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## A *relax and cut* approach using the multi-commodity flow formulation for the traveling salesman problem

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### Abstract

In this paper we explore the multi-commodity flow formulation for the Asymmetric Traveling Salesman Problem (ATSP) to obtain dual bounds. The procedure employed is a variant of a relax and cut procedure proposed in the literature that computes the Lagrangean multipliers associated to the subtour elimination constraints preserving the optimality of the multipliers associated to the assignment constraints. The results obtained by the computational study are encouraging and show that the proposed algorithm generated good dual bounds for the ATSP with a low execution time.

**Keywords:** traveling salesman problem; relax and cut; Lagrangean relaxation.

## Un enfoque *relax and cut* usando una formulación de flujo multiproductos para el problema del agente viajero

### Resumen

En este artículo nosotros exploramos una formulación de flujo multiproductos para el Problema del Agente Viajero Asimétrico (ATSP) en la obtención de cotas duales de este problema. El procedimiento empleado es una variante del método *relax and cut* propuesto en la literatura que computa los multiplicadores lagrangianos asociados a las restricciones de eliminación de subrutras preservando la optimalidad de los multiplicadores asociados a las restricciones de asignación. Los resultados obtenidos con la experimentación computacional son alentadores y muestran que el algoritmo propuesto genera buenas cotas duales con un tiempo de ejecución bajo.

**Keywords:** problema del agente viajero; *relax and cut*; relajación lagrangiana.

### 1. Introduction

The Traveling Salesman Problem (TSP) has been the subject of many works beginning with the seminal paper of Dantzig, Fulkerson, and Johnson in 1954 [3]. The applications can vary from everyday routing problems e.g. [1,17], to production planning problems [13]. Many of these works also discuss models and solution approaches for the TSP. For the majority of solution approaches, it is important to have good primal and dual bounds. The latter can be obtained by exploring different types of relaxations.

The linear relaxation of a Mixed Integer formulation to an optimization problem can provide a dual bound and its quality

depends on how close the formulation is to the convex hull of solutions. Öncan et al. [14] review several mathematical formulations for the ATSP and discuss the quality of the associated bounds. The difference among the formulations is how the subtour elimination constraints are formulated. The formulation presented in [3], known as DFJ, provides a stronger dual bound and has been the basis for several solution methods for the ATSP [e.g. 16]. However it has an exponential number of subtour elimination constraints. Another formulation is the multi-commodity flow formulation (MC-ATSP) and it uses a polynomial number of constraints to eliminate subtours. It is as strong as the DFJ formulation; however, it might present difficulties when it comes to solving the associated linear relaxation.

Due to the computational effort necessary to solve the linear relaxation of the MC-ATSP, Rocha, Fernandes, and Soares [16] apply a Lagrangean relaxation to derive dual bounds for the ATSP. In the present paper, we also explore Lagrangean bounds for the MC-ATSP formulation. However, instead of dualizing all the subtour elimination constraints at once, we dualize only the ones that are violated by the current solution of the relaxed problem. This idea has been denominated the Relax and Cut procedure. It was introduced in the works of Balas and Christofides [2] and Gavish [7], and it has been the subject of many studies in recent years, although it has not always been referred to as such [2,4,-6,8,9, 16,18].

To briefly describe the relax and cut method, consider an integer optimization problem (IP) defined by (1)-(4).

$$\min f(x) = cx \quad (1)$$

$$\text{subject to } Ax \leq b \quad (2)$$

$$Cx \leq d \quad (3)$$

$$x \in X \quad (4)$$

A relaxation of (IP) can be obtained by removing the constraints (2), and is denominated (RP). Let  $\bar{x}$  be the optimal solution of (RP), and let  $a_i x \leq b_i$  be a constraint of (IP) that is violated by  $\bar{x}$ . A Lagrangean type relaxation of problem (IP) can be built by dualizing the violated constraint using  $\lambda \in \mathbb{R}_+^n$  as stated in problem (LRP) defined by (5)-(7).

$$g(\lambda) = \min f(x) = cx - \lambda(a_i x - b_i) \quad (5)$$

$$\text{subject to } Cx \leq d \quad (6)$$

$$x \in X \quad (7)$$

Fixing the value of  $\lambda$  it is possible to obtain a dual bound for problem (IP). The best bound that can be obtained by the relaxation (LRP) is found by solving the associated dual problem stated in (8).

$$\max_{\lambda \geq 0} \{g(\lambda)\}. \quad (8)$$

Having solved the problem (8), it might be possible to improve the dual bound obtained so far by identifying new valid inequalities for (IP) that are not satisfied by the current solution of (8), reformulating the relaxation (LRP) by dualizing them in a Lagrangean fashion, and solving the new Lagrangean dual problem. This procedure has been coined by Lucena [11] as a *Delayed Relax and Cut* method. A different procedure, coined as *Non-Delayed Relax and Cut* in [12], identifies new violated valid inequalities and reformulates the relaxation each time a multiplier is updated.

