Opuscula Mathematica • Vol. 26 • No. 3 • 2006

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POLYNOMIAL QUASISOLUTIONS OF LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

Abstract. The paper discusses a linear differential-difference equation of neutral type with linear coefficients, when at the initial time moment $t = 0$ the value of the desired function $x(t)$ is known. The authors are not familiar with any results which would state the solvability conditions for the given problem in the class of analytical functions. A polynomial of some degree N is introduced into the investigation. Then the term "polynomial quasisolution" (PQ-solution) is understood in the sense of appearance of the residual $\Delta(t) = O(t^N)$, when this polynomial is substituted into the initial problem. The paper is devoted to finding PQ-solutions for the initial-value problem under analysis.

Keywords: differential-difference equations, neutral type, initial value problem, polynomial quasisolution.

Mathematics Subject Classification: 34K15.

1. INTRODUCTION

Application of mathematical modeling methods in investigation of various processes often leads to investigation of differential equations of diverse structure. Unlike ordinary differential equations, functional differential equations admit taking account of the process prehistory. Linear differential-difference equations (LDDEs) with constant delay are among the most studied functional differential equations. In the first turn, it is reasonable to note the fundamental works of A.A. Myshkis [1], E. Pinni [2], R. Bellman and K.L. Cooke [3], N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina [4]. In numerous works, the problem in which the initial function was given for the initial set one way or anther was considered as the main one. As far as LDDEs are concerned, it was shown that assigning the initial function guarantees the existence of the unique solution in both the positive direction of the axis of the independent variable and the negative one. But, as a rule, these solutions were not analytical on an interval whose length exceeded the delay. Some LDDEs, like those where the problem's parameters are constant or have some special representation, form an exception. In this case, it is possible to apply Euler's classical method, which represents the desired solution in the form of an exponential function and presumes finding the roots of the characteristic quasi-polynomial generated by the initial problem. In this case, it is possible to find the particular solutions having analytical structure on the total domain of the independent variable. In the more general case, when the problem's parameters are time-dependent, no results concerning the existence of analytical solutions of LDDEs are known. The paper considers a method of polynomial quasisolutions for the investigation of neutral type LDDEs.

2. STATEMENT OF THE PROBLEM

Consider the following initial problem for a neutral type LDDE

$$d\bar{x}(t)/dt + p(t)d\bar{x}(t-1)/dt = a(t)\bar{x}(t-1) + \bar{f}(t), \tag{2.1}$$

$$t \in J = (-\infty, \infty),\ \bar{x}(0) = x_0.$$

Here

$$a(t) = a_0 + a_1 t,\ p(t) = p_0 + p_1 t,\ \bar{f}(t) = \sum_{n=0}^{F} \bar{f}_n t^n. \tag{2.2}$$

Let

$$\bar{x}(t) = \sum_{n=0}^{\infty} \bar{x}_n t^n \tag{2.3}$$

be a formal solution of problem (2.1). In this case, it is not possible to apply the classical method of undetermined coefficients, since it is impossible to construct the recurrent formula for defining the undetermined coefficients $\bar{x}_n$ in (2.3). The infinite-dimensional linear system of equations expressed in terms of $\bar{x}_n$ obtained in this case cannot nowadays be analyzed in the aspect of obtaining the coefficients $\bar{x}_n$.

Introduce the polynomial

$$x(t) = \sum_{n=0}^{N} x_n t^n. \tag{2.4}$$

In this case,

$$\dot{x}(t) = \sum_{n=0}^{N} n x_n t^{n-1},\ x(t-1) = \sum_{n=0}^{N} x_n (t-1)^n = \sum_{n=0}^{N} \tilde{x}_n t^n, \tag{2.5}$$

where

$$\tilde{x}_n = \sum_{i=0}^{N-n} (-1)^i C_{n+i}^i x_{n+i},\ C_n^m = \frac{n!}{m!(n-m)!}.$$

