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Explosion Tests for Stochastic Integral Equations Related to Interest Rate Models

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Abstract. In the present paper a class of stochastic integral equations is studied. It is closely related to the interest rate model proposed by Ritchken and Sankarasubramanian [8] [9]. Explosion tests for these equations are given.

1. Introduction

Let $(\Omega, \mathcal{F})$ be an appropriate measurable space and let us consider the following stochastic integral equation.

$$X_t = x + \eta t + \xi \int_0^t \int_0^s a(X_u) du ds + \int_0^t (a(X_s))^{\frac{1}{2}} dW_s, \tag{1.1}$$

where $x > 0$ and $a(\cdot)$ denotes a measurable non-negative function on the real line satisfying $a(0) = 0$, η and ξ are positive constants, $W.$ denotes a one dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \in [0,\infty)})$ and $\{\mathcal{F}_t\}_{t \in [0,\infty)}$ is an augmented filtration.

In this paper we study pathwise uniqueness and the global existence of the solution for (1.1). Our results are the following.

Theorem 1.1. *Assume that for each integer n, there exists a constant $K^n > 0$ such that*

$$|a(x) - a(y)| \leq K^n |x - y| \tag{1.2}$$

holds for every $|x| \leq n, |y| \leq n$ and that for every $x > 0$ and $y > 0$,

$$|\sqrt{a(x)} - \sqrt{a(y)}|^2 \leq \rho(|x - y|) \tag{1.3}$$

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where ρ is a non-decreasing Borel function from $(0,\infty)$ *to* $(0,\infty)$ *such that*

$$\int_{0+} \frac{da}{\rho(a)} = +\infty. \tag{1.4}$$

Then pathwise uniqueness up to the explosion time holds for (1.1).

Here we say that $\mathfrak{e}$ is the explosion time of the solution when we have $\mathfrak{e} = \lim_{n\to\infty} \tau_n$ where $\tau_n = \inf\{t, X_t > n\}$.

Theorem 1.2. *Let σ and γ be positive constants and $\frac{1}{2} < \gamma \leq 1$ and let $a(x) = \sigma^2 x^{2\gamma}$. Then the solution for* (1.1) *explodes almost surely.*

Theorem 1.3. *Let $\xi \equiv 1$ and let $a(x) = xL(x)$ where $L(x)$ is a slowly varying function on $[0,\infty)$; i.e. it is real valued, positive, measurable and $L(\lambda x) \sim L(x)$ for each $\lambda > 0$. Here by $f(x) \sim g(x)$ we mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.*

Assume that

(a) $\lim_{x\to\infty} L(x) = +\infty$,

(b) $\inf_{x>0} L(x) > 0$,

(c) $(L(\sqrt{x}))^{-1}$ *satisfies global Lipschitz condition,*

and

(d) $L(0) > 2\eta$.

Let $\mathfrak{f}$ be the right continuous inverse of the map $x \longmapsto x\sqrt{\bar{L}(x)}$ where $\bar{L}(x) \overset{def}{=} \sup_{0\leq y\leq x} L(y)$, more precisely,

$$\mathfrak{f}(x) = \inf\{y > 0,\ y\sqrt{\bar{L}(y)} > x\}.$$

Then we have the following.

(i) If $\int_1^{+\infty} \frac{dx}{x\sqrt{L(\mathfrak{f}(x))}} = +\infty$, *then* $P(\mathfrak{e} < \infty) = 0$.

(ii) If $\int_1^{+\infty} \frac{dx}{x\sqrt{L(\mathfrak{f}(x))}} < +\infty$, *then* $P(\mathfrak{e} < \infty) = 1$.

Since $\bar{L}$ is non-decreasing, $\mathfrak{f}$ is well defined

Theorem 1.1 is proved in section 2 as a corollary of a more general theorem. We give a proof of theorem 1.2 in section 3. Theorem 1.3 is proved in section 4.

The above equation is closely related to the interest rate model proposed by Ritchken and Sankarasubramanian [8] [9]. We describe this relationship in section 5.

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2. Pathwise Uniqueness of a Stochastic Integral Equation

In this section we shall study the pathwise uniqueness for somewhat more general equations.

Let $(\Omega, \mathcal{F})$ be an appropriate measurable space and let us consider the following stochastic integral equation.

$$X_t = x + \eta(t) + \int_0^t \xi(s) \left(\int_0^s \mu(u, X_u) du \right) ds + \int_0^t \sigma(s, X_s) dW_s, \tag{2.1}$$

Assume

$$x > 0,\ \eta(\cdot),\ \xi(\cdot) : [0, \infty) \to [0, \infty), \text{ continuous} \tag{2.2}$$

and

$$\sigma,\ \mu : [0, \infty) \times [0, \infty) \to [0, \infty), \text{ jointly measurable} \tag{2.3}$$

and satisfies the following conditions.

