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FURTHER PROPERTIES OF THE RATIONAL RECURSIVE SEQUENCE $x_{n+1} = \frac{ax_{n-1}}{b+cx_nx_{n-1}}$

Abstract. In this paper we consider the difference equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_nx_{n-1}}, \quad n = 0, 1, \ldots \tag{E}$$

with positive parameters a and c, negative parameter b and nonnegative initial conditions. We investigate the asymptotic behavior of solutions of equation (E).

Keywords: difference equation, explicit formula, positive solutions, asymptotic stability.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

In this paper we consider the following rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_nx_{n-1}}, \quad n = 0, 1, \ldots \tag{E}$$

where b is a negative real number and a and c are positive real numbers and the initial conditions x_{-1}, x_0 are nonnegative real numbers such that at least one of them is positive. Eq. (E) in the case of positive b was considered in [1].We use the explicit formula for solutions of Eq. (E) in investigating their behavior.

There has been a lot of work concerning the asymptotic behavior of solutions of rational difference equations. Second order rational difference equations were investigated, for example, in [1–13]. This paper is motivated by the short notes [4], where the author studied the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{-1 + x_nx_{n-1}}, \quad n = 0, 1, \ldots.$$

2. MAIN RESULTS

Let $p = \frac{b}{a}, q = \frac{c}{a}$. Then Eq. (E) can be rewritten as

$$x_{n+1} = \frac{x_{n-1}}{p + qx_n x_{n-1}}, \quad n = 0, 1, \dots. \tag{E1}$$

The change of variables $x_n = \frac{1}{\sqrt{q}} y_n$ reduces the above equation to

$$y_{n+1} = \frac{y_{n-1}}{p + y_n y_{n-1}}, \quad n = 0, 1, \dots \tag{E2}$$

where p is a negative real number, the initial conditions y_{-1}, y_0 are nonnegative real numbers such that at least one of them is positive. We will also assume $y_0 y_{-1} \neq \frac{p^n(1-p)}{p^n-1}$ for $n = 1, 2, \dots$, $p \neq -1$ and $y_0 y_{-1} \neq 1$ for $p = -1$ (which ensures that the denominator in Eq. (E2) is not equal to zero). Hereafter, we focus our attention on Eq. (E2) instead of Eq. (E). Note, that the solution $\{y_n\}$ of Eq. (E2) with $y_{-1} = 0$ or $y_0 = 0$ is oscillatory. In fact, in this case there is

$$\{y_n\} = \left\{0, y_0, 0, \tfrac{y_0}{p}, 0, \tfrac{y_0}{p^2}, \dots\right\} \text{ or } \{y_n\} = \left\{y_{-1}, 0, \tfrac{y_{-1}}{p}, 0, \tfrac{y_{-1}}{p^2}, 0, \dots\right\}.$$

Obviously, if $p = -1$, these solutions are 4-periodic.

Here, we review some results which will be useful in our investigation of the behavior of solutions of Eq. (E2).

Let I be some interval on the real line and let $f : I \times I \to I$ be a continuous function.

Definition 1. ([8]) *For every pair of initial conditions $(x_{-1}, x_0) \in I \times I$, the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{E3}$$

has the unique solution $\{x_n\}_{n=-1}^{\infty}$, which is called a recursive sequence. An equilibrium point of (E3) *is a point $\alpha \in I$ with $f(\alpha, \alpha) = \alpha$; it is also called a trivial solution of Eq.* (E3).

Definition 2. ([13]) *Let α be an equilibrium point of Eq.*(E3)*:*

(i) *α is stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any initial conditions $(x_{-1}, x_0) \in I \times I$ with $|x_{-1} - \alpha| + |x_0 - \alpha| < \delta$, the inequality $|x_n - \alpha| < \varepsilon$ holds for $n = 1, 2, \dots$;*

(ii) *α is a local attractor if there exists $\gamma > 0$ such that $x_n \to \alpha$ holds for any initial conditions $(x_{-1}, x_0) \in I \times I$ with $|x_{-1} - \alpha| + |x_0 - \alpha| < \gamma$;*

