

A Vertical Resonance of g-Mode Oscillations in Warped Disks and QPOs in Low-Mass X-Ray Binaries

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Abstract

Resonant oscillations in warped disks are examined in order to explain high-frequency QPOs and horizontal-branch QPOs in low-mass X-ray binaries. Different from our previous work, addressed to the same subjects, we relax in this paper the assumption that the disks are isothermal in the vertical direction. That is, the pressure, density, and temperature are assumed to be distributed in the vertical direction with a polytropic relation, and the polytropic index changes as the disk state changes. By this generalization and by some subsidiaries we can qualitatively explain, within the framework of our resonance model, observed large frequency variations in neutron-star QPOs and little variations in black-hole QPOs. We consider vertical resonances of g-mode oscillations, since they are most appropriate to explain observations.

Key words: accretion, accretion disks — quasi-periodic oscillations — resonance — warp — X-rays: stars

1. Introduction

In this paper we are interested in possible origins of high-frequency quasi-periodic oscillations (QPOs) in neutron-star and black-hole low-mass X-ray binaries (LMXBs), and additionally in horizontal-branch QPOs in luminous neutron-star LMXBs (Z sources). One of the important characteristics of kHz QPOs in neutron-star LMXBs is that they usually occur in a pair, and their frequencies change with time (see, e.g., van der Klis 2000, 2004 for reviews). The time change occurs in such a way that their frequency ratio decreases with an increase of frequencies. In the case of black-hole LMXBs, high-frequency QPOs usually also occur in a pair, but their frequencies change little with time, the ratio being kept close to 2 : 3 (see van der Klis 2004 for a review). In Z sources, another important characteristic of QPOs is present. It is a strong correlation between kHz QPOs and horizontal-branch QPOs in their frequencies (Psaltis et al. 1999). That is, the frequency of the lower kHz QPO, ν_L , and that of the horizontal-branch QPOs, ν_{HBO} , are correlated in each object so that $\nu_{\text{HBO}} \sim 0.08\nu_L$.

Paying attention to the closeness to 2 : 3 of the frequency ratio in high-frequency QPOs in black-hole LMXBs, Abramowicz and Kluźniak (2001) and Kluźniak and Abramowicz (2001) presented a resonance model of disk oscillations to explain the QPOs. On the other hand, the importance of an external deformation of disks on resonance processes has been emphasized by Kato (2003, 2004a, b), Kluźniak et al. (2004), and Lee, Abramowicz, and Kluźniak (2004). Kato (2003, 2004a, b) especially considered models of resonant oscillations on warped disks. The models were extended to a case where the warp had precession (Kato 2005). In these warp models, the temperature is assumed to always distribute isothermally in the vertical direction. In real situations, however, this will not be the case. The vertical structure of disks will change by the difference of the disk

state resulting from a difference of the mass-accretion rate. Considering this, we assume in this paper that the pressure, density, and temperature distributions in the vertical direction are polytropic, and that the polytropic index changes with time. By adopting this model, we can easily obtain the time variations of the resonant oscillations as a result of changes of the polytropic index.¹ The purpose of this paper is to demonstrate this generalization and to apply the results to QPOs.

In warped disks there are four types of resonant oscillations due to the four possible combinations of two types of oscillations (g-mode and p-mode oscillations) and two types of resonances (vertical and horizontal resonances). Without considering all cases, we consider here only a case where the time variations of QPOs can be well explained. This is the case where disk oscillations are g-modes, and the resonances occur through vertical motions.²

2. Vertical Resonances of g-Mode Oscillations in Warped Disks

Details of the non-linear resonant oscillations on warped disks are presented by Kato (2003, 2004a, b). An overview of the model is sketched in figure 1 of Kato (2004a).

We consider geometrically thin relativistic disks rotating with angular velocity, $\Omega(r)$. The epicyclic frequency on the disk is denoted by $\kappa(r)$. The oscillations on geometrically thin disks are classified into g-mode and p-mode oscillations (see, e.g., Kato et al. 1998; Kato 2001). In simplified disks the oscillations are further sub-classified by the set (m, n) , where $m = (0, 1, 2, \dots)$ is the number of nodes in the azimuthal direction, and $n = (0, 1, 2, \dots)$ is the number related to nodes in the vertical direction. That is, n represents the number of nodes

¹ In Kato (2005), time variation of QPOs was interpreted as a result of time change of precession of warps.