(a) For each $t > 0$,

$$|\sigma(s, x) - \sigma(s, y)|^2 \leq \rho(|x - y|) \tag{2.4}$$

holds for every $s \leq t$. Here ρ is a non-decreasing Borel function from $(0, \infty)$ to $(0, \infty)$ such that

$$\int_{0+} \frac{da}{\rho(a)} = +\infty. \tag{2.5}$$

(b) (Local Lipschitz Condition) For each $t > 0$ and integer n, there exists a constant $K_t^n > 0$ such that

$$|\mu(s,x) - \mu(s,y)| \le K_t^n |x-y| \tag{2.6}$$

holds for every $s \le t$ and $|x| \le n, |y| \le n$.

We define solutions for (2.1) as usual in the weak sense but up to the explosion time. The existence of weak solutions up to the explosion time follows from Skorohod's results [11] by slight modifications. (See Ikeda and Watanabe [2].) By Yamada-Watanabe's theory [14] we are also able to obtain the unique strong solution from weak existence and pathwise uniqueness.

In this paper we say that pathwise uniqueness up to the explosion time holds for (2.1) if for any two weak solutions (up to the explosion time) of common initial value and common Brownian motion (relative to possibly different filtrations) (X, W) and $(\tilde{X}, W)$,

$$P[X_t = \tilde{X}_t\,;\, 0 \le {}^{\forall} t\, < \mathfrak{e}] = 1.$$

THEOREM 2.1. *Under the conditions* (2.2)–(2.6) *pathwise uniqueness up to the explosion time holds for* (2.1).

PROOF. Let $X^1$and X^2 be two solutions (with respect to the same Brownian motion and $X_0^1 = X_0^2$ *a.s.*) of (2.1) under the conditions (2.2)–(2.6). Let

$$\tau_n^i \stackrel{def}{=} \inf\{t \,|\, |X_t^i| \ge n\}, \quad i = 1, 2, \ n \in \mathbb{N},$$

and

$$\tau_n \stackrel{def}{=} \tau_n^1 \wedge \tau_n^2.$$

To prove the theorem we use the following lemma from Revuz and Yor [7].

LEMMA 2.2 (See e.g. Revuz and Yor [7]). *Fix an integer n. Then*

$$L_t^0(X^1 - X^2) = 0, \quad 0 \le {}^{\forall} t \le \tau_n.$$

Here we denote by L^0 the local time at 0.

We will show that

$$P[X_t^1 = X_t^2\,;\, 0 \le {}^\forall t \le \tau_n] = 1, \quad {}^\forall n \in \mathbb{N}.$$

By virtue of Tanaka's formula and lemma 2.2, for $t > 0$,

$$\begin{aligned} |X^1_{t\wedge\tau_n} - X^2_{t\wedge\tau_n}| &= \int_0^{t\wedge\tau_n} \mathrm{sgn}(X_s^1 - X_s^2)\left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right) dW_s \\ &\quad + \int_0^{t\wedge\tau_n} \mathrm{sgn}(X_s^1 - X_s^2)\xi_s \\ &\quad \cdot \left(\int_0^s (\mu(u, X_u^1) - \mu(u, X_u^2))du\right) ds. \end{aligned} \tag{2.8}$$

Since the stochastic integral term of (2.8) is bounded,

$$\begin{aligned} &E\,|\,X^1_{t\wedge\tau_n} - X^2_{t\wedge\tau_n}\,| \\ &\le E\left[\int_0^{t\wedge\tau_n} \xi_s \left(\int_0^s |\mu(u, X_u^1) - \mu(u, X_u^2)|du\right) ds\right] \end{aligned} \tag{2.9}$$

(by integration by parts)

$$\begin{aligned} &= E\left[\left(\int_0^{t\wedge\tau_n} \xi_s ds\right)\left(\int_0^{t\wedge\tau_n} |\mu(s, X_s^1) - \mu(s, X_s^2)|ds\right)\right] \\ &\quad - E\left[\int_0^{t\wedge\tau_n} \left(\int_0^s \xi_u du\right) |\,\mu(s, X_s^1) - \mu(s, X_s^2)\,|\,ds\right] \end{aligned} \tag{2.10}$$