The remainder of this paper is organized as follows. The *non-delayed relax and cut* method applied to a generic formulation of the ATSP is presented in section 2. In Section 2.1, the procedure presented in [2] for the DFJ formulation is briefly described followed by the description of our proposal to adapt it

for MC-ATSP formulation presented in Section 2.2. A numerical study comparing the two procedures is presented in Section 3 and concluding remarks are given in Section 4. This paper is an extension of work presented at the CILOG 2014 [10].

## 2. The non-delayed relax and cut method applied to the ATSP

Consider a Graph  $G(V, A)$  with  $|V| = n$ ,  $|A| = n^2$  and a cost  $c_{ij}$  for each arc  $(i, j) \in A$ . A generic mathematical formulation for the ATSP problem is stated in (9)-(13) and is denominated (GATSP).

$$v(ATSP) = \min \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} \quad (9)$$

$$\text{Subject to } \sum_{j \in V} x_{ij} = 1, i \in V \quad (10)$$

$$\sum_{i \in V} x_{ij} = 1, j \in V \quad (11)$$

$$\sum_{i \in V} \sum_{j \in V} a_{ij}^t x_{ij} \geq a_t^0, t \in T \quad (12)$$

$$x_{ij} = 0/1, \quad i, j \in V. \quad (13)$$

The variable  $x_{ij}$  defines whether city  $j$  succeeds city  $i$  in the Hamiltonian cycle. The objective function (9) states the search for the minimum cost Hamiltonian cycle. Constraints (10)-(11) guarantees that each city is included exactly once in the Hamiltonian cycle. The constraints set (12) state the usual subtour elimination constraints in a generic format [2].

If constraints (12) are dropped we obtain a relaxation for the ATSP and this problem is known as the Assignment Problem (AP). Given the properties of the constraint matrix of (AP), it can be solved as a continuous linear optimization problem. Let  $\bar{x}$  be the optimal primal solution to the continuous version of (AP),  $(u, v)$  be the associated optimal dual solution and  $B$  be the associated optimal basis. If  $\bar{x}$  is feasible to the GATSP, we are done. Otherwise, it is of interest to compute strong primal and dual bound for the GATSP. In what follows, the non-delayed relax and cut method will be applied to the GATSP formulation in order to obtain dual bounds.

Let  $\lambda_t, t \in T$ , be the Lagrangean multipliers associated to constraints (12) and  $X$  be the feasible set associated to the relaxation (AP). Dualizing the constraints (12) we get the Lagrangean function (14).

$$\begin{aligned} L(\lambda) &= \min_{x \in X} \left\{ \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} - \sum_{t \in T} \lambda_t \left( \sum_{i \in V} \sum_{j \in V} a_{ij}^t x_{ij} - a_t^0 \right) \right\} \\ &= \min_{x \in X} \left\{ \sum_{i \in V} \sum_{j \in V} \left( c_{ij} - \sum_{t \in T} \lambda_t a_{ij}^t \right) x_{ij} \right\} + \sum_{t \in T} \lambda_t a_t^0 \quad (14) \end{aligned}$$

A dual bound (DL) for the ATSP can be obtained by solving the associated Lagrangean dual (15).

$$DL = \max_{\lambda \geq 0} \{L(\lambda)\}. \quad (15)$$

Several methods can be used to solve (15), among them the subgradient method and the Volume algorithm (e.g. [8], [16]). Balas and Christofides [2] propose a non-delayed relax and cut algorithm in which the search for the optimal Lagrangean multipliers is done by searching for  $\lambda$  values that maintains the optimality of the primal solution  $\bar{x}$  for the relaxation (AP). That is, the search for  $\lambda \geq 0$  that improve the dual bound given by the relaxation (AP),  $v(AP)$ , is limited to dual solutions such that conditions (16) and (17) are satisfied.

$$\text{If } \bar{x}_{ij} = 1, \text{ then } u_i + v_j + \sum_{t \in T} \lambda_t a_{ij}^t = c_{ij} \quad (16)$$

$$\text{If } \bar{x}_{ij} = 0, \text{ then } u_i + v_j + \sum_{t \in T} \lambda_t a_{ij}^t \leq c_{ij} \quad (17)$$

That is, conditions (16) and (17) impose the search for  $\lambda_t$  among the values that guarantee feasible solutions to the dual of the problem (AP), problem (DAP) defined by (18)-(20).