² In Kato (2005), horizontal resonances of g-mode oscillations were considered.

that u_r (the radial component of velocity associated with oscillations) has in the vertical direction. It is noted, however, that u_z (the vertical component of velocity associated with oscillations) has $(n - 1)$ nodes in the vertical direction. In the case of $n = 0$, $u_z = 0$ and a series of g-modes starts from $n = 1$.

A warp is generally a global pattern on disks with $m = n = 1$. The warp may have precession, but here we assume that it has no precession in order to consider idealized cases. On a disk deformed by the warp, we superpose g-mode oscillations with arbitrary m and n (n is taken to be 1 or 2, later). A g-mode oscillation with the frequency ω and (m, n) has a relatively large amplitude, global pattern only around the radius where

$$(\omega - m\Omega)^2 - \kappa^2 = 0 \quad (1)$$

is satisfied. This can be understood if the dispersion relation for local perturbations is considered. That is, the dispersion relation shows that the region of $(\omega - m\Omega)^2 - \kappa^2 > 0$ is an evanescent region of the g-mode oscillations. In the region where $(\omega - m\Omega)^2$ is smaller than κ^2 , on the other hand, the oscillations have very short wavelengths in the radial direction when geometrically thin disks are considered.

A non-linear interaction of a g-mode oscillation with the warp produces an oscillation with ω , \tilde{m} , and \tilde{n} , where $\tilde{m} = m \pm 1$ and $\tilde{n} = n \pm 1$ (these oscillations are called hereafter intermediate oscillations). These intermediate oscillations resonantly interact with the disk at the radius where the dispersion relation for these intermediate oscillations is satisfied [see Kato (2004b) for detailed discussions]. There are two types of resonances, corresponding to two types of wave modes. One is resonances that occur through motions in the vertical direction (vertical resonances); the other is those through motions in the radial direction (horizontal resonances) (see, e.g., Kato 2004a, b). In this paper we are interested in vertical resonances. They occur around the radius where

$$(\omega - \tilde{m}\Omega)^2 - \Psi_{\tilde{n}}\Omega^2 \sim 0 \quad (2)$$

is satisfied (see, e.g., Kato 2004a, b, and the Appendix in this paper), where $\Psi_{\tilde{n}}$ is a number related to \tilde{n} and N (a polytropic index specifying the vertical structure of disks, see the Appendix). Hereafter, we are particularly interested in cases where $\tilde{n} = 2$ (assuming $n = 1$) and $\tilde{n} = 3$ (assuming $n = 2$). The quantities $\Psi_{\tilde{n}}$ in these cases are

$$\Psi_2 = 2 + \frac{1}{N} \quad \text{and} \quad \Psi_3 = 3 + \frac{3}{N}, \quad (3)$$

as shown in the Appendix. In the limit where the disk is isothermal in the vertical direction, $N = \infty$ and we have $\Psi_2 = 2$ and $\Psi_3 = 3$. They are cases that have been considered in previous papers (Kato 2003, 2004a, b).

Combining equations (1) and (2), we find that the resonances occur at radii of

$$\kappa = (\Psi_2^{1/2} - 1)\Omega \quad \text{and} \quad \kappa = (\Psi_3^{1/2} - 1)\Omega, \quad (4)$$

for $\tilde{n} = 2$ and $\tilde{n} = 3$, respectively. After this resonance the intermediate oscillations feedback to the original oscillations by non-linear interaction with the original oscillations themselves. The results are amplification or dampening of the original oscillations, depending on the types of oscillations and resonances (Kato 2004b).