Write out the relations

$$a(t)x(t-1) = (a_0 + a_1 t)\sum_{n=0}^{N} \tilde{x}_n t^n = \sum_{n=0}^{N+1} \bar{a}_n t^n, \tag{2.6}$$

$$p(t)\dot{x}(t-1) = (p_0 + p_1 t)\sum_{n=0}^{N} n\tilde{x}_n t^{n-1} = \sum_{n=0}^{N} \bar{p}_n t^n, \tag{2.7}$$

where

$$\bar{a}_n = \begin{cases} a_0\tilde{x}_0, & n = 0, \\ a_0\tilde{x}_n + a_1\tilde{x}_{n-1}, & 1 \le n \le N, \\ a_1\tilde{x}_N, & n = N+1. \end{cases} \tag{2.8}$$

$$\bar{p}_n = \begin{cases} p_0\tilde{x}_1, & n = 0, \\ (n+1)p_0\tilde{x}_(n+1) + np_1\tilde{x}_n, & 1 \le n \le N-1, \\ p_1 N\tilde{x}_N, & n = N. \end{cases} \tag{2.9}$$

Let us conduct the dimension analysis of the polynomials obtained as a as a result of substitution of polynomial (2.4) into equation (2.1). The derivative derivative $\dot{x}(t)$ is represented as a polynomial of degree F. Hence, to ensure that – after substitution of (2.4)–(2.7) into (2.1) and comparison of the degrees for the similar powers of t – the last coefficient x_N in (2.4) is defined by the given coefficient $\bar{f}_F$ in (2.2) it is necessary that $N = F + 1$. In this case, the degree of the polynomial in (2.6) is equal to $F + 2$.

Define a function $f(t)$ of the form

$$f(t) = \sum_{n=0}^{F+2} f_n t^n, \tag{2.10}$$

where $f_i = \bar{f}_i,\ i = \overline{0, F},\ f_{F+i},\ i = 1, 2$, are some unknown coefficients.

Definition 2.1. *The problem*

$$\dot{x}(t) + p(t)\dot{x}(t-1)/dt = a(t)x(t-1) + f(t), t \in J,\ x(0) = \bar{x}(0) = x_0. \tag{2.11}$$

is said to be coordinated with respect to degrees of the polynomials for problem (2.1).

Assuming in (2.4)–(2.7) that $N = F + 1$ and substituting these expressions (as well as (2.10)) into (2.11), we obtain

$$\sum_{n=0}^{F+1} nx_n t^{n-1} + \sum_{n=0}^{F+1} \bar{p}_n t^n = \sum_{n=0}^{F+2} \bar{a}_n t^n + \sum_{n=0}^{F+2} f_n t^n.$$

By equating coefficients for equal powers of t, we obtain

$$\begin{aligned} nx_n &= \bar{a}_{n-1} - \bar{p}_{n-1} + f_{n-1},\ 1 \le n \le F+1, \\ 0 &= \bar{a}_{F+1} - \bar{p}_{F+1} + f_{F+1},\ n = F+2, \\ 0 &= \bar{a}_{F+2} + f_{F+2},\ n = F+3. \end{aligned}$$

Remark 2.1. *According to formula* (2.10)*, the first F coefficients of the polynomial $f(t)$ are determined by the coefficients of the polynomial $\bar{f}(t)$ of problem* (2.1)*. Hence, the cases are admitted when some leading or even all coefficients $\bar{f}_n$ in* (3.2) *are zero.*

Since the degree of the polynomial $x(t)$ is $F+1$, by virtue of this remark it is possible to choose a degree of the polynomial $\bar{f}(t)$ depending on the desired degree of the polynomial $x(t)$ by adding the corresponding number of zero terms to $\bar{f}(t)$, i.e., to find solutions of the given problem in the form of polynomials (2.4) of diverse degrees.

