial value problem

$$y^{\Delta}(t)=-a(t)y(t), \qquad y(t_0)=1\,.$$

Now we define a mesh of points $t_m:=\tau^{-m}(t_0)$, where τ^{-m} means the m-th iterate of the inverse function τ^{-1} and denote $I_{m+1}=:\langle t_m,t_{m+1}\rangle\cap\mathbb{T}$, $m=0,1,2,\dots$. Then $\bigcup_{i=1}^{\infty}I_m=\langle t_0,\infty)\cap\mathbb{T}$ and I_m is nonempty for any $m=0,1,2,\dots$. Indeed, if I_m is empty for a fixed m, then I_{m+1} must be empty (otherwise if $\bar t\in I_{m+1}$, then $\tau(\bar t)\in I_m$) and this would contradict the assumption $\sup\mathbb{T}=\infty$.

Further, denote $B_m:=\sup\{|z(t)|,\ t\in I_m\}$ and consider any $t^*\in I_{m+1}$, $m\ge 1$. Integrating (2.3) (in the sense of the integration on time scales) from t_m to t^*, we get

$$\int_{t_m}^{t^*}\Big[\frac{z(t)\varphi(t)}{\mathrm{e}_{-a}(t,t_0)}\Big]^{\Delta}\Delta t=\int_{t_m}^{t^*}\frac{b(t)z(\tau(t))\varphi(\tau(t))}{\mathrm{e}_{-a}(\sigma(t),t_0)}\Delta t,$$

i.e., by the definition of the Cauchy integral,

$$\frac{z(t^*)\varphi(t^*)}{\mathrm{e}_{-a}(t^*,t_0)}-\frac{z(t_m)\varphi(t_m)}{\mathrm{e}_{-a}(t_m,t_0)}=\int_{t_m}^{t^*}\frac{b(t)z(\tau(t))\varphi(\tau(t))}{\mathrm{e}_{-a}(\sigma(t),t_0)}\Delta t.$$

From here we can determine $z(t^*)$ as

$$z(t^*)=\frac{\varphi(t_m)\mathrm{e}_{-a}(t^*,t_0)}{\varphi(t^*)\mathrm{e}_{-a}(t_m,t_0)}z(t_m)+\frac{\mathrm{e}_{-a}(t^*,t_0)}{\varphi(t^*)}\int_{t_m}^{t^*}\frac{b(t)z(\tau(t))\varphi(\tau(t))}{\mathrm{e}_{-a}(\sigma(t),t_0)}\Delta t,$$

hence

$$\begin{aligned}|z(t^*)|\le& B_m\frac{\varphi(t_m)\mathrm{e}_{-a}(t^*,t_0)}{\varphi(t^*)\mathrm{e}_{-a}(t_m,t_0)}+B_m\frac{\mathrm{e}_{-a}(t^*,t_0)}{\varphi(t^*)}\int_{t_m}^{t^*}\frac{|b(t)|\varphi(\tau(t))}{\mathrm{e}_{-a}(\sigma(t),t_0)}\Delta t\le\\ \le& B_m\frac{\varphi(t_m)\mathrm{e}_{-a}(t^*,t_0)}{\varphi(t^*)\mathrm{e}_{-a}(t_m,t_0)}+B_m\frac{\mathrm{e}_{-a}(t^*,t_0)}{\varphi(t^*)}\int_{t_m}^{t^*}\frac{a(t)\varphi(t)}{\mathrm{e}_{-a}(\sigma(t),t_0)}\Delta t\end{aligned} \tag{2.4}$$

by use of (2.1). Applying the integration by parts to the last integral,

$$\int_{t_m}^{t^*}f^{\Delta}(t)g(t)\Delta t=(fg)(t^*)-(fg)(t_m)-\int_{t_m}^{t^*}f(\sigma(t))g^{\Delta}(t)\Delta t$$

we obtain

$$\begin{aligned}\int_{t_m}^{t^*} \frac{a(t)\varphi(t)}{\mathrm{e}_{-a}(\sigma(t),t_0)}\Delta t = \int_{t_m}^{t^*} \Big[\frac{1}{\mathrm{e}_{-a}(t,t_0)}\Big]^{\Delta}\varphi(t)\Delta t = \\ = \frac{\varphi(t^*)}{\mathrm{e}_{-a}(t^*,t_0)} - \frac{\varphi(t_m)}{\mathrm{e}_{-a}(t_m,t_0)} - \int_{t_m}^{t^*} \varphi^{\Delta}(t)\frac{1}{\mathrm{e}_{-a}(\sigma(t),t_0)}\Delta t \le \\ \le \frac{\varphi(t^*)}{\mathrm{e}_{-a}(t^*,t_0)} - \frac{\varphi(t_m)}{\mathrm{e}_{-a}(t_m,t_0)}\,.\end{aligned}$$