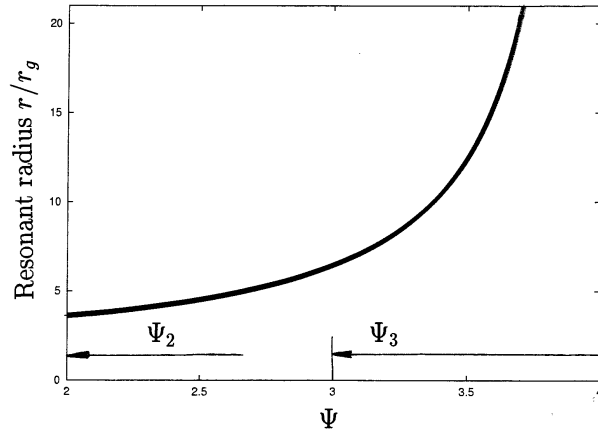


Fig. 1. r/r_g - Ψ relation obtained by solving the resonance condition $\kappa = (\Psi^{1/2} - 1)\Omega$. The disk is Keplerian with the Schwarzschild metric. The ranges of variations of Ψ_2 and Ψ_3 are shown.

Detailed examination shows that when $\tilde{m} = m - 1$, the frequencies of the g-mode oscillations that resonantly interact with the disk at $\kappa = (\Psi_{\tilde{n}}^{1/2} - 1)\Omega$ are $m\Omega - \kappa$ at the radius. On the other hand, when $\tilde{m} = m + 1$ the frequencies of the g-mode oscillations that resonantly interact with the disk at $\kappa = (\Psi_{\tilde{n}}^{1/2} - 1)\Omega$ are $m\Omega + \kappa$ there (e.g., Kato 2004a). The most observable oscillations are those with small m 's; otherwise, they are phase-mixed. Hence, the frequencies, ω , of typical non-axisymmetric resonant oscillations are $\Omega - \kappa$ (i.e., $m = 1$, $\tilde{m} = 0$), $\Omega + \kappa$ (i.e., $m = 1$, $\tilde{m} = 2$), and $2\Omega - \kappa$ (i.e., $m = 2$, $\tilde{m} = 1$). As axisymmetric resonant oscillations we have κ (i.e., $m = 0$, $\tilde{m} = 1$). Axisymmetric oscillations, however, will be less observable. Hence, we think that the above non-axisymmetric oscillations are related to the horizontal-branch QPOs, upper and lower kHz QPOs. Considering this, we introduce

$$\omega_H = \Omega + \kappa, \quad \omega_L = 2\Omega - \kappa, \quad \text{and} \quad \omega_{HBO} = \Omega - \kappa, \quad (5)$$

and examine their frequencies at the resonance radius.

3. Frequencies and Their Variations of Resonant Oscillations

In the limit of isothermal disks in the vertical direction, i.e., $N = \infty$, the resonance occurs at $\kappa = (\sqrt{2} - 1)\Omega$ when $\tilde{n} = 2$ and at $\kappa = (\sqrt{3} - 1)\Omega$ when $\tilde{n} = 3$. These radii are $3.62r_g$ and $6.46r_g$, respectively (e.g., Kato 2004a). The temperature distribution in the vertical direction in disks is, however, generally not isothermal. Furthermore, it changes with time, depending on the state of the disks. In figure 1, the r/r_g - Ψ relation given by $\kappa = (\Psi^{1/2} - 1)\Omega$ is shown for the range of $\Psi = 2.0$ – 4.0 . In the case of $\tilde{n} = 2$, Ψ (i.e., Ψ_2) changes from 2 (for $\gamma = 1$ or $N = \infty$) to 2.67 ($\gamma = 5/3$ or $N = 1.5$), while Ψ_3 changes from 3 ($\gamma = 1$) to 5 (for $\gamma = 5/3$) in the case of $\tilde{n} = 3$. The ranges of variations of Ψ_2 and Ψ_3 are also shown in figure 1.