Definition 2.2. *If there exists a polynomial of degree $F+1$*

$$x(t) = \sum_{n=0}^{F+1} x_n t^n, \ t \in J, \tag{2.12}$$

which identically satisfies problem (2.11)*, then this polynomial will be called a polynomial quasisolution (PQ-solution) of problem* (2.1)*.*

3. MAIN RESULTS

Rewrite problem (2.11), which has been coordinated with respect to the degree of polynomials, in the form

$$\dot{x}(t) + p(t)\dot{x}(t-1)/dt = a(t)x(t-1) + f(t), t \in J, \ x(0) = x_0. \tag{3.1}$$

Let us find the PQ-solution of form (2.4)

$$x(t) = \sum_{n=0}^{N} x_n t^n, \ t \in J. \tag{3.2}$$

Then, according to (2.10) and Remark 2.1, for the function $f(t)$ there is

$$f(t) = \sum_{n=0}^{F} f_n t^n + \Delta(t), \ (N > F+1), \tag{3.3}$$

where $f_i = \bar{f}_i, \ i = \overline{0, F}$ are known coefficients; $\Delta(t) = f_n t^N + f_{N+1} t^{N+1}$, f_N and f_{N+1} are unknown coefficients.

Substituting (3.2) and (3.3) into (3.1) and taking account of (2.8) and (2.9), with the use of the method of undetermined coefficients we obtain

$$\begin{cases} x_1 = a_0\tilde{x}_0 - p_0\tilde{x}_1 + f_0, \\ 2x_2 = a_0\tilde{x}_1 + a_1\tilde{x}_0 - (2p_0\tilde{x}_2 + p_1\tilde{x}_1) + f_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ nx_n = a_1\tilde{x}_{n-2} + [a_0 - (n-1)p_1]\tilde{x}_{n-1} - p_0 n\tilde{x}_n + f_{n-1}, \ 3 \le n \le N. \end{cases} \tag{3.4}$$

Let us transform the coefficient $\tilde{x}_n$ in (2.5) as follows

$$\tilde{x}_n = \sum_{i=0}^{N-n} (-1)^i C_{n+i}^i x_{n+i} = x_n + \sum_{i=n+1}^{N} \bar{C}_i^n x_i,\ n = \overline{0, N-1};\ \tilde{x}_N = x_N, \tag{3.5}$$

where $\bar{C}_{n+i}^i = \bar{C}_{n+i}^n = (-1)^i C_{n+i}^i$.

Substituting (3.5) into (3.4) for $n = N+2-k,\ k = 0, 1, \dots$ we have:

— for $n = N+2$,

$$0 = a_1 x_N + f_{N+1};$$

— for $n = N+1$,

$$0 = a_1 x_{N-1} + (a_1 \bar{C}_N^1 + [a_0 - N p_1]) x_N + f_N;$$

— for $n = N$,

$$\begin{aligned} 0 = a_1 x_{N-2} &+ (a_1 \bar{C}_{N-1}^1 + [a_0 - (N-1) p_1]) x_{N-1} + \\ &+ (a_1 \bar{C}_N^2 + [a_0 - (N-1) p_1] \bar{C}_N^1 - N(p_0 + 1)) x_N + f_N; \end{aligned}$$

. . . .

— for $n = N-s$,

$$\begin{aligned} 0 = a_1 x_{N-s-2} &+ (a_1 \bar{C}_{N-s-1}^1 + [a_0 - (N-s-1) p_1] x_{N-s-1} + \\ &+ (a_1 \bar{C}_{N-s}^2 + [a_0 - (N-s-1) p_1] \bar{C}_{N-s}^1 - \\ &- (N-s)(p_0 + 1)) x_{N-s} + \\ &+ \sum_{i=1}^{s} (a_1 \bar{C}_{N-s+i}^{2+i} + [a_0 - (N-s-1) p_1] \bar{C}_{N-s+i}^{i+1} - (N-s) p_0 \bar{C}_{N-s+i}^i) x_{N-s+i} + \\ &+ f_{N-s-1}. \end{aligned}$$