$$\max \sum_{i \in V} u_i + \sum_{j \in V} v_j + \sum_{\lambda \in T} \lambda_t a_0^t \quad (18)$$

Subject to

$$u_i + v_j + \sum_{\lambda \in T} \lambda_t a_{ij}^t \leq c_{ij}, i, j \in V \quad (19)$$

$$\lambda \geq 0. \quad (20)$$

Let

$$\Lambda = \{(u, v, \lambda) : (16); (17)\}. \quad (21)$$

The Lagrangean dual problem (15) can be restated as (22).

$$\max_{\lambda \in \Lambda} \{L(\lambda)\}. \quad (22)$$

Note 1. We say that a constraint admits a positive multiplier if there is  $\lambda > 0$  such that B is optimal to (AP) and when it is dualized in a Lagrangean fashion it gives a better dual bound.

Proposition 1. [2] If only a subset  $T' \subseteq T$  of constraints (12) are used to build the Lagrangean function  $L(\lambda)$ , then:

$$\begin{aligned} v(ATSP) &\geq v'(ATSP) \\ &= \sum_{i \in V} u_i + \sum_{j \in V} v_j + \sum_{t \in T'} \lambda_t a_0^t. \end{aligned} \quad (23)$$

The procedure proposed in [2] attempts to improve the dual bounds to the (ATSP) iteratively while keeping the

solutions  $\bar{x}$  and  $(u, v)$ , respectively, primal and dual optimal for (AP). The procedure identifies valid inequalities that:

$$\text{are violated by } \bar{x}, \text{ and} \quad (24)$$

$$\text{admit positive multipliers.} \quad (25)$$

Once valid inequalities that satisfy (24) and (25) are identified, they are included in the (AP) formulation and dualized in a Lagrangean fashion with the maximum possible value that satisfies (16) and (17). The addition of new constraints to the reformulated (AP) implies in addition of new variables to the dual problem (18)-(20), in the term  $\sum_{t \in T} \lambda_t a_0^t$  of (23), so improving the dual bound given by the partial Lagrangean function  $L(\lambda)$ . Given that violated constraints are identified (cut) and used to build a new Lagrangean relaxation to the problem it is a relax and cut procedure. Moreover, since the cuts are identified each time a new Lagrangean solution is found, the procedure proposed in [2] can be called a non-delayed relax and cut procedure, or simply RCP. In what follows, we will specify how to obtain valid inequalities that satisfy (24) and (25).

## 2.1. The relax and cut procedure for the formulation DFJ-ATSP

Balas and Christofides [2] develop the RCP procedure for three types of subtour elimination constraints (cut set, clique and articulation point). To identify violated inequalities that admit positive multipliers, they consider an auxiliary spanning graph  $G_0(V, A_0)$  in which there is an arc in  $A_0$  for each variable with zero reduced cost (i.e. an arc for each variable that satisfies (19) at equality).

In the case of the cut set constraints, if  $Q_t = (S_t, \bar{S}_t)$  is the cut set associated to  $S_t \subset V, \forall t \in T$ . Then the cut constraint (26) admits a positive multiplier if and only if condition (27) is satisfied.

$$\sum_{(i,j) \in Q_t} x_{ij} \geq 1. \quad (26)$$

$$Q_t \cap A_0 = \emptyset. \quad (27)$$

More details of how to identify violated cut set constraints can be found in [2] and [10].

## 2.2. The Relax and Cut procedure for the ATSP multi-commodity formulation

A strong formulation for the ATSP, with a polynomial number of constraints, is based on the multi-commodity network flow problem (e.g. [14]). In this formulation, the subtour elimination constraints are formulated in terms of a flow of  $(n - 1)$  commodities in a network. The reasoning is based on the assumption that there are  $(n - 1)$  commodities available at city 1 and a demand of one unit of commodity  $j, j \neq 1$  at city  $j$ . The formulation is an extended one in the sense that besides the binary assignment variables  $x_{ij}$ , a set of continuous variables

$y_{kij} \in R$  are defined to represent the flow of a commodity  $k$  through the arc  $(i, j)$ . The multi-commodity subtour elimination constraints are defined by (28)-(31).