Substituting this back to inequality (2.4), we get

$$|z(t^*)| \le B_m \frac{\varphi(t_m)\mathrm{e}_{-a}(t^*,t_0)}{\varphi(t^*)\mathrm{e}_{-a}(t_m,t_0)} + B_m \frac{\mathrm{e}_{-a}(t^*,t_0)}{\varphi(t^*)}\Big(\frac{\varphi(t^*)}{\mathrm{e}_{-a}(t^*,t_0)} - \frac{\varphi(t_m)}{\mathrm{e}_{-a}(t_m,t_0)}\Big) = B_m\,.$$

The fact that $t^* \in I_{m+1}$ was arbitrary implies $B_{m+1} \le B_m$ for any $m = 1, 2, \dots,$ i.e., the sequence (B_m) is bounded as $m \to \infty$, which implies that also $z(t)$ is bounded as $t \to \infty$. The proof is complete. □

3. APPLICATIONS

Corollary 1. *Let $\mathbb{T} = \mathbb{R}$ and consider the delay differential equation*

$$y'(t) = -a(t)y(t) + b(t)y(\tau(t))\,, \tag{3.1}$$

where $t \ge t_0$, $a, b, \tau \in C([t_0,\infty))$, $a(t) > 0$, $b(t) \ne 0$, $\tau(t) < t$, $\tau(t)$ is increasing for all $t \ge t_0$ and $\underset{t\to\infty}{\to} \lim \tau(t) = \infty$. Let φ be a positive, nondecreasing and differentiable function φ fulfilling (2.1) *on $[t_0,\infty)$. Then*

$$y(t) = O(\varphi(t)) \quad \text{as } t \to \infty$$

for any solution y of (3.1).

Proof. We verify the validity of the assumptions of Theorem 1. First we note that for $\mathbb{T} = \mathbb{R}$ the forward jump operator $\sigma(t)$ is the identity function, hence the graininess function is the zero function on $\mathbb{T}$ and the property $1 - a(t)\mu(t) > 0$ for all $t \in \mathbb{T}$ (following from the assumption $-a \in \mathcal{R}^+$) is satisfied trivially. Further, the assumption of rd-continuity of a, b and τ follows immediately from their continuity. The statement now follows from Theorem 1. □

Below, we show that the previous result generalizes some known asymptotic results concerning delay differential equations.

Example 1. *To illustrate this generalization, we consider the differential equation with rescaling of the form*

$$y'(t) = -ay(t) + by(\lambda t), \quad t \ge 0, \tag{3.2}$$

where $|b| \geq a > 0$ and $0 < \lambda < 1$ are scalars. Then inequality (3) becomes

$$-a\varphi(t) + |b|\varphi(\lambda t) \leq 0, \quad t \geq 0 \tag{3.3}$$

and it is easy to check that the function

$$\varphi(t) = t^{\alpha}, \quad \alpha = \frac{\log(|b|/a)}{\log \lambda^{-1}}$$

fulfils (3.3) *for all $t > 0$ (even in the form of the equality). Hence, by Corollary 1,*

$$y(t) = O(t^{\alpha}) \quad \text{as } t \to \infty$$

for any solution y of (3.2). *We note that this result is just a part of Theorem 3 of Kato and McLeod [12].*

Similarly we can prove this generalization to hold for differential equations with other types of delayed argument.

Corollary 2. *Let $\mathbb{T} = \mathbb{Z}$ and let y be a solution of the difference equation*

$$\triangle y(n) = -a(n)y(n) + b(n)y(n-k), \quad n \in \mathbb{Z},\, k \in \mathbb{N}\,,$$

where $0 < a(n) < 1$, $b(n) \neq 0$ and let φ be a positive and nondecreasing solution of

$$-a(n)\varphi(n) + |b(n)|\varphi(n-k) \leq 0\,.$$

Then

$$y(n) = O(\varphi(n)) \quad \text{as } n \to \infty\,.$$

Proof. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(n) = n + 1$, hence $\mu(n) \equiv 1$ for all $n \in \mathbb{Z}$. Now the condition $-a \in \mathcal{R}^{+}$ (especially the property $1 - \mu(n)a(n) > 0$) is equivalent to the inequality $a(n) < 1$ involved in the assumptions of Corollary 2. The verification of remaining assumptions of Theorem 1 is easy. □

Remark 1. *Note that the assumption $a(n) < 1$ also appears in a similar asymptotic investigation done by Péics [17]. Following the proof of Corollary 2, this assumption seems to be a consequence of the positive regressivity of $-a$.*

Example 2. *Consider the difference equation*

$$\Delta y(n) = -a(n)y(n) + ra(n)y(n-k), \quad n \in \mathbb{Z},\, k \in \mathbb{N},\, r \in \mathbb{R},\, |r| \geq 1, \tag{3.4}$$

where $0 < a(n) < 1$ for all $n \in \mathbb{Z}$. Then inequality (2.1) *becomes*

$$-a(n)\varphi(n) + |r|a(n)\varphi(n-k) \leq 0 \tag{3.5}$$

and it is easy to check that the function $\varphi(n) = |r|^{n/k}$ fulfills (3.5) *(even in the form of the equality). Hence,*

$$y(n) = O\left(|r|^{\frac{n}{k}}\right) \quad \text{as } n \to \infty$$

for any solution y of (3.4).