Rewrite these equalities in the form

$$\begin{aligned} &a_{NN} x_N + f_{N+1} = 0, \\ &a_{N-1,N-1} x_{N-1} + a_{N-1,N} x_N + f_N = 0, \\ &a_{N-2,N-2} x_{N-2} + a_{N-2,N-1} x_{N-1} + a_{N-2,N} x_N + f_{N-1} = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{N-s,N-s} x_{N-s} + a_{N-s,N-s+1} x_{N-s+1} + \ldots + a_{N-s,N} x_N + f_{N-s-1} = 0, \end{aligned} \tag{3.6}$$

where

$$
\begin{aligned}
&a_{NN} = a_1;\\
&a_{N-1,N-1} = a_1,\\
&a_{N-1,N} = a_1\bar{C}^1_N + [a_0 - Np_1];\\
&a_{N-2,N-2} = a_1,\\
&a_{N-2,N-1} = a_1\bar{C}^1_{N-1} + [a_0 - (N-1)p_1],\\
&a_{N-2,N} = a_1\bar{C}^2_N + [a_0 - (N-1)p_1]\bar{C}^1_N - N(p_0+1);\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_{N-s,N-s} = a_1,\\
&a_{N-s,N-s+1} = a_1\bar{C}^1_{N-s+1} + [a_0 - (N-s+1)p_1],\\
&a_{N-s,N-s+2} = a_1\bar{C}^2_{N-s+2} + [a_0 - (N-s+1)p_1]\bar{C}^1_{N-s+2} - (N-s+2)(p_0+1),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_{N-s,N-s+k} = a_1\bar{C}^k_{N-s+k} + [a_0 - (N-s+1)p_1]\bar{C}^{k-1}_{N-s+k} - (N-s+2)p_0\bar{C}^{k-2}_{N-s},\\
&\hfill k \geq 3.
\end{aligned}
\tag{3.7}
$$

The analysis of equalities (3.7) renders the following lemma true.

Lemma 3.1. *The general term of the sequence* $\{x_n\}_{n=1}^N$*, generated by system* (3.6)*, is defined by the formula*

$$x_{N-s} = \sum_{i=0}^{s} K_{N-s,N-s+i} f_{N-s+i+1}, \tag{3.8}$$

where

$$
\begin{aligned}
&K_{N-s,N-s} = -\frac{1}{a_{N-s,N-s}},\\
&K_{N-s,N-r} = -\frac{1}{a_{N-s,N-s}} \sum_{i=1}^{s-r} a_{N-s,N-s+i} K_{N-s+i,N-r},\ s > r.
\end{aligned}
$$

Proof. Express the unknown coefficients x_n of PQ-solution (3.2) in terms of the unknown coefficients f_N and f_{N+1} of polynomial (3.3).

$$a_{NN}x_N = -f_{N+1} \Rightarrow\ x_N = K_{NN}f_{N+1},\ K_{NN} = -\frac{1}{a_{NN}};$$

$$a_{N-1,N-1} + a_{N-1,N}x_N = -f_N \Rightarrow x_{N-1} = K_{N-1,N-1}f_N + K_{N-1,N}f_{N+1}.$$

Here

$$K_{N-1,N-1} = -\frac{1}{a_{N-1,N-1}},\ K_{N-1,N} = -\frac{a_{N-1,N}}{a_{N-1,N-1}}K_{NN};$$

$$a_{N-2,N-2} + a_{N-2,N-1}x_{N-1} + a_{N-2,N}x_N = -f_{N-1} \quad \Rightarrow$$
$$x_{N-2} = K_{N-2,N-2}f_{N-1} + K_{N-2,N-1}f_N + K_{N-2,N}f_{N+1},$$

where

$$K_{N-2,N-2} = -\frac{1}{a_{N-2,N-2}},\ K_{N-2,N-1} = \frac{a_{N-2,N-1}}{a_{N-2,N-2}}K_{N-1,N-1},$$
$$K_{N-2,N} = \frac{a_{N-2,N-1}}{a_{N-2,N-2}}K_{N-1,N} + \frac{a_{N-2,N}}{a_{N-2,N-2}}K_{N,N}.$$

....