(iii) *α is locally asymptotically stable if it is stable and is a local attractor;*

(iv) *α is a repeller if there exists $\gamma > 0$ such that for each $(x_{-1}, x_0) \in I \times I$ with $|x_{-1} - \alpha| + |x_0 - \alpha| < \gamma$, there exists N such that $|x_N - \alpha| \geq \gamma$.*

Assume α is an equilibrium point of Eq. (E3). Let $r = -\frac{\partial f(\alpha,\alpha)}{\partial x_n}$, $s = -\frac{\partial f(\alpha,\alpha)}{\partial x_{n-1}}$. Then the linearized equation associated with Eq. (E3) about the equilibrium α is

$$z_{n+1} + rz_n + sz_{n-1} = 0. \tag{E4}$$

Theorem A ([7])(Linearized stability theorem).

(i) *If* $|r| < 1 + s$ *and* $s < 1$*, then* α *is locally asymptotically stable.*
(ii) *If* $|r| < |1 + s|$ *and* $|s| > 1$ *then* α *is a repeller.*

The equilibria of Eq. (E2) are the solutions of the equation

$$\bar{y} = \frac{\bar{y}}{p + \bar{y}^2}.$$

So, equilibrium points of Eq. (E2) are $\bar{y} = 0$ and $\bar{y} = \pm\sqrt{1-p}$. The local asymptotic behavior of the zero equilibrium of Eq. (E2) is characterized by the following result.

Theorem 1. *The following statements are true:*

(i) *if* $p \in (-\infty, -1)$*, then* $\bar{y} = 0$ *is locally asymptotically stable;*
(ii) *if* $p \in (-1, 0)$*, then* $\bar{y} = 0$ *is a repeller.*

Proof. For Eq. (E2), there is

$$\frac{\partial f}{\partial y_n} = -\frac{y_{n-1}^2}{(p + y_n y_{n-1})^2},$$
$$\frac{\partial f}{\partial y_{n-1}} = \frac{p}{(p + y_n y_{n-1})^2}.$$

Therefore, for $\bar{y} = 0$ we get $r = 0$, $s = -\frac{1}{p}$ and the linearized equation associated with Eq. (E2) about the equilibrium $\bar{y} = 0$ is

$$z_{n+1} - \frac{1}{p} z_{n-1} = 0.$$

(i) The result follows from Theorem A(i) and the following relations

$$|r| - (1 + s) = -1 + \frac{1}{p} < 0,$$

and

$$s = -\frac{1}{p} < 1.$$

(ii) The result follows from Theorem A(ii) and the following relations

$$|r| - |1 + s| = -\left|\frac{p-1}{p}\right| = \frac{1-p}{p} < 0$$

and

$$-\frac{1}{p} > 1.$$

This completes the proof. □

It is easy to see that the method use in the proof of Theorem 1 in [1] can be use in our case too. Thus the following formula

$$y_n = \begin{cases} y_{-1}\dfrac{\prod\limits_{i=0}^{\frac{n+1}{2}-1}[p^{2i}+y_0y_{-1}\sum\limits_{k=0}^{2i-1}p^k]}{\prod\limits_{i=0}^{\frac{n+1}{2}-1}[p^{2i+1}+y_0y_{-1}\sum\limits_{k=0}^{2i}p^k]} & \text{for } n \text{ odd},\\ y_0\dfrac{\prod\limits_{i=0}^{\frac{n}{2}-1}[p^{2i+1}+y_0y_{-1}\sum\limits_{k=0}^{2i}p^k]}{\prod\limits_{i=0}^{\frac{n}{2}-1}[p^{2i+2}+y_0y_{-1}\sum\limits_{k=0}^{2i+1}p^k]} & \text{for } n \text{ even}, \end{cases} \tag{1}$$

holds for all solutions of Eq. (E2) with positive initial conditions y_{-1}, y_0 such that $y_0y_{-1} \neq \frac{p^n(1-p)}{p^n-1}$ for $n = 1, 2, \ldots$, $p \neq -1$ and $y_0y_{-1} \neq 1$ for $p = -1$.
If all parameters and initial conditions in Eq. (E) are positive, then all solutions of Eq. (E) are positive, too. It is not true in the case of negative b. In the next theorem we give sufficient conditions for every solution of Eq. (E2) to be positive.