If the value of Ψ is specified, the resonant radius is obtained. Then, from equation (5), the frequencies of the resonant oscillations (ω_H , ω_L , and ω_{HBO}) are derived as functions of Ψ . The relations among these frequencies are shown in figure 2 as functions of ω_H . The relation between $2\omega_{HBO}$ and ω_H is also

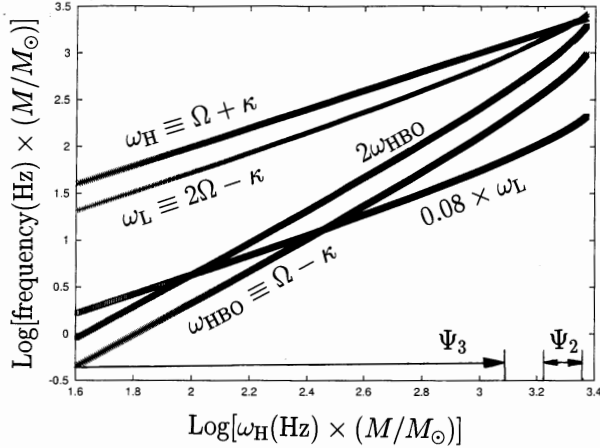


Fig. 2. Frequencies (ω_H , ω_L , ω_{HBO} , and $2\omega_{HBO}$) as functions of ω_H . For comparison, the $0.08\omega_L$ – ω_H relation is also shown. The ranges of variations of Ψ_2 and Ψ_3 are shown near the horizontal scale.

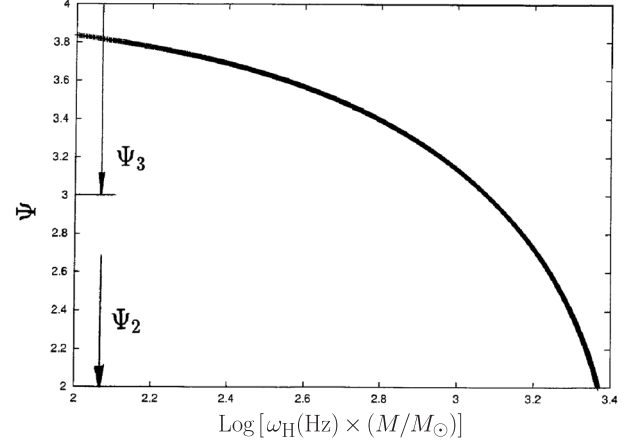


Fig. 3. Ψ – $\text{Log}[\omega_H(\text{Hz}) \times (M/M_\odot)]$ relation. The disk is Keplerian with the Schwarzschild metric. The ranges of variations of Ψ_2 and Ψ_3 are also shown.

shown. Important points are that: i) figure 2 is free from the mass of the central source, if the both horizontal and vertical scales are normalized by $(M/M_\odot)^{-1}$, ii) the curves in figure 2 are universal as long as g-mode oscillations are concerned. They are free from N (and Ψ), and even free from the presence or absence of precession (see Kato 2005). That is, the frequencies of the resonant oscillations change along the curves when N (and thus Ψ) is changed. The relation between ω_H and Ψ is shown in figure 3. In general, an increase of Ψ decreases ω_H . The range of the variation of ω_H by a change of Ψ is also shown in figure 2.

4. Summary and Numerical Estimate

The main results of this paper are summarized in figure 2. The figure should be compared with figure 2.6 of van der Klis (2004), which summarizes observational data concerning QPO frequencies on a frequency–frequency diagram. This comparison suggests that the resonant oscillations specified by ω_H , ω_L , and ω_{HBO} well correspond to the upper and lower kHz QPOs and the horizontal-branch QPOs in neutron-star LMXBs, respectively. Adopting these identification, we qualitatively explain some basic observational characteristics of the QPOs.

The kHz QPOs in neutron-star LMXBs usually appear in a pair, and the separation frequency of the twin peaks decreases as the peak frequencies increase. This observational characteristic is derived in our model, as shown in figure 2 (see the curves of ω_L and ω_H). In our model the frequency change of QPOs is the result of a change of the temperature distribution in the vertical direction. As the temperature distribution in the vertical direction approaches to an isothermal one [i.e., Ψ_2 (the case of $\tilde{n} = 2$) tends to 2.0 or Ψ_3 (the case of $\tilde{n} = 3$) tends to 3.0], the resonant radius becomes smaller (see figure 1) and the frequencies of resonant oscillations increase. Observations show that the frequencies of the pair QPOs increase with an increase of the mass-accretion rate (van der Klis et al. 1997). Hence, if our model is correct, it suggests that the temperature distribution in the vertical direction approaches to the isothermal one as the mass-accretion rate increases. Here, let

us make a quantitative estimate. Figure 2 shows that the 3 : 2 ratio of ω_H and ω_L is realized when $\text{log}[\omega_H(M/M_\odot)] \sim 2.93$, i.e., $\omega_H = 851(M/M_\odot)^{-1}$. This occurs for $\Psi_3 = 3.24$ (i.e., $\tilde{n} = 3$), as shown in figure 3. This gives $1/N = 0.08$ and $\gamma = 1.08$. The resonant radius is then at $8.33r_g$, as shown in figure 1.