Finally, by the mathematical induction method we obtain the formula

$$x_{N-s} = \sum_{i=0}^{s} K_{N-s,N-s+i}f_{N-s+i+1},$$

where

$$K_{N-s,N-s} = -\frac{1}{a_{N-s,N-s}};$$
$$K_{N-s,N-r} = -\frac{1}{a_{N-s,N-s}}\sum_{i=1}^{s-r} a_{N-s,N-s+i}K_{N-s+i,N-r},\ s > r,$$

which proves the statement of the lemma. □

Return to formula (3.4). Taking into account (3.5), rewrite the first equality in the form

$$x_1 = a_0\tilde{x}_0 - p_0\tilde{x}_1 + f_0 =$$
$$= a_0(x_0 - x_1 + x_2 - \ldots + (-1)^N x_N) - p_0(x_1 - 2x_2 + 3x_3 - \ldots + N(-1)^{N+1}x_N) + f_0.$$

From that there follows:

$$a_0x_0 = (a_0+p_0+1)x_1 - (a_0+2p_0)x_2 + (a_0+3p_0)x_3 - \ldots + (-1)^{N+1}(a_0+Np_0)x_N - f_0$$

or

$$x_0 = (1+\frac{p_0+1}{a_0})x_1 - (1+2\frac{p_0}{a_0})x_2 + \ldots - (-1)^{N+1}(1+N\frac{p_0}{a_0})x_N - \frac{f_0}{a_0}.$$

Introduce the notations:

$$V_1 = 1+\frac{p_0+1}{a_0},\ V_2 = -(1+2\frac{p_0}{a_0}),\ldots,\ V_n = (-1)^{n+1}(1+n\frac{p_0}{a_0}).$$

Then

$$x_0 = V_1 x_1 + V_2 x_2 + \ldots + V_N x_N - \frac{f_0}{a_0}. \tag{3.9}$$

Taking into account (3.8), express the coefficients x_n in terms of the coefficients $f_i,\ i = \overline{1, N+1}$:

$$x_0 = \sum_{i=0}^{N} K_{0,i} f_{i+1} = K_{00} f_1 + K_{01} f_2 + K_{02} f_3 + \ldots + K_{0,N-1} f_N + K_{0,N} f_{N+1} =$$
$$= \sum_{k=0}^{N} f_{N+1-k} K_{0,N-k},$$
$$x_1 = \sum_{i=0}^{N-1} K_{1,i+1} f_{i+2} = K_{11} f_2 + K_{12} f_3 + \ldots + K_{1,N-1} f_N + K_{1,N} f_{N+1} =$$
$$= \sum_{k=0}^{N-1} f_{N+1-k} K_{1,N-k},$$
$$x_2 = \sum_{i=0}^{N-2} K_{2,i+2} f_{i+3} = K_{22} f_3 + \ldots + K_{2,N-1} f_N + K_{2,N} f_{N+1} =$$
$$= \sum_{k=0}^{N-2} f_{N+1-k} K_{2,N-k},$$

$$x_m = \sum_{i=0}^{N-m} K_{m,i+m} f_{i+m+1} = K_{m,m} f_{m+1} + \ldots + K_{m,N-1} f_N + K_{m,N} f_{N+1} =$$
$$= \sum_{k=0}^{N-m} f_{N+1-k} K_{m,N-k},$$

$$x_N = K_{NN} f_{N+1}.$$

Substitute the coefficients x_n thus obtained into (3.9)

$$x_0 = V_1 \sum_{k=0}^{N-1} f_{N+1-k} K_{1,N-k} + V_2 \sum_{k=0}^{N-2} f_{N+1-k} K_{2,N-k} + \ldots$$
$$+ V_m \sum_{k=0}^{N-m} f_{N+1-k} K_{m,N-k} + V_N K_{NN} f_{N+1} - \frac{f_0}{a_0}. \tag{3.10}$$

Transform this formula, groupping the terms with the same coefficients $f_{N+1-i},\ i = \overline{1, N-2}$.

— For f_{N+1},

$$V_1K_{1,N} + V_2K_{2,N} + \ldots + V_mK_{m,N} + \ldots + V_NK_{NN};$$

— for f_N,

$$V_1K_{1,N-1} + V_2K_{2,N-1} + \ldots + V_mK_{m,N-1} + \ldots + V_{N-1}K_{N-1,N-1};$$

— for f_{N-1},

$$V_1K_{1,N-2} + V_2K_{2,N-2} + \ldots + V_mK_{m,N-2} + \ldots + V_{N-2}K_{N-2,N-2};$$

....