Quite similarly we can formulate the asymptotic estimates of solutions of (1.1) for other choices of $\mathbb{T}$. In the final part, we consider the case $\mathbb{T} = q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ appearing in the q-calculus.

Corollary 3. *Let $\mathbb{T} = q^{\mathbb{Z}}$ with $q > 1$ and consider the delay equation*

$$y^{\Delta}(t) = -a(t)y(t) + b(t)y(q^{-1}t), \quad t \in \mathbb{T}$$

leading (under the choice $\mathbb{T} = q^{\mathbb{Z}}$) to the q-difference equation

$$\frac{y(q^{n+1}) - y(q^n)}{q-1} = -a(q^n)q^n y(q^n) + b(q^n)q^n y(q^{n-1}), \quad n \in \mathbb{Z}, \tag{3.6}$$

where $0 < a(q^n) < q^{-n}/(q-1)$, $b(q^n) \neq 0$ for all $n \in \mathbb{Z}$. Further, let φ be a positive and nondecreasing function fulfilling

$$-a(q^n)\varphi(q^n) + |b(q^n)|\varphi(q^{n-1}) \leq 0, \quad n \in \mathbb{Z}\,.$$

Then

$$y(q^n) = O(\varphi(q^n)) \quad \text{as } n \to \infty$$

for any solution y of (3.6)*.*

Proof. If $\mathbb{T} = q^{\mathbb{Z}}$, then $q^{-1}t \in \mathbb{T}$ for any $t \in \mathbb{T}$, i.e., the delayed argument $\tau(t) = q^{-1}t$ is mapping $\mathbb{T}$ into itself. Further,

$$\sigma(q^n) = q^{n+1}, \qquad \mu(q^n) = (q-1)q^n \quad \text{and} \quad y^{\Delta}(q^n) = \frac{y(q^{n+1}) - y(q^n)}{(q-1)q^n}.$$

Then the assumptions and conclusions of Corollary 3 are the reformulation of those presented in Theorem 1. □

Example 3. Let $\mathbb{T} = q^{\mathbb{Z}}, q > 1$ and consider the equation

$$y^{\Delta}(t) = -\frac{a}{t}y(t) + \frac{b}{t}y(q^{-1}t), \quad t \in q^{\mathbb{Z}}, \tag{3.7}$$

i.e.,

$$\frac{y(q^{n+1}) - y(q^n)}{q-1} = -ay(q^n) + by(q^{n-1}), \quad n \in \mathbb{Z},$$

where $0 < a < 1/(q-1)$, $b \neq 0$, $|b| \geq a$. Then for any solution y there holds

$$y(q^n) = O\left((\frac{|b|}{a})^n \right) \quad \text{as } n \to \infty.$$

Remark 2. *Equation* (3.7) *can be solved (under the choice $\mathbb{T} = q^{\mathbb{Z}}$) explicitly. Hence, we can also use the previous example as the illustration of the sharpness of the condition $|b| \geq a$ (following from the assumption that a solution φ of* (2.1) *must be nondecreasing).*

Put $a = 1/2$, $q = 2$ in (3.7) and consider the equation

$$y(2^{n+1}) - y(2^n) = -\frac{1}{2}y(2^n) + by(2^{n-1}), \quad n \in \mathbb{Z}, \tag{3.8}$$

where $|b| \geq 1/2$. Then, by Corollary 3,

$$y(2^n) = O\left(|2b|^n\right) \quad \text{as } n \to \infty \tag{3.9}$$

for any solution y of (14). On the other hand, for $b > 0$, equation (14) admits the solution

$$y(2^n) = \left(\frac{1 + \sqrt{1 + 16b}}{4}\right)^n, \quad n \in \mathbb{Z}$$

and it is easy to check that for $0 < b < a = 1/2$ this function does not satisfy asymptotic property (15).

Finally we note that the validity of Theorem 1 can also be preserved in the "stable" case (i.e., when the upper bound function φ fulfilling (2.1) is decreasing), but under the introduction of some additional requirements. The research on that subject is currently under way.

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