Theorem 2. *Assume that $p \in (-1, 0)$. Let $\{y_n\}$ be a solution of Eq.* (E2) *with positive initial conditions y_{-1}, y_0 such that $y_0y_{-1} \neq \frac{p^n(1-p)}{p^n-1}$ for $n = 1, 2, \ldots$. If $y_0y_{-1} > -p$ then $\{y_n\}$ is positive.*

Proof. Let $\{y_n\}$ be a solution of Eq. (E2). From (1), for the subsequence $\{y_{2n-1}\}$ there follows

$$y_{2n-1} = y_{-1}\frac{\prod\limits_{i=0}^{n-1}[p^{2i}+y_0y_{-1}\sum\limits_{k=0}^{2i-1}p^k]}{\prod\limits_{i=0}^{n-1}[p^{2i+1}+y_0y_{-1}\sum\limits_{k=0}^{2i}p^k]}.$$

Obviously, for $p \in (-1, 0)$,

$$p^{2i} + y_0y_{-1}\sum_{k=0}^{2i-1}p^k > 0$$

for all $i = 0, 1, \ldots$.
On the other hand, if $y_0y_{-1} > -p$, then

$$p^{2i+1} + y_0y_{-1}\sum_{k=0}^{2i}p^k > 0 \tag{2}$$

for all $i = 0, 1, \ldots$.
Therefore, all terms of the sequence $\{y_{2n-1}\}$ are positive. For n even the proof is similar. $\square$

Remark 1. *If $y_0y_{-1} = 1-p$ then from* (E2) *we get $y_{n+1} = \frac{y_{n-1}}{p+y_ny_{n-1}} = y_{n-1}$. Hence $\{y_{2n}\} = \{y_0, y_0, y_0, \ldots\}$ and $\{y_{2n-1}\} = \{y_{-1}, y_{-1}, y_{-1}, \ldots\}$.*

Theorem 3. *Assume that* $p \in (-1, 0)$. *Let* $\{y_n\}$ *be a solution of Eq.* (E2) *with positive initial conditions* y_{-1}, y_0 *such that* $y_0y_{-1} \neq \frac{p^n(1-p)}{p^n-1}$ *for* $n = 1, 2, \ldots$. *If* $-p < y_0y_{-1} < 1-p$ *then the subsequence* $\{y_{2n}\}$ *is decreasing and subsequence* $\{y_{2n-1}\}$ *is increasing.*

Proof. Let $\{y_n\}$ be a solution of Eq. (E2). From (1), for the subsequence $\{y_{2n}\}$ there follows

$$y_{2n} = y_0 \frac{\prod_{i=0}^{n-1} [p^{2i+1} + y_0y_{-1} \sum_{k=0}^{2i} p^k]}{\prod_{i=0}^{n-1} [p^{2i+2} + y_0y_{-1} \sum_{k=0}^{2i+1} p^k]}.$$

Thus for $n \geq 1$

$$\begin{aligned} \frac{y_{2n+2}}{y_{2n}} &= \frac{\prod_{i=0}^{n} [p^{2i+1} + y_0y_{-1} \sum_{k=0}^{2i} p^k] \prod_{i=0}^{n-1} [p^{2i+2} + y_0y_{-1} \sum_{k=0}^{2i+1} p^k]}{\prod_{i=0}^{n} [p^{2i+2} + y_0y_{-1} \sum_{k=0}^{2i+1} p^k] \prod_{i=0}^{n-1} [p^{2i+1} + y_0y_{-1} \sum_{k=0}^{2i} p^k]} = \\ &= \frac{p^{2n+1} + y_0y_{-1} \sum_{k=0}^{2n} p^k}{p^{2n+2} + y_0y_{-1} \sum_{k=0}^{2n+1} p^k}. \end{aligned} \tag{3}$$