$$\sum_{i \in V} y_{k1i} - \sum_{i \in V} y_{ki1} = 1, k \in V \setminus \{1\} \quad (28)$$

$$\sum_{i \in V} y_{kik} - \sum_{i \in V} y_{kki} = 1, k \in V \setminus \{1\} \quad (29)$$

$$\sum_{i \in V \setminus \{1\}} y_{kij} - \sum_{i \in V \setminus \{1\}} y_{kji} = 0, j, k \in V \setminus \{1\}, j \neq k \quad (30)$$

$$0 \leq y_{kij} \leq x_{ij}, i, j, k \in V, k \neq 1. \quad (31)$$

The constraints (28) guarantee that there is one unit of commodity  $k$  available at node 1 and this product cannot flow back to node 1. For each node  $k \in V \setminus \{1\}$ , constraint (29) imposes that the demand for product  $k$  in node  $k$  is met. Constraints (30) are the flow conservation constraints. Finally, constraints (31), impose that the flow of product  $k$  goes through arc  $(i, j)$  only if this arc is included in a path from node 1 to node  $k$ . The multi-commodity formulation for the ATSP, *MC-ATSP*, is given by (9)-(11), (13) and (28)-(31). In what follows, we detail the RCP procedure defined in Section 2 to obtain dual bounds for the ATSP considering the *MC-ATSP* formulation.

In an optimal solution to the Assignment Problem (*AP*) that includes subtours, the multi-commodity flow constraints (28)-(31) are violated for any node  $w \in V$  that is not in the same subtour that includes node 1, denoted hereafter as  $S_1$ . That is, condition (24) is satisfied for every  $k \in \bar{S}_1 = V \setminus S_1$ . Let us now derive the conditions to identify among the violated constraints the ones that admit positive multipliers.

In order to do that, let us build the dual problem associated to the linear relaxation of *MC-ATSP*. For each commodity  $k \in V \setminus \{1\}$ , define  $\alpha_k$ ,  $\beta_k$  and  $\gamma_j^k$  as the dual variable associated to (28), (29) and (30) respectively. For each node  $j \in V \setminus \{1, k\}$ ,  $\varphi_{kij}$  is the dual variable associated to (31). Let  $a_{kij}^\alpha$ ,  $a_{kij}^\beta$ ,  $a_{kij}^{\gamma_j^k}$  and  $a_{kij}^{\gamma_i^k}$  be the nonzero coefficients of the flow variable  $y_{kij}$  in constraints (28), (29), (30) and (31) respectively. A column of the constraints' matrix of the *MC-ATSP* associated to the  $y_{kij}$  variable is represented in Fig. 1, in which the coefficients are defined according to (32)-(35).

$$a_{kij}^\alpha = \begin{cases} 1, & \text{if } i = 1; \\ -1, & \text{if } j = 1; \\ 0, & \text{otherwise.} \end{cases} \quad (32)$$

$$a_{kij}^\beta = \begin{cases} -1, & \text{if } i = k; \\ 1, & \text{if } j = k; \\ 0, & \text{otherwise.} \end{cases} \quad (33)$$

$$a_{kij}^{\gamma_j^k} = \begin{cases} 1, & \text{if } j \neq 1, j \neq k; \\ 0, & \text{otherwise.} \end{cases} \quad (34)$$

Dual Variables			
	$y_{kij}$	$y_{425}$	$y_{413}$
$\alpha_k$	0	0	0
$\beta_k$	0	0	0
$\gamma_j^k$	0	0	0
	$\vdots$	$\vdots$	$\vdots$
$a_{kij}^\alpha$	0	0	1
$a_{kij}^\beta$	0	0	0
$a_{kij}^{\gamma_j^k}$	1	1	1
$a_{kij}^{\gamma_i^k}$	-1	-1	0
	$\vdots$	$\vdots$	$\vdots$
	0	0	0
	-1	-1	-1

$$a_{kij}^{\gamma_i^k} = \begin{cases} -1, & \text{if } i \neq 1, i \neq k; \\ 0, & \text{otherwise.} \end{cases} \quad (35)$$

Figure 1. The columns of the flow variables  $y_{kij}$ ,  $y_{425}$ ,  $y_{413}$ .

Source: The authors

The dual problem associated to the *MC-ATSP* formulation is defined by (36)-(39).

$$\max \sum_{i \in V} u_i + \sum_{j \in V} v_j + \sum_{k=2}^n (\alpha_k + \beta_k) \quad (36)$$

Subject to

$$u_i + v_j + \sum_{k=2}^n \varphi_{kij} \leq \hat{c}_{ij}, i, j \in V \quad (37)$$

$$a_{kij}^\alpha \alpha_k + a_{kij}^\beta \beta_k + a_{kij}^{\gamma_j^k} \gamma_j^k + a_{kij}^{\gamma_i^k} \gamma_i^k - \varphi_{kij} \leq 0, i, j, k \in V, k \neq 1 \quad (38)$$

$$\varphi_{kij} \geq 0, i, j, k \in V, k \neq 1. \quad (39)$$

The main idea of the RCP is to dualize only the multi-commodity constraints that have a positive multiplier and therefore guarantee an improvement in the quality of the Lagrangean dual bound. As we can see in the objective function (36), to obtain better bounds it is necessary to identify constraints such that condition (40) is met, in which  $T'$  is the set of dualized constraints.