(since ξ is non-negative and continuous, there exists a positive constant C_t^n)

$$\le C_t^n\, E\left[\int_0^{t\wedge\tau_n} |\,\mu(s, X_s^1) - \mu(s, X_s^2)\,|\,ds\right] \tag{2.11}$$

$$\le C_t^n E\left[\int_0^{t\wedge\tau_n} K_t^n\,|\,X_s^1 - X_s^1\,|\,ds\right] \tag{2.12}$$

$$\le C_t^n K_t^n E\left[\int_0^t |\,X^1_{s\wedge\tau_n} - X^2_{s\wedge\tau_n}\,|\,ds.\right] \tag{2.13}$$

By Gronwall's lemma,

$$X^1_{t\wedge\tau_n} = X^2_{t\wedge\tau_n},\ t \ge 0, \quad a.s.$$

Letting $n \uparrow \infty$, we get the desired result. □

Proof of Theorem 1.1. A direct consequence of theorem 2.1 □

3. Proof of Theorem 1.2

Let X_t be the unique solution of (1.1) and

$$Y_t \overset{def}{=} \sigma^2 \xi \int_0^t X_s^{2\gamma} ds. \tag{3.1}$$

Then

$$dX_t = (Y_t + \eta)\, dt + \sigma X_t^{\gamma} dW_t. \tag{3.2}$$

We set a 'scale function' as follows:

$$S(x,y) = \frac{1}{1-\gamma} x^{1-\gamma} + y^{\frac{1}{\delta}} \quad x > 0,\ y > 0. \tag{3.3}$$

Then by Itô's formula,

$$S(X_t, Y_t) \overset{def}{=} \sigma W_t + \int_0^t \mathcal{L} S(X_s, Y_s) ds \tag{3.4}$$

where

$$\mathcal{L} \overset{def}{=} \frac{1}{2} \sigma^2 x^{2\gamma} \frac{\partial^2}{\partial x^2} + (y + \eta) \frac{\partial}{\partial x} + \xi \sigma^2 x^{2\gamma} \frac{\partial}{\partial y}.$$

In our stting

$$\mathcal{L} S(x,y) = -\frac{1}{x^{1-\gamma}} \left(\frac{\sigma^2 \gamma}{2} \right) + \frac{y+\eta}{x^{\gamma}} + \xi \sigma^2 x^{2\gamma} \left(\frac{1}{\delta} y^{\frac{1}{\delta} - 1} \right). \tag{3.5}$$

Set

$$G(R) \overset{def}{=} \inf_{S(x,y)=R} \mathcal{L} S(x,y), \qquad R > 0. \tag{3.6}$$

We will show that

Lemma 3.1. *there exist positive constants C_1, C_2, C_3 such that for all $R > 0$,*

$$G(R) > C_1 R^{\frac{\delta+1}{3}} - C_2 R^{\delta - \frac{\gamma}{1-\gamma}} - C_3. \tag{3.7}$$

To estimate (3.6) we use the following elementary lemmas.

LEMMA 3.2. *(i) Let* $k_1, k_2 > 0,\ \delta_1 < \delta_2 < 0$ *and*

$$f(x) = k_1 x^{\delta_1} - k_2 x^{\delta_2}. \quad x > 0.$$

Then for $x > 0$,

$$f(x) \geq k_1^{\frac{\delta_2}{\delta_2-\delta_1}} k_2^{-\frac{\delta_1}{\delta_2-\delta_1}} \left(\frac{\delta_1}{\delta_2}\right)^{\frac{\delta_1}{\delta_2-\delta_1}} \left(1 - \frac{\delta_1}{\delta_2}\right). \tag{3.8}$$

(ii) Let $k_1, k_2 > 0,\ \delta_1 < 0 < \delta_2$ *and*

$$f(x) = k_1 x^{\delta_1} + k_2 x^{\delta_2}. \quad x > 0.$$

Then for $x > 0$,

$$f(x) \geq k_1^{\frac{\delta_2}{\delta_2-\delta_1}} k_2^{-\frac{\delta_1}{\delta_2-\delta_1}} \left(\frac{-\delta_1}{\delta_2}\right)^{\frac{\delta_1}{\delta_2-\delta_1}} \left(1 + \frac{\delta_1}{\delta_2}\right). \tag{3.9}$$

Equality holds when

$$x = \left(\frac{k_1\,|\delta_1\,|}{k_2\delta_2}\right)^{\frac{1}{\delta_2-\delta_1}}. \tag{3.10}$$