One of the other prominent correlations among QPO frequencies in neutron-star LMXBs is that the frequencies of the horizontal-branch QPOs, ν_{HBO} , are correlated with the frequencies of lower kHz QPOs, ν_L , by $\nu_{HBO} \sim 0.08\nu_L$. In order to compare our results with the observations, the curve of the $0.08\omega_L$ – ω_H relation is drawn in figure 2. The curve crosses the curve labelled by ω_{HBO} around $\text{log}[\omega_H(M/M_\odot)] \sim 2.46$, i.e., $\omega_H = 288(M/M_\odot)^{-1}$. Figure 3 then shows $\Psi_3 \sim 3.66$, which means $3/N \sim 0.66$ or $\gamma \sim 1.22$. This implies that the observed correlation can be explained if the temperature distribution in the vertical direction is, on the average, around $\gamma \sim 1.2$. From figure 1 we see that the resonance occurs around $r = 18.0r_g$ when $\Psi_3 = 3.66$. As shown above, both $\omega_H : \omega_L = 3 : 2$ and $\omega_{HBO} = 0.08\omega_L$ cannot be simultaneously satisfied in a rigorous sense. A slight larger coefficient than 0.08, say 0.09, is compatible with the ratio 3 : 2.

In the case of black-hole LMXBs, observations show that the frequencies of pair QPOs change little, and their ratio is kept to be close to 3 : 2, unlike the case of neutron-star kHz QPOs. As discussed in the case of neutron-star QPOs, one possibility of explaining the observed 3 : 2 is that it represents $\omega_H : \omega_L$. Another one, which is better, is to consider the resonances of $\tilde{n} = 2$ and to regard the ratio as $\omega_L : 2\omega_{HBO}$. As discussed in Kato (2004b), if we consider the resonance at $4.0r_g$, ω_H is equal to ω_L and the ratio ω_L (or ω_H) : $2\omega_{HBO}$ is just 3 : 2. Figure 1 shows that the resonance at $4.0r_g$ is realized for $\tilde{n} = 2$ when $\Psi_2 = 2.25$, which means $1/N = 0.25$ (i.e., $\gamma = 1.25$) in the present model. Figure 2 (see also figure 3) shows that this occurs when $\text{log}[\omega_H(M/M_\odot)] = 3.33$, i.e., $\omega_H = 2.14 \times 10^3(M/M_\odot)^{-1}$. It is noted that in the case where the resonance occurs through intermediate oscillations of $\tilde{n} = 2$ (not $\tilde{n} = 3$), the frequency variation for a change of N (or γ) is weak (see the variation range of Ψ_2 in figure 2). Furthermore, $\omega_H = 2.14 \times 10^3(M/M_\odot)^{-1}$ is compatible with observed results derived by McClintock and

Remillard (2003). It is, however, unclear why the black-hole QPOs are the resonances of $\tilde{n} = 2$ ($n = 1$), while the neutron-star QPOs are the resonances of $\tilde{n} = 3$ ($n = 2$).

More quantitative comparisons of our model with observations would be premature at the present stage, since the frequencies of vertical resonances are sensitive to the vertical structure of the disks (the vertical structure is not always polytropic). In other words, if our present model is correct, a comparison of our results with observations would give good information concerning the vertical structure of disks.