$$\sum_{t \in T'} (\alpha_k + \beta_k) > 0. \quad (40)$$

The multiplier  $\varphi = (\varphi_{kij})$ , associated to (27), has sign constraints. Therefore to guarantee (39), let  $\varphi_{kij}$  be fixed to the current value of the reduced cost,  $\hat{c}_{ij}$ , associated to the primal variable  $x_{ij}$ , see expression (41).

$$\varphi_{kij} = c_{ij} - u_i - v_j - \sum_{k' \in T'} \varphi_{k'ij}. \quad (41)$$

To simplify the notation, fixing  $\varphi_{kij}$  to the value defined in

(41), constraints (37) and (38) can be replaced by (42) in the dual problem.

$$u_i + v_j + a_{kij}^\alpha \alpha_k + a_{kij}^\beta \beta_k + a_{kij}^\gamma \gamma_j^k + a_{kij}^{\gamma_i} \gamma_i^k \leq c_{ij}, \quad i, j \in V. \quad (42)$$

Now, it is necessary to derive feasible values to the dual variables  $\alpha_k, \beta_k$  and  $\gamma_j^k$ , for  $j \in V \setminus \{1, k\}$ . There are two cases. In the first case, we consider  $x_{ij} = 1$ . To simplify the exposition, suppose that no subtour elimination constraints have been dualized yet,  $T = \emptyset$ , and take a subtour  $S_k$  that contains node  $k$ . Let  $l_1 \in S_k = \{k, l_1, l_2, \dots, l_{|S_k|}\}$  such that  $x_{kl_1} = 1$ . Then the dual constraint associated to the variable  $x_{kl_1}$  (42) is:

$$u_k + v_{l_1} - \beta_k + \gamma_{l_1}^k = c_{kl_1}. \quad (43)$$

The associated slack variable is zero and so for  $(k, l_1)$ ,  $u_k + v_{kl_1} = c_{kl_1}$  and we obtain (44) and (45).

$$-\beta_k + \gamma_{l_1}^k = 0 \quad (44)$$

$$\beta_k = \gamma_{l_1}^k. \quad (45)$$

Consider now a second node  $l_2 \in S_k$  such that  $x_{l_1 l_2} = 1$ , similarly we have:

$$u_{l_1} + v_{l_2} + \gamma_{l_1}^k - \gamma_{l_2}^k = c_{l_1 l_2},$$

obtaining  $\gamma_{l_1}^k = \gamma_{l_2}^k$ . Continuing this reasoning we have:

$$\gamma_{l_i}^k = \beta_k, \forall l_i \in S_k. \quad (46)$$

Consider now a node in the same subtour as node 1,  $m_1 \in S_1 = \{1, m_1, m_2, \dots, m_{|S_1|}\}$  such that  $x_{1m_1} = 1$ . According to (42):

$$u_1 + v_{m_1} + \alpha_k + \gamma_{m_1}^k \leq c_{1m_1}. \quad (47)$$

As the associated slack, reduced cost, is zero ( $\hat{c}_{1m_1} = 0$ ) we get the equality  $\alpha_k = -\gamma_{m_1}^k$ , which is also valid for the other nodes  $m_i \in S_1$ , that is:

$$\alpha_k = -\gamma_{m_i}^k, \forall m_i \in S_1. \quad (48)$$

And so,  $\gamma_{m_i}^k = \gamma_{m_j}^k$  for  $m_i, m_j \in S_1, m_i, m_j \neq 1, m_i \neq m_j$ . In general, for a subtour  $S_q, S_q \neq S_1$  and  $S_q \neq S_k$ , and any distinct nodes  $q_i, q_j \in S_q$  we have (50).

$$\gamma_{q_i} = \gamma_{q_j}. \quad (49)$$

Consider now the case when  $x_{ij} = 0$ . Let  $S_q$  and  $S_r$  be two distinct subtours and the nodes  $q_i$  and  $r_j$ , both different from 1 and  $k$ . Consider also the arc  $(q_i, r_j)$  and the respective constraint (42):

$$u_{q_i} + v_{r_j} + \gamma_{r_j}^k - \gamma_{q_i}^k \leq c_{q_i r_j}. \quad (50)$$