Since $y_0y_{-1} < 1 - p$, there is

$$y_0y_{-1}p^{2n+1} > p^{2n+1} - p^{2n+2}.$$

Hence

$$y_0y_{-1}(\sum_{k=0}^{2n+1} p^k - \sum_{k=0}^{2n} p^k) > p^{2n+1} - p^{2n+2},$$

and therefore

$$p^{2n+1} + y_0y_{-1} \sum_{k=0}^{2n} p^k < p^{2n+2} + y_0y_{-1} \sum_{k=0}^{2n+1} p^k.$$

From the above inequality, by (2) and (3) it follows that the subsequence $\{y_{2n}\}$ is decreasing. Similarly we prove that the subsequence $\{y_{2n-1}\}$ is increasing. This completes the proof. □

Theorem 4. *Assume that* $p \leq -2$. *Let* $\{y_n\}$ *be a solution of Eq.* (E2) *with positive initial conditions* $y_{-1}, y_0 \in (0, 1)$. *Then the subsequences* $\{y_{4n-1}\}$ *and* $\{y_{4n}\}$ *are both positive and decreasing, while subsequences* $\{y_{4n+1}\}$ *and* $\{y_{4n+2}\}$ *are both negative and increasing.*

Proof. Let $y_{-1}, y_0 \in (0,1)$. Then $y_1, y_2 \in (0,1)$ and $y_3, y_4 \in (-1,0)$. By induction we can prove that $\{y_{4n-1}\}$, $\{y_{4n}\} \in (0,1)$ and $\{y_{4n+1}\}$, $\{y_{4n+2}\} \in (-1,0), n = 0,1,\dots$. Since, by (1),

$$\frac{y_{4n+4}}{y_{4n}} = \frac{(p^{4n+1} + y_0 y_{-1}\frac{1-p^{4n+1}}{1-p})(p^{4n+3} + y_0 y_{-1}\frac{1-p^{4n+3}}{1-p})}{(p^{4n+2} + y_0 y_{-1}\frac{1-p^{4n+2}}{1-p})(p^{4n+4} + y_0 y_{-1}\frac{1-p^{4n+4}}{1-p})} < 1,$$

we have

$$y_{4n+4} < y_{4n}, \quad n = 0,1,\dots$$

Similarly we can see that $y_{4n+3} < y_{4n-1}$, and $y_{4n+5} > y_{4n+1}$, $y_{4n+6} > y_{4n+2}$ for $n = 0,1,\dots$ and the result follows. $\square$

3. NUMERICAL RESULTS

Example 1. *Let $y_{-1} = \frac{3}{4}$, $y_0 = 1$ be the initial conditions of Eq.* (E2) *with $p = -\frac{1}{2}$. Then, by Theorem 2, the solution is positive.*

Table 1 sets forth the values of y_n for selected small n's.

Table 1

n	$y(n)$	n	$y(n)$
1	3	2	0.4
3	4.285714285	4	0.3294117647
5	4.700460829	6	0.3142081447
7	4.811495337	8	0.3105403454
9	4.839742863	10	0.3096314468
27	4.849202586	28	0.3093292089

Table 2

n	$y(n)$	n	$y(n)$
-1	1.333333333	0	1
1	2	2	0.75
3	2.4	4	0.6617647058
5	2.604255319	6	0.6262337149
7	2.700933010	8	0.6111095799
9	2.745131352	10	0.6045146869
11	2.765024171	12	0.6016082844
13	2.773915175	14	0.6003213855
15	2.777876608	16	0.5997503824
17	2.779639200	18	0.5994967911
19	2.780422961	20	0.5993841209
21	2.780771375	22	0.5993340526
23	2.780926241	24	0.5993118014
25	2.780995074	26	0.5993019123
27	2.781025666	28	0.5992975172
29	2.781039263	30	0.5992955638

Example 2. *Let $p = -2/3, y(-1) = 4/3, y(0) = 1$. Thus the condition $-p < y(0)y(-1) < 1-p$ holds and by Theorem 3, the subsequence $\{y_{2n}\}$ is decreasing and subsequence $\{y_{2n-1}\}$ is increasing.*
Table 2 sets forth the values of y_n for selected small n's.