PROOF OF LEMMA 3.1. We first remark that under the constraint that

$$R = \frac{1}{1-\gamma} x^{1-\gamma} + y^{\frac{1}{\delta}}, \quad x > 0,\ y > 0,\ R > 0,$$

there are bounds for both x and y, i.e.;

$$0 < x < ((1-\gamma)R)^{\frac{1}{1-\gamma}}, \quad 0 < y < R^{\delta}. \tag{3.11}$$

Then we have

$$\frac{y}{x^\gamma} = \left(R - \frac{1}{1-\gamma}x^{1-\gamma}\right)^\delta \geq \frac{R^\delta}{x^\gamma} - \frac{\delta}{1-\gamma}\frac{R^{\delta-1}}{x^{2\gamma-1}}. \tag{3.12}$$

And by (3.11)

$$\frac{1}{\delta}y^{\frac{1}{\delta}-1} > \frac{1}{\delta}R^{1-\delta}. \tag{3.13}$$

Let c_1, c_2 be positive constants such that $c_1 + c_2 = 1$. By lemma 3.2 (ii), and since $1 + \left(\frac{-\gamma}{2\gamma}\right) > 0$,

$$c_1\frac{R^\delta}{x^\gamma} + \xi\sigma^2 x^{2\gamma}\frac{1}{\delta}R^{1-\delta} \geq {}^\exists C_1 R^{\frac{\delta+1}{3}}. \tag{3.14}$$

Similarly by lemma 3.2 (i),

$$c_2\frac{R^\delta}{x^\gamma} - \frac{\delta}{1-\gamma}\frac{R^{\delta-1}}{x^{2\gamma-1}} \geq -{}^\exists C_2 R^{\delta-\frac{\gamma}{1-\gamma}}. \tag{3.15}$$

and

$$\frac{\eta}{x^\gamma} - \frac{1}{x^{1-\gamma}}\left(\frac{\sigma^2\gamma}{2}\right) \geq -{}^\exists C_3. \tag{3.16}$$

By (3.12)–(3.16) we get (3.7). $\square$

Proof of Theorem 1.2. Let $\hat{R}_.$ be the solution of the following stochastic differential equation for $\lambda \geq 1$.

$$\begin{cases} d\hat{R}_t = \sigma dW_t + (C_1\hat{R}_t^\lambda - C_2\hat{R}_t - C_3)\,dt, \\ \hat{R}_0 = \dfrac{1}{1-\gamma}x^{1-\gamma}. \end{cases} \tag{3.17}$$

Denote its explosion time by $\mathfrak{t}$. By Feller's explosion test, (See e.g. Ikeda and Watanabe [2])

$$P[\mathfrak{t} < \infty] = \begin{cases} 1, & \text{if } \lambda > 1, \\ 0, & \text{if } \lambda = 1. \end{cases} \tag{3.18}$$

Then by the comparison theorem (See Yamada [13], and Ikeda and Watanabe [2]) and lemma 3.1,

$$S(X_t, Y_t) > \hat{R}_t, \ a.s. \tag{3.19}$$

Since $S(X_t, Y_t) \uparrow \infty$ implies $X_t \uparrow \infty$, we have

$$P(\mathfrak{e} < \mathfrak{t}) = 1.$$

We can take $\dfrac{\delta+1}{3} > 1$ and $\delta - \dfrac{\gamma}{1-\gamma} = 1$ if and only if $\gamma > \dfrac{1}{2}$. This implies

$$P(\mathfrak{e} < \infty) = 1, \ \text{ if } \gamma > \frac{1}{2}. \ \square$$

4. Proof of Theorem 1.3

The key idea of the following proof is *time change*. Note that $X.$ is rewritten as

$$X_t = x + \eta t + \int_0^t [M]_s ds + M_t \tag{4.1}$$

where $M_t = \int_0^t (a(X_s))^{\frac{1}{2}} dW_s$.

First let us consider the following equation on $(\Omega, \mathcal{G}, P)$.

$$V_t = x^2 + 2\int_0^t \sqrt{V_s} dB_s + \int_0^t \left(\frac{2(s+\eta)}{L(\sqrt{V_s})} + 1 \right) ds$$

where B denotes a P-Brownian motion with respect to a new filtration $\{\mathcal{G}_t\}$. By the assumption (c), the above has the unique solution (in the strong sense). Moreover by the assumption (d) and continuity of L, we see that $\inf_{t>0} V_t > 0$ a.s.

Let $Y_t \overset{def}{=} \sqrt{V_t}$. By Ito's formula we have

$$Y_t = x + \int_0^t \frac{s+\eta}{a(Y_s)} ds + B_t. \tag{4.2}$$

Let

$$A_t = \int_0^t \frac{ds}{a(Y_s)}, \quad 0 \le t \le \infty \tag{4.3}$$

and C_t be its right continuous inverse, i.e.

$$C_t = \inf\{s, A_s > t\}, \quad 0 \le t \le A_\infty. \tag{4.4}$$

Since A_t is strictly increasing and continuous, so is C_t.