As  $q_i$  and  $r_j$  do not belong to the same subtour, then  $x_{q_i r_j} = 0$ . Moreover, as  $\bar{x}$  is the optimal solution to (AP), it is possible to say that  $\hat{c}_{q_i r_j} \geq 0$ , that is,  $u_{q_i} + v_{r_j} \leq c_{q_i r_j}$ . As  $\hat{c}_{q_i r_j} = c_{q_i r_j} - u_{q_i} - v_{r_j}$ , to keep the dual feasibility and according to (51):

$$\gamma_{r_j}^k - \gamma_{q_i}^k \leq \hat{c}_{q_i r_j}. \quad (51)$$

Since (51) and picking one node in each subtour  $q_1 \in S_q$  and  $r_1 \in S_r$  we have:

$$\gamma_{r_1}^k - \gamma_{q_1}^k \leq \hat{c}_{q_i r_j}. \quad (52)$$

The inequality (53) is valid for any  $q \in S_q$  and  $r \in S_r$ . Then:

$$\gamma_{r_1}^k - \gamma_{q_1}^k \leq \min\{\hat{c}_{qr}, q \in S_q, r \in S_r\}. \quad (53)$$

Similarly, taking the arc  $(r_j, q_i)$  we have:

$$\gamma_{q_i}^k - \gamma_{r_j}^k \leq \hat{c}_{r_j q_i} \quad (54)$$

$$\gamma_{q_1}^k - \gamma_{r_1}^k \leq \hat{c}_{r_j q_i} \quad (55)$$

$$\gamma_{q_1}^k - \gamma_{r_1}^k \leq \min\{\hat{c}_{rq}, r \in S_r, q \in S_q\} \quad (56)$$

$$\gamma_{r_1}^k - \gamma_{q_1}^k \geq -\min\{\hat{c}_{rq}, r \in S_r, q \in S_q\} \quad (57)$$

We can restate (58) as,

$$\max\{-\hat{c}_{rq}, r \in S_r, q \in S_q\} \leq \gamma_{r_1}^k - \gamma_{q_1}^k. \quad (58)$$

From (54) and (59) we have:

$$\max\{-\hat{c}_{rq}, r \in S_r, q \in S_q\} \leq \gamma_{r_1}^k - \gamma_{q_1}^k \leq \min\{\hat{c}_{qr}, q \in S_q, r \in S_r\}. \quad (59)$$

To obtain (60), we supposed that  $S_r$  and  $S_q$  were different from both  $S_1$  and  $S_k$ . Using a similar argument and considering node  $m_1$  as representing the subtour  $S_1$  we get  $\gamma_{m_1}^k = -\alpha_k$ , and for  $l_1 \in S_k, \gamma_{l_1}^k = \beta_k$ . Then we can derive:

$$\max\{-\hat{c}_{ml}, m \in S_1, l \in S_k\} \leq \alpha_k + \beta_k \leq \min\{\hat{c}_{lm}, l \in S_k, m \in S_1\}. \quad (60)$$

which gives bounds to  $\alpha_k + \beta_k$  that keeps the dual feasibility of the optimal basis  $B$ . If  $\min\{\hat{c}_{lm}, l \in S_k, m \in S_1\} \leq 0$  then the constraints associated to  $k$  do not admit positive multipliers. Otherwise, the maximum possible values for  $(\alpha_k + \beta_k)$  can led to an improved dual bound.

To summarize the optimization problem associated to the definition of the best values for  $\alpha_k$  and  $\beta_k$ , let:

$$\gamma_k^k = \beta_k \quad (61)$$

$$\gamma_1^k = -\alpha_k. \quad (62)$$

From (47) and (62) we have that:

$$\gamma_k^k = \gamma_{l_i}^k, \forall l_i \in S_k. \quad (63)$$

Similarly, from (49) and (63) we get:

$$\gamma_1^k = \gamma_{m_i}^k, \forall m_i \in S_1. \quad (64)$$

The problem to identify violated multi-commodity inequalities with positive multipliers is given by (66)-(70).

$$\max \alpha_k + \beta_k \quad (65)$$

Subject to

$$\alpha_k = -\gamma_1^k \quad (66)$$

$$\beta_k = \gamma_k^k \quad (67)$$

$$h_k^{rq} \leq \gamma_r^k - \gamma_q^k \leq H_k^{rq}, S_r, S_q \in \mathcal{S}, S_r \neq S_q. \quad (68)$$

$$\alpha_k, \beta_k, \gamma_k \text{ free.} \quad (69)$$

In (69),  $h_k^{rq} := \max\{-\varphi_{kr'q'}, (r', q') \in (S_r, S_q)\}$ ;

$H_k^{rq} := \min\{\varphi_{kq'r'}, (q', r') \in (S_q, S_r)\}$ ;  $\mathcal{S}$  is a collection of subtours associated to  $\bar{x}$ ;  $r$  and  $q$  are nodes in  $S_r$  and  $S_q$ , respectively, with  $S_r \neq S_q$ .