5. Discussion

The present resonance model naturally explains some basic characteristics of the kHz QPOs of neutron-star LMXBs and the high-frequency QPOs of black-hole LMXBs. It is especially noted that the observed frequency change of kHz QPOs, which is observationally related to a change of the mass-accretion rate, is naturally explained if the vertical structure of disks changes with the change of the mass-accretion rate. In this sense, the present model seems to be superior to a precession model of warps (Kato 2005). In the latter model, a time variation of precession is required to explain the time variation of the observed kHz QPO frequencies, and it is unclear whether such a variation of precession is generally expected theoretically. The latter model with precession, however, can naturally explain the observed hectohertz QPOs as a manifestation of the precession of warps. In the present model of the vertical resonances, on the other hand, there is a problem concerning excitation. The vertical resonances do not excite the g-mode oscillations, but rather dampen them in the limit of $N = \infty$ (Kato 2004b). We should carefully study in the future whether our previous results concerning the stability of resonance are correct and relevant, even when $N \neq \infty$. Comparisons of the characteristics of the present model with those of the precession model are given in table 1.

Many QPO models have been proposed so far. Some of them connect the observed QPO frequencies with the characteristic frequencies of disks, such as the orbital frequency, radial and vertical epicyclic frequencies and others, or their combinations. For example, in their precession model, Stella and Vietri (1998) identify the periastron precession frequency, $\Omega - \kappa$ [see the third in equation (5)], with the lower frequency of the kHz QPO. An important issue in such models is where preferred radii to produce specific frequencies exist. Concerning this point, these models are classified into various types, e.g., precession model (Stella, Vietri 1998), resonance model (Kluźniak, Abramowicz 2001; Abramowicz, Kluźniak 2001), and beat-frequency model (Miller et al. 1998). Our warp model is based on hydrodynamical wave phenomena and

different from the models mentioned above. However, if we discuss the present model in connection with them, a warp can be regarded as a process to select preferred radii.

Finally, the applicability of the present model to cataclysmic variables (CVs) is briefly discussed. Mauche (2002) pointed out that the frequency correlation $\nu_{\text{HBO}} \sim 0.08\nu_{\text{L}}$ can be extended to CVs. This has recently been confirmed by Warner and Woudt (2004). That is, the DNO (dwarf novae oscillation)–QPO relation in CVs is on the line of extension of the HBO–kHz QPO relation in X-ray stars. Furthermore, it is known that the DNOs in CVs change their frequencies, accompanying harmonics (Warner, Woudt 2004).

The disks of CVs are Keplerian and Ω and κ are almost equal. Hence, at a glance, the resonance conditions, i.e., equation (4), seem not to be realized anymore. This is, however, not the case. The observations show that DNOs usually occur in the phase of outbursts. Near the transition front of outbursts, the disk thickness changes sharply. In such region, the derivation of the resonance condition, $\kappa = (\Psi_3^{1/2} - 1)\Omega$, will be inaccurate, since the assumption of slow radial change of H/r is involved in the derivation. If we assume, however, that the resonance condition, $\kappa = (\Psi_3^{1/2} - 1)\Omega$, is still valid, the resonances occur even in the Newtonian Kepler disks, if $\Psi_3 \sim 4$. Since $\Psi_3 = 3 + 3/N$, $\Psi_3 = 4$ is realized when $1/N = 1/3$, i.e., $\gamma = 4/3$. (As mentioned before, $\Psi_3 = 3$ for $\gamma = 1$ and $\Psi_3 = 5$ for $\gamma = 5/3$.) A vertical disk structure with $N = 3$ will not be unrealistic. This consideration suggests that the harmonic structure of DNOs can be interpreted as the oscillations of $2\Omega + \kappa$, $\Omega + \kappa$, and κ , at the resonant radius. Their frequency ratios are 3 : 2 : 1 in the Newtonian Kepler disks. The observed frequency changes of DNOs are interpreted as being the result of a change of the resonant radius by a change of the disk structure. [See also Kluźniak et al. (2005) for an interpretation of DNOs.] The observed correlation between DNOs and QPOs in CVs, however, cannot be explained by the present model. Further considerations are needed.

Appendix. Vertical Oscillations of Polytropic Disks

We employ cylindrical coordinates (r, φ, z) , whose origin is at the center of the central object and the z -axis is in the vertical direction perpendicular to the disk plane. Let us consider a disk where the pressure, p , and density, ρ , are related in the vertical direction with a polytropic relation, i.e., $p = K\rho^{1+1/N}$, where N is the polytropic index. Then, the vertical integration of the hydrostatic balance in the vertical direction gives

$$T_0 = T_{00}(r) \left(1 - \frac{z^2}{H^2} \right), \quad (\text{A1})$$

Table 1. Comparison of two resonant models of g-mode oscillations.