Let $(\hat{\Omega}, \mathcal{F}, \hat{P}, \{\mathcal{F}_t\}_{t\in[0,\infty)})$ be an enlargement of $(\Omega, \mathcal{G}, P, \{\mathcal{G}_{C_t}\}_{t\in[0,\infty)})$ and define a new Brownian motion with respect to this new filtration as follows:

$$W_t = \begin{cases} \displaystyle\int_0^{C_t} \frac{dB_s}{\sqrt{a(Y_s)}}, & \text{for } t < A_\infty \\ \displaystyle\int_0^{\infty} \frac{dB_s}{\sqrt{a(Y_s)}} + \hat{\beta}_{(t-A_\infty)}, & \text{for } A_\infty \le t < \infty, \text{ if } A_\infty < \infty \end{cases} \tag{4.5}$$

where β is a Brownian motion on $\hat{\Omega}$ independent of $\displaystyle\int \frac{dB}{\sqrt{a(Y)}}$.

Then $\hat{X}_t \overset{def}{=} Y_{C_t}$ is defined on $(\hat{\Omega}, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,\infty)})$ as a weak solution for (1.1). Theorem 1.1 ensures that $\hat{X} = X$ is the unique strong solution.

Moreover we shall see that $X_{A_\infty} = Y_\infty = \infty$, so that if $A_\infty < \infty$, the solution explode in finite time, otherwise the global existence is ensured. The key estimation is given bellow as lemma 4.1. Instead of (4.2) we will consider the following (random) ordinary differential equation for each $\omega \in \Omega$.

$$Z_t = x + \int_0^t \frac{s+\eta}{a(Z_s + B_s)}\, ds \tag{4.6}$$

which in turn means $Z_t = Y_t - B_t > 0 \quad a.s.$ by (4.2).

First we shall have

$$Y_t = Z_t + B_t \sim Z_t \to +\infty \quad \text{as} \quad t \to \infty. \tag{4.7}$$

This is done by the following

LEMMA 4.1. *Let* $\bar{\Omega} = \{\omega \in \Omega, \lim_{t\to+\infty} \frac{B_t(\omega)}{t} = 0\}$ *and fix* $\omega \in \bar{\Omega}$.

(i) There exists a positive constant $M(\omega)$ *such that*

$$Z_t(\omega) < x + M(\omega)\, t. \tag{4.8}$$

(ii) There exists a slowly varying function $L^* \sim L$ *and positive constant* $K(\omega)$ *such that*

$$Z_t(\omega) > x + \frac{t}{M+K} \left(L^*(t)\right)^{-1}. \tag{4.9}$$

PROOF OF LEMMA 4.1. Since $\lim_{t\to+\infty} \frac{B_t(\omega)}{t} = 0$, there exists a positive constant K such that $\left|\frac{B_t(\omega)}{t}\right| < K$. Set $\inf_{x>0} L(x) = c$ and let Z_t^1 be the solution for

$$Z_t^1 = x + \int_0^t \frac{s+\eta}{(Z_s^0 + B_s)c}\, ds, \tag{4.10}$$

Then we have $Z_t^1 \geq Z_t$ for all $t \in [0, \infty)$.

Set $M \stackrel{def}{=} \left(K + \frac{\eta}{cx} + \frac{x}{\eta}\right)$ and $Z_t^2 \stackrel{def}{=} x + Mt$. Then we have

$$\begin{aligned}
(Z_t^2)' &= M \\
&= M \frac{t+\eta}{Z_t^2 - Kt} \frac{Z_t^2 - Kt}{t+\eta} \\
&= \frac{t+\eta}{(Z_t^2 - Kt)c} \frac{x + (\frac{x}{\eta} + \frac{\eta}{cx})t}{t+\eta} \left(K + \frac{x}{\eta} + \frac{\eta}{cx}\right) c \\
&= \frac{t+\eta}{(Z_t^2 - Kt)c} \left\{\left(\frac{x}{\eta} + \frac{\eta}{cx}\right) - \left(\frac{1}{t+\eta}\right)\frac{\eta^2}{cx}\right\} \left(K + \frac{x}{\eta} + \frac{\eta}{cx}\right) c \\
&> \frac{t+\eta}{(Z_t^2 - Kt)c} \left(K + \frac{x}{\eta} + \frac{\eta}{cx}\right) \frac{cx}{\eta} > \frac{t+\eta}{(Z_t^2 - Kt)c} > \frac{t+\eta}{(Z_t^2 + B_t(\omega))}.
\end{aligned} \tag{4.11}$$

By the comparison theorem we have $Z_t^2 > Z_t^1$ which proves the first part of Lemma 4.1.