— for f_{N-m},

$$V_1K_{1,N-m+1} + V_2K_{2,N-m-1} + \ldots + V_{N-m-1}K_{N-m-1,N-m-1};$$

....

when f_2,

$$V_1K_{11}.$$

These transformations lead to the formula

$$x_0 = \sum_{k=0}^{N-1}(\sum_{i=1}^{N-k} V_iK_{i,N-k})f_{N+1-k} - \frac{f_0}{a_0}. \tag{3.11}$$

Denote

$$\bar{K}_{0,N-k} = \sum_{i=1}^{N-k} V_iK_{i,N-k}$$

and rewrite (3.11) as follows:

$$x_0 = \sum_{k=0}^{N-1} \bar{K}_{0,N-k}f_{N+1-k} - \frac{f_0}{a_0}.$$

Theorem 3.1. *For the initial value problem*

$$\frac{dx(t)}{d(t)} + p(t)\frac{dx(t-1)}{dt} = a(t)x(t-1) + f(t),\ t \in J = (-\infty, \infty),\ x(0) = x_0, \tag{3.12}$$

let the following conditions hold:

$$a(t) = a_0 + a_1t,\ p(t) = p_0 + p_1t,\ \bar{f}(t) = \sum_{n=0}^{F} \bar{f}_nt^n.$$

Then this problem has the unique PQ-solution of the form $x(t) = \sum\limits_{n=0}^{N} x_n t^n$ $(N > F+1)$ *with the residual* $\Delta(t) = f_N t^N + f_{N+1} t^{N+1}$ *if the determinant*

$$D = \begin{vmatrix} K_{0,N} & K_{0,N-1} \\ \bar{K}_{0,N} & \bar{K}_{0,N-1} \end{vmatrix}$$

is nonzero.

Proof. Consider formula (3.8) when $s = N$

$$x_0 = \sum_{k=0}^{N} K_{0,N-k} f_{N+1-k}. \tag{3.13}$$

Write equalities (3) and (3.13) separating the terms with the unknown coefficients f_N and f_{N+1}

$$\begin{aligned} x_0 &= K_{0,N} f_{N+1} + K_{0,N-1} f_N + \sum_{k=2}^{N} K_{0,N-k} f_{N+1-k}, \\ x_0 &= \bar{K}_{0,N} f_{N+1} + \bar{K}_{N-1,N} f_N + \sum_{k=2}^{N-1} \bar{K}_{0,N-k} f_{N+1-k} - \tfrac{f_0}{a_0}. \end{aligned} \tag{3.14}$$

Rewrite these relations in the form of a linear system with respect to coefficients f_N and f_{N+1}

$$\begin{cases} K_{0,N} f_{N+1} + K_{0,N-1} f_N = x_0 - \sum\limits_{k=2}^{N} K_{0,N-k} f_{N+1-k}, \\ \bar{K}_{0,N} f_{N+1} + \bar{K}_{0,N-1} f_N = x_0 - \sum\limits_{k=2}^{N-1} \bar{K}_{0,N-k} f_{N+1-k} + \frac{f_0}{a_0}. \end{cases} \tag{3.15}$$

Denote

$$D_1 = \begin{vmatrix} x_0 - \sum\limits_{k=2}^{N} K_{0,N-k} f_{N+1-k} & K_{0,N-1} \\ x_0 - \sum\limits_{k=2}^{N-1} \bar{K}_{0,N-k} + \frac{f_0}{a_0} & \bar{K}_{0,N-1} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} K_{0,N} & x_0 - \sum\limits_{k=2}^{N} K_{0,N-k} f_{N+1-k} \\ \bar{K}_{0,N} & x_0 - \sum\limits_{k=2}^{N-1} \bar{K}_{0,N-k} f_{N+1-k} + \frac{f_0}{a_0} \end{vmatrix}.$$