Horizontal resonances (Kato 2005)		Vertical resonances (present paper)
Time variation	change of precession	change of vertical disk structure
Hectohertz QPOs	precession	precession ?
Excitation	○ (probably)	?

$$\rho_0 = \rho_{00}(r) \left(1 - \frac{z^2}{H^2}\right)^N, \quad (\text{A2})$$

and

$$p_0 = p_{00}(r) \left(1 - \frac{z^2}{H^2}\right)^{N+1}, \quad (\text{A3})$$

where subscript 0 represents the quantities in the equilibrium state, and 00 represents those on the equatorial plane (e.g., Kato et al. 1998). The half-thickness of the disk, $H(r)$, is related to p_{00} and ρ_{00} by

$$\Omega_K^2 H^2 = 2(N+1) \frac{p_{00}}{\rho_{00}}. \quad (\text{A4})$$

We consider here a vertical oscillation of the disk, assuming, for simplicity, that it has no velocity in the horizontal direction. The perturbation associated with the oscillation is assumed to be proportional to $\exp[i(\omega t - m\varphi)]$, where ω is the frequency of the oscillation and m is the number of arms in the azimuthal direction. If the oscillation occurs under the polytropic relation given above, the vertical component of the equation of motion gives

$$i(\omega - m\Omega)u_z = -\frac{\partial h_1}{\partial z}, \quad (\text{A5})$$

where $h_1 = c_s^2 \rho_1 / \rho_0$, u_z and ρ_1 are the vertical velocity and the perturbation density associated with the oscillation, respectively, and $c_s^2 = \gamma p_0 / \rho_0$ with $\gamma = 1 + 1/N$. The equation of continuity is written as

$$i(\omega - m\Omega)\rho_1 + \frac{\partial}{\partial z}(\rho_0 u_z) = 0. \quad (\text{A6})$$

Elimination of u_z from the two above equations gives an equation for h_1 as

$$\frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\rho_0 \frac{\partial h_1}{\partial z} \right) + \frac{(\omega - m\Omega)^2}{c_s^2} h_1 = 0, \quad (\text{A7})$$

which is reduced to

$$\frac{\partial^2 h_1}{\partial x^2} - \frac{2Nx}{1-x^2} \frac{\partial h_1}{\partial x} + \frac{2N}{1-x^2} \frac{(\omega - m\Omega)^2}{\Omega_K^2} h_1 = 0 \quad (\text{A8})$$

by changing the independent variable from z to a dimensionless variable x defined by $x = z/H$. This equation should be solved with a boundary condition that the Lagrangian change of pressure is zero at the disk surface to obtain eigen-values ω 's.

The solutions of equation (A8) are easily obtained by expressing h_1 in terms of a finite power series of x . In the fundamental mode of oscillations, $h_1 \propto x$ and $(\omega - m\Omega)^2 = \Omega^2$. In the first overtone, $h_1 \propto 1 - (1 + 2N)x^2$ and $(\omega - m\Omega)^2 = (2 + 1/N)\Omega^2$. In the second overtone we have $h_1 \propto x - (1 + 2N/3)x^3$ and $(\omega - m\Omega)^2 = (3 + 3/N)\Omega^2$. In this way we obtain the eigen-frequencies of vertical oscillations as

$$(\omega - m\Omega)^2 = \Psi_n \Omega^2, \quad (\text{A9})$$

where for the fundamental ($n = 1$), first overtone ($n = 2$), and second overtone ($n = 3$) we have

$$\Psi_1 = 1, \quad (\text{A10})$$

$$\Psi_2 = 2 + \frac{1}{N}, \quad (\text{A11})$$

and

$$\Psi_3 = 3 + \frac{3}{N}. \quad (\text{A12})$$

For $\gamma = 5/3$ and $\gamma = 4/3$, we have $1/N = 2/3$ and $1/3$, respectively. In the case of isothermal disks, $1/N = 0$.

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