If the optimal solution  $(\alpha_k^*, \beta_k^*, \gamma_k^*)$  of (66)-(70) is greater than zero, the set of constraints (28)-(31) associated to  $k$  is violated by  $\bar{x}$  and dualized with positive multipliers. The reduced costs can be updated according to (71).

$$\hat{c}_{rq} = \hat{c}_{rq} + \gamma_{q_1}^k - \gamma_{r_1}^k, S_r, S_q \in \mathcal{S}, S_r \neq S_q. \quad (70)$$

The MC-RCP procedure consists in iteratively evaluating all the subtours through the nodes  $k' \in V \setminus \{1, k\}$ . According to (36), at the end of the procedure an improved dual bound ( $LB$ ) for the ATSP is given by (72).

$$LB = v(AP) + \sum_{k \in T'} \alpha_k + \beta_k. \quad (71)$$

The pseudocode of the MC-RCP procedure is shown in Fig. 2.

### 3. Computational Study

In this section, we present results of the computational implementation of the procedure relax and cut considering the cut set constraints (Procedure CS-RCP described in

Section 2.1) and the multi-commodity constraints (Procedure MC-RCP described in Section 2.2). The multi-commodity

```

1 Input: ATSP instance  $(G(V, A), |V| = n, |A| = m, c_{ij}, \forall (i, j) \in A)$ 
2 Output: Lower Bound, multi-commodity violated constraints.
3 Begin
4   Solve the AP associated to the ATSP
5    $LB = v(AP)$ 
6   Identify the set of subtours associated to  $\bar{x}$ ,  $\mathcal{S}$ .
7   If  $|\mathcal{S}| = 1$ , stop.  $\bar{x}$  is an optimal solution.
8    $S_1$  is the subtour containing vertex 1.
9    $T' = \emptyset$ 
10   $N = \emptyset$ 
11   $W = V \setminus S_1$ 
12  While  $W \neq \emptyset$ 
13    Begin
14      Select  $k \in W = V \setminus (S_1 \cup T' \cup N)$ 
15      If  $\min\{\hat{c}_{lm}, l \in S_k, m \in S_1\} = 0$ , then  $N = N \cup \{k\}$ 
16    Else
17      Begin
18        Solve problem (66)-(70)
19        If  $(\alpha_k^*, \beta_k^*, \gamma_k^*) \leq 0$  then  $N = N \cup \{k\}$ 
20      Else
21        Begin
22          For  $(r, q) \in A$  do
23             $\hat{c}_{rq} = \hat{c}_{rq} + \gamma_{q_1}^k - \gamma_{r_1}^k, S_r, S_q \in \mathcal{S}, S_r \neq S_q$ .
24             $LB = LB + \alpha_k + \beta_k$ 
25             $T' = T' \cup \{k\}$ 
26             $N = \emptyset$ 
27             $W = W \setminus \{k\}$ 
28          End
29        End
30       $\bar{x}$  is a solution to the AP,  $LB$  is a lower bound for the ATSP
      and the constraints in  $T'$  were dualized to find  $LB$ 
31    End

```

Figure 2. Pseudocode of MC-RCP procedure.

Source: The authors

formulation for the ATSP, model MC-ATSP, was written in the syntax of the AMPL modeling language. The CS-RCP and the MC-RCP algorithms were coded in the C++ programming language, using the CPLEX 12.5 libraries and run on a machine with Intel Core i5 2.67 GHz with 3.80 GB of RAM, operating system Windows 7 Ultimate. A maximum of 30 minutes (1800 seconds) of CPU time was allowed in each run. Thirteen instances of the TSPLIB library [15] were used in the tests (br17, ftv33, ftv35, ftv38, p43, ftv44, ftv47, ft53, ftv55, ftv64, ft70, ftv70, ftv170) ranging from 17 to 171 nodes. The instances size and the corresponding optimal solution values are given in [15].

As described in section 2, the relax and cut algorithms start from a solution of the Assignment Problem (AP) and search for violated valid inequalities that admits positive multipliers. For the cut set subtour elimination constraints (procedure CS-RCP), identifying valid inequalities is related to the existence of a reachable set  $R(i) \neq V$  of a node  $i$  in the admissible graph  $G_0$  [10]. As for the multi-commodity subtour elimination constraints (procedure MC-RCP) the search is undertaken through all the subtours that do not include node 1 (see Fig. 2).