Example 3. *Let $p = -11$, $y(-1) = 0.2$, $y(0) = 0.5$. Then, by Theorem 4, the subsequences $\{y_{4n-1}\}$ and $\{y_{4n}\}$ are both positive and decreasing, while the subsequences $\{y_{4n+1}\}$ and $\{y_{4n+2}\}$ are both negative and increasing.*

Table 3 sets forth the values of y_n for selected small n's.

Table 3

n	$y(n)$	n	$y(n)$
-1	0.2	0	0.5
3	0.001668183	4	0.004128759
7	$1.3786645E-5$	8	$3.4121976E-5$
11	$1.1393922E-7$	12	$2.8199980E-7$
15	$9.4164646E-10$	16	$2.3305769E-9$
19	$7.7822021E-12$	20	$1.9260966E-11$
23	$6.4315720E-14$	24	$1.5918154E-13$
27	$5.3153487E-16$	28	$1.3155499E-15$
1	-0.018348623	2	-0.045416666
5	-0.0001516531	6	-0.000375341
9	$-1.2533314E-6$	10	$-3.1019978E-6$
13	$-1.0358111E-8$	14	$-2.5636346E-8$
17	$-8.5604223E-11$	18	$-2.1187063E-10$
21	$-7.0747292E-13$	22	$-1.7509969E-12$
25	$-5.8468836E-15$	26	$-1.4471049E-14$
29	$-4.8321352E-17$	30	$-1.1959544E-16$

REFERENCES

[1] A. Andruch-Sobiło, M. Migda, *On the rational recursive sequence* $x_{n+1} = \frac{ax_{n-1}}{b+cx_nx_{n-1}}$ (submitted).

[2] C. Cinar, *On the positive solutions of the difference equation* $x_{n+1} = \frac{x_{n-1}}{1+x_nx_{n-1}}$, Appl. Math. and Comp. 150 (2004), 21–24.

[3] C. Cinar, *On the positive solutions of the difference equation* $x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$, Appl. Math. and Comp. 156 (2004), 587–590.

[4] C. Cinar, *On the positive solutions of the difference equation* $x_{n+1} = \frac{x_{n-1}}{-1+x_nx_{n-1}}$, Appl. Math. and Comp. 158 (2004), 813–816.

[5] D.C. Chang, S. Stević, *On the recursive sequence* $x_{n+1} = \alpha + \frac{\beta x_{n-1}}{1+g(x_n)}$, Appl. Anal. 82 (2003), 145–156.

[6] H.M. El-Owaidy, A.M. Ahmed, M.S. Moousa, *On the recursive sequences* $x_{n+1} = \frac{-\alpha x_{n-1}}{\beta \pm x_n}$, Appl. Math. Comp., 145 (2003), 747–753.

[7] V.L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer-Academic Publishers, Dordrecht, 1993.

[8] W.A. Kosmala, M.R.S. Kulenovic, G. Ladas, C.T. Teixeira, *On the recursive sequence* $y_{n+1} = \frac{p+y_{n-1}}{qy_n+y_{n-1}}$, J. Math. Anal. Appl. 251 (2000), 571–586.

[9] W.T. Li, H.R. Sun, *Dynamics of rational difference equation*, Appl. Math. Comput. 163 (2005), 577–591.

[10] Y.S. Huang, P.M. Knopf, *Bondedness of positive solutions of second-order rational difference equations*, J. Difference Equ. Appl. 10 (2004), 935–940.

[11] S. Stević, *More on a rational recurrence relation*, Appl. Math. E-Notes 4 (2004), 80–84.

[12] X. Yan, W. Li, *Global attractivity in the recursive sequence* $x_{n+1} = \frac{\alpha-\beta x_n}{\gamma - x_{n-1}}$, Appl. Math. Comput. 138 (2003), 415–423.

[13] X. Yang, W. Su, B. Chen, G. Megson, D. Evans, *On the reecursive sequences* $x_n = \frac{ax_{n-1}+bx_{n-2}}{c+dx_{n-1}x_{n-2}}$, Appl. Math. Comp. 162 (2005), 1485–1497.

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