To prove the latter part, we first observe that

$$Z_t + B_t = t\left(\frac{Z_t}{t} + \frac{B_t}{t}\right) \le x + t(M+K). \tag{4.12}$$

Note that $\bar{L} = \sup_{0\le y\le x} L(y)$ is non-decreasing, $\bar{L} \ge L$ and $\bar{L} \sim L$. (See Seneta [10].) Since $\bar{L} \ge L$ and by (4.12) and monotonicity of $\bar{L}$ we have

$$\begin{aligned} \frac{t+\eta}{a(Z_t+B_t)} &= \frac{t+\eta}{Z_t+B_t}\,\frac{1}{L(Z_t+B_t)} \ge \frac{t+\eta}{Z_t+B_t}\,\frac{1}{\bar{L}(Z_t+B_t)} \\ &> \frac{t+\eta}{x+t(M+K)}\,\frac{1}{\bar{L}(x+t(M+K))} \sim \frac{(L(t))^{-1}}{M+K}. \end{aligned} \tag{4.13}$$

Let

$$Z_t^3 \overset{def}{=} x + \int_0^t \frac{s+\eta}{x+s(M+K)}\,\frac{1}{\bar{L}(x+s(L+K))}\,ds. \tag{4.18}$$

Then by the comparison theorem we see that $Z_t^3 < Z_t$. On the other hand, it is well known that $\int^x N(y)\,dy \sim xN(x)$ holds for arbitrary slowly varying function N. (See Seneta [10].) Hence we have $Z_t^3 \sim \frac{t}{M+K}(L(t))^{-1}$. This proves (ii) of Lemma 4.1. □

Then we have for arbitrary $\epsilon > 0$, there exists T_ϵ such that for all $t > T_\epsilon$

$$(1-\epsilon)\frac{t+\eta}{a(Z_t)} \le \frac{dZ}{dt} \quad = \frac{t+\eta}{a(Z_t+B_t)} \le (1+\epsilon)\frac{t+\eta}{a(Z_t)}. \tag{4.14}$$

Consider the following differential equations.

$$\frac{d\overline{Z}}{dt} = \frac{(1+\epsilon)(t+\eta)}{a(\overline{Z}_t)}\ ,T_\epsilon < t < \infty,\ \overline{Z}_{T_\epsilon} = Z_{T_\epsilon}. \tag{4.15}$$

$$\frac{d\underline{Z}}{dt} = \frac{(1-\epsilon)(t+\eta)}{a(\underline{Z}_t)}\ ,T_\epsilon < t < \infty,\ \underline{Z}_{T_\epsilon} = Z_{T_\epsilon}. \tag{4.16}$$

Then by comparison theorems and (4.14),

$$\underline{Z}_t \le Z_t \le \overline{Z}_t, \quad {}^\forall t \ge T_\epsilon. \tag{4.17}$$

On the other hand we have

$$\begin{aligned} \frac{1}{2}(t - T_\epsilon)(t + T_\epsilon + 2\eta) &= \int_{T_\epsilon}^{t} a(\underline{Z}_s)\frac{d\underline{Z}}{ds}\,ds = \int_{\underline{Z}_{T_\epsilon}}^{\underline{Z}_t} a(\underline{Z}_s)d\underline{Z}_s \\ &= \int_{\underline{Z}_{T_\epsilon}}^{\underline{Z}_t} \underline{Z}_s L(\underline{Z}_s)d\underline{Z}_s \sim \frac{1}{2}\underline{Z}_t^2 L(\underline{Z}_t). \end{aligned} \tag{4.18}$$

The last relation is a consequence of the following lemma and (4.7).