Since the determinant of system (3.15) is nonzero by assumption, we find the solution using Cramer's rule

$$f_N = \frac{D_2}{D},\ f_{N+1} = \frac{D_1}{D}.$$

Then from the chain of equalities (3.6) we obtain all the coefficients x_n, the PQ-solution in form (3.2) and the residual

$$\Delta(t) = f_N + f_{N+1}t^{N+1} = \frac{D_2}{D}t^N + \frac{D_1}{D}t^{N+1}.$$

This proves the theorem. □

Let $F < N - 1$. Introduce a coefficient f_{N-1} and consider it as a free parameter. Rewrite linear system (3.15) in the form

$$\begin{cases} K_{0,N}f_{N+1} + K_{0,N-1}f_N + K_{0,N-2}f_{N-1} = W_1, \\ \bar{K}_{0,N}f_{N+1} + \bar{K}_{0,N-1}f_N + \bar{K}_{0,N-2}f_{N-1} = W_2, \end{cases} \tag{3.16}$$

where

$$W_1 = x_0 - \sum_{k=3}^{N} K_{0,N-k}f_{N+1-k}, \; W_2 = x_0 - \sum_{k=3}^{N-1} \bar{K}_{0,N-k}f_{N+1-k} + \frac{f_0}{a_0}.$$

Transform (3.16) as follows:

$$\begin{cases} K_{0,N}f_{N+1} + K_{0,N-1}f_N = W_1 - K_{0,N-2}f_{N-1}, \\ \bar{K}_{0,N}f_{N+1} + \bar{K}_{0,N-1}f_N = W_2 - \bar{K}_{0,N-2}f_{N-1}. \end{cases} \tag{3.17}$$

This system has a solution if the determinant

$$D = \begin{vmatrix} K_{0,N} & K_{0,N-1} \\ \bar{K}_{0,N} & \bar{K}_{0,N-1} \end{vmatrix} \tag{3.18}$$

is nonzero.

Let $D \neq 0$. Hence (3.17) is an underdetermined linear system, and it has an infinite number of solutions.

Denote

$$D_1 = \begin{vmatrix} W_1 - K_{0,N-1} & K_{0,N-1} \\ W_2 - \bar{K}_{0,N-2} & \bar{K}_{0,N-1} \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} K_{0,N} & W_1 - K_{0,N-2} \\ \bar{K}_{0,N} & W_2 - \bar{K}_{0,N-2} \end{vmatrix}.$$

Since the determinant of system (3.17) is nonzero, using Cramer's rule we find the coefficients f_N and f_{N+1} as functions of f_{N-1}:

$$f_N = \frac{D_2}{D} = \varphi_1(f_{N-1}), \;\; f_{N+1} = \frac{D_1}{D} = \varphi_2(f_{N-1}).$$

Then for the residual $\Delta(t)$ we derive

$$\Delta(t) = f_{N-1}t^{N-1} + \varphi_1(f_{N-1})t^N + \varphi_2(f_{N-1})t^{N+1}. \tag{3.19}$$

These results prove the following theorem.

Theorem 3.2. *For initial value problem* (3.12)*, let the conditions of Theorem 3.1 hold.*

Then this problem has an infinite set of PQ-solutions of the form $x(t) = \sum_{n=0}^{N} x_n t^n$ *with the residuals* $\Delta(t) = f_{N-1}t^{N-1} + \varphi_1(f_{N-1})t^N + \varphi_2(f_{N-1})t^{N+1}$*, if*

$$D = \begin{vmatrix} K_{0,N} & K_{0,N-1} \\ \bar{K}_{0,N} & \bar{K}_{0,N-1} \end{vmatrix} \tag{3.20}$$

is nonzero.

Since some of the coefficients $f_{N-i},\ i = 1, 2\dots$ may be chosen as free parameters, from Theorems 3.1 and 3.2, we may derive the following result.

Corollary 3.1. *There exist PQ-solutions of degree N for problem* (3.1)*, having different residuals* $\Delta(t)$ *with estimates* $O(t^{N-k}),\ k = 1, 2, \dots$.

Some numerical results on PQ-solutions for LDDEs of delay type have been published in [6].

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Received: October 3, 2005.