Lemma 4.2 (Karamata [5]). *If N is slowly varying on $[c, \infty)$, then for each $k > -1$*

$$\lim_{x \to +\infty} \frac{x^{k+1}N(x)}{\displaystyle\int_c^x y^k N(y)\,dy} = k + 1. \tag{4.19}$$

We also have in the similar way

$$\overline{Z}_t^2 L(\overline{Z}_t) \sim (1 + \epsilon)(t - T_\epsilon)^2. \tag{4.20}$$

Consequently we have

$$Z_t \sim \frac{t}{\sqrt{L(Z_t)}}. \tag{4.21}$$

Then we have

$$a(Y_t) = Y_t L(Y_t) \sim Z_t L(Z_t). \tag{4.22}$$

Since we have by definition of $\mathfrak{f}$ and (4.21), it follows that $\mathfrak{f}(t) \sim Z_t$. Then by (4.3) and (4.22), we have

$$A_t \approx \int^t \frac{ds}{Z_s L(Z_s)} = \int^t \frac{ds}{Z_s\sqrt{L(Z_s)}\sqrt{L(Z_s)}} \approx \int^t \frac{ds}{s\sqrt{L(\mathfrak{f}(s))}} \tag{4.23}$$

Here we use the notation $f \approx g$ meaning $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} < \infty$. Then proof is complete since the above relationship holds almost surely.

5. A Generalized RS Model

In this section we explain first the interest rate model which admits no arbitrage and then give a rather general model that includes the one proposed by Ritchken and Sankarasubramanian [8] [9]. It is also explained how the these models are related to the stochastic integral equation studied so far in the present paper.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, P)$ be a filtered probability space with the usual conditions.

A zero-coupon bond price process with maturity T, denoted by $p(\cdot, T)$, is an $\{\mathcal{F}_t\}$- adapted continuous semimartingale up to time $T \in (0,\infty)$ with $p(T,T) = 1$. If we assume that $\frac{\partial}{\partial T}\log p(t,T)$ exists for every T and that for fixed t it is uniformly bounded, then the spot rate process is given by

$$r_t \overset{def}{=} -\frac{\partial}{\partial T}\log p(t,T)\bigg|_{T\downarrow t}. \tag{5.1}$$

It is well known that absence of arbitrage in the financial market is almost equivalent to the existence of so-called equivalent martingale measure $Q \approx P$ and under Q we must have

$$p(t,T) = E^Q\left(e^{-\int_t^T r_s\,ds}\,|\mathcal{F}_t\right). \tag{5.2}$$

where $E^Q(\cdot)$ denotes the expectation with respect to Q. (See e.g. Duffie [1].)

Here we give a so-called no-arbitrage model of interest rates by specifying the dynamics of the spot rate process.

Before giving the model, we shall have an elementary lemma. Let $M.$ be a continuous Q-local martingale such that $M_0 = 0$ and ξ is a deterministic continuous function.

Set

$$X_t \overset{def}{=} \int_0^t \xi_s[M]_s ds + M_t. \tag{5.3}$$

and

$$\bar{X}_t \overset{def}{=} \xi_t X_t. \tag{5.4}$$

where for a martingale $N.$, $[N].$ denotes its quadratic variation process.

LEMMA 5.1. *Then we have*

$$-\int_0^u \bar{X}_t dt = N_u^u - \frac{1}{2}[\ N^u]_u \tag{5.5}$$

where

$$N_t^u \stackrel{def}{=} -\int_0^t \left(\int_s^u \xi_v dv\right) dM_s. \tag{5.6}$$

PROOF. Since ξ is continuous, a version of Fubini's theorem can be applied. So we have

$$\begin{aligned}\int_0^u \bar{X}_t dt &= \int_0^u \xi_t M_t dt + \int_0^u \xi_t \left(\int_0^t \xi_s [M]_s ds\right) dt \\ &= \int_0^u \left(\int_s^u \xi_t dt\right) dM_s + \int_0^u \xi_t dt \int_0^t \left(\int_v^t \xi_s ds\right) d[M]_v \\ &= \int_0^u \left(\int_s^u \xi_t dt\right) dM_s + \frac{1}{2}\int_0^u \left(\int_v^u \xi_s ds\right)^2 d[M]_v. \ \square\end{aligned} \tag{5.7}$$

Now we set the spot rate process as

$$r_t \stackrel{def}{=} \eta_t + \sum_{i=1}^n \xi_t^i X_t^i \tag{5.8}$$

where $\eta.$ and $\xi^i, i = 1, ..., n$ are linearly independent deterministic continuous functions, and $X^i, i = 1, ..., n$ are defined by (5.3) through independent M^i and ξ^i, that is

$$X_t^i = \int_0^t \xi_s^i [M^i]_s ds + M_t^i, i = 1, ..., n. \tag{5.9}$$

Then we can see that the zero-coupon bond price processes are described as the functions of X^i's and $[M^i]$'s. More precisely we have

Proposition 5.2. *Assume that for all $T > 0$ and $i = 1, ..., n$*

$$E^Q\left[\exp\frac{1}{2}\int_0^t\int_s^T \xi_u^i dud[M^i]_s\right] < \infty \tag{5.10}$$

holds for all $0 \le t \le T$. Then the bond prices are given by the following.

$$p(t,T) = \exp\sum_{i=1}^n\left\{-\left(\int_t^T {\xi^i}_u du\right)X_t^i - \frac{1}{2}\left(\int_t^T {\xi^i}_u du\right)^2[M^i]_t\right\} \cdot e^{-\int_t^T \eta_s ds}. \tag{5.11}$$

Proof. By (5.2), it suffices to calculate

$$E^Q\left(e^{-\int_t^T r_s\,ds}|\mathcal{F}_t\right) = e^{\int_0^t r_s ds}E^Q\left(e^{-\int_0^T r_s\,ds}|\mathcal{F}_t\right). \tag{5.12}$$

By (5.3)–(5.6), (5.9) we have

$$e^{-\int_0^u r_s ds} = \prod_{i=1}^n \mathcal{E}(\,N^{i,u})_u e^{-\int_0^u \eta_s ds}. \tag{5.13}$$

where $N_t^{i,u} = -\int_0^t\left(\int_s^u \xi_u^i du\right)dM_s^i$ and $\mathcal{E}(\cdot)$ denotes its exponential semimartingale. By (5.10), $\mathcal{E}(\,N^{i,u})_\cdot$ is a martingale and by (5.12) and (5.13),

$$p(t,T) = \prod_{i=1}^n \frac{\mathcal{E}(\,N^{i,T})_t}{\mathcal{E}(\,N^{i,t})_t}e^{-\int_t^T \eta_s ds}. \tag{5.14}$$

An easy calculation leads to (5.11). □

Indeed Ritchken and Sankarasubramanian's interest rate model [8] [9] is included in the above one. To see this, we first give the original RS model.

Define the forward rate processes as

$$f(t,T) \stackrel{def}{=} -\frac{\partial}{\partial T}\log p(t,T). \tag{5.15}$$

They assume that $f(\cdot, T)$ be an Ito process, that is,

$$df(t,T) = \sigma(t,T,\omega)dW_t + \mu(t,T,\omega)dt \tag{5.16}$$

where $W.$ denotes a Q-Brownian motion. Then we have by (5.2)

$$dr_t = \sigma(t,t,\omega)dW_t + \left(\mu(t,t,\omega) + \frac{\partial}{\partial T} f(t,T)\Big|_{T\downarrow t}\right) dt. \tag{5.17}$$

By no-arbitrage argument (e.g. Duffie [1]) we see that

$$\mu(t,T,\omega) = \sigma(t,T,\omega) \int_t^T \sigma(t,u)du. \tag{5.18}$$

In [8] [9], they claim that the constraint

$$\sigma(t,T,\omega) = \sigma(t,t,\omega)e^{-\int_t^T \kappa(x)dx}, \kappa \in C_b[0,\infty) \tag{5.19}$$

leads to the conclusion that

$$\mu(t,t,\omega) + \frac{\partial}{\partial T} f(t,T)\Big|_{T\downarrow t} = \kappa(t)(f(0,t) - r_t) + \phi_t + \frac{d}{dt} f(0,t) \tag{5.20}$$

where ϕ_t is given by

$$d\phi_t = (\sigma(t,t,\omega)^2 - 2\kappa(t)\phi_t)dt. \tag{5.21}$$

It follows that if we put $\sigma(t,t,\omega) = \hat{\sigma}(r_t,t)$, r is a (2-dim) Markovian.

One can easily see that if we set in (5.8) $i = 1$,

$$\xi_t = \xi_0 e^{-\int_0^t \kappa(x)dx}, \tag{5.22}$$

$$\eta_t = f(0,t), \tag{5.23}$$

and

$$M_t = \int_0^t \xi_s^{-1} \hat{\sigma} dW_s, \tag{5.24}$$

we have the Markovian spot rate process of Ritchken and Sankarasubramanian's model.

Remarks. To the best of our knowledge, Jamshidian [4] is the first to consider a class of forward rate processes given by (5.19)–(5.21) which he called Quasi Gaussian.

A multi-factor version of the RS model is discussed in Inui and Kijima [3]. They construct a whole-yield model while ours is a spot rate model. They pointed out that n-factor the RS model allows one to use the 2n-state Markovian spot rate. This also coincides with ours since we can set (ξ^i, M^i) as 2 dimensional diffusion processes.

Takahashi [12] reported that SIE models have advantages as price processes of financial assets compared with one dimensional diffusion models. Here by SIE models we mean that the processes are given by general stochastic differential equations including those of non-Markov types. His model is slightly different from ours. See also Kannan et al.[6].

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