At first the instances of the AP and the MC-ATSP models, as well as the linear relaxation of the model MC-ATSP (RMC-ATSP) were solved by the solver CPLEX using the default parameters, except for the instances of the relaxation RMC-

*ATSP* that were solved by the barrier method, as suggested in [14]. It was not possible to find feasible solution for 6 instances (p43, ftv44, ftv55, ftv70, ftv70 and ftv170) of the model *MC-ATSP* in 30 minutes (the allowed execution time). Also, the solver runs out of memory when solving instance ftv170 of the *MC-ATSP* model. The linear relaxation of the multi-commodity formulation is indeed very strong, it provided an average gap of 1.03%. The gap associated to the relaxation *AP* of the instances br17 and p43 is very high, 100% and 97% respectively, which influences the results of the algorithms relax and cut as will be discussed next.

To compare two bounds  $\kappa$  and  $\rho$  we compute their relative value ( $vr(\kappa, \rho)$ ) as in (73).

$$vr(\kappa, \rho) = 100 \frac{(\kappa - \rho)}{\kappa}. \quad (72)$$

Table 1 presents comparisons between the *RMC-ATSP* relaxation (RMC) and the *MC-RCP* procedure (MC), presenting the obtained dual bounds, computational time, in seconds, and the relative value of the *RMC* and *MC* dual

bounds, for each instance.

For most instances (9 out of 13) the MC-RCP procedure provided dual bounds with relative value of less than 10% of the bound given by the linear relaxation of the MC-ATSP model. For all but one instance, the average CPU time taken to solve the linear relaxation RMC-ATSP was 258.71 seconds while the average time to run the MC-RCP procedure was 4.96 seconds. The linear relaxation of the ftv170 instance of model MC-ATSP could not be solved in the allowed execution time for the solver (1800 seconds). Taken the optimal value for the instance ftv170 given in the TSPLIB, the MC-RCP provided a dual bound with a gap of 4.39% in 16.53 seconds.

We also compared the bounds given by the two relax and cut procedures presented in this work. Table 2 shows, for each instance, the dual bounds associated to the procedures CS-RCP and MC-RCP, CS and MC, respectively, and the corresponding CPU time (in seconds). For each bound (CS and MC), we also show the number of violated inequalities that are dualized in each procedure (cut), the integer gap and their relative value  $vr(CS, MC)$ .

Table 1.  
Linear Relaxation and Relax and Cut results.

Instance	Dual Bounds		Time (sec.)		$vr(RMC, MC)$	
	RMC	MC	RMC	MC	(%)	
br17	39.00	14.00	0.39	0.56		64.10
ftv33	1286.00	1185.00	9.67	1.38		7.85
ftv35	1457.33	1384.00	16.99	1.76		5.03
ftv38	1514.33	1441.00	25.40	1.92		4.84
p43	5611.00	501.00	55.16	12.26		91.07
ftv44	1584.87	1530.00	49.17	3.67		3.46
ftv47	1748.61	1708.00	111.49	8.77		2.32
ft53	6905.00	5979.00	183.91	2.10		13.41
ftv55	1584.00	1459.00	267.93	4.82		7.89
ftv64	1807.50	1756.00	680.41	6.77		2.85
ftv70	38652.50	38194.00	728.10	6.43		1.19
ftv70	1909.00	1794.00	975.85	9.14		6.02
ftv170	*	2634.00	*	16.53		*

\*No value found within the allowed execution time (1800 sec.)

Source: The authors

Table 2.  
Procedures CS-RCP and MC-RCP - computational results.

Instance	Dual Bounds		Time(sec.)		Cuts		gap(%)		$vr(CS, MC)$
	CS	MC	CS	MC	CS	MC	CS	MC	
br17	37.00	14.00	0.96	0.56	11.00	3.00	5.13	64.10	62.16
ftv33	1204.00	1185.00	1.10	1.38	2.00	0.00	6.38	7.85	1.58
ftv35	1398.00	1384.00	1.48	1.76	3.00	1.00	5.09	6.04	1.00
ftv38	1465.00	1441.00	1.76	1.92	3.00	1.00	4.25	5.82	1.64
p43	5583.00	501.00	11.16	12.68	24.00	8.00	0.68	91.09	91.02
ftv44	1538.00	1530.00	2.50	3.67	4.00	2.00	4.65	5.15	0.52
ftv47	1664.00	1708.00	2.31	8.77	3.00	7.00	6.31	3.83	-2.64
ft53	6693.00	5979.00	7.81	2.10	10.00	1.00	3.07	13.41	10.67
ftv55	1451.00	1459.00	2.98	4.82	3.00	4.00	9.76	9.27	-0.55
ftv64	1735.00	1756.00	4.18	6.77	3.00	5.00	5.66	4.51	-1.21
ftv70	38311.00	38194.00	8.36	6.43	5.00	1.00	0.94	1.24	0.31
ftv70	1773.00	1794.00	3.77	9.14	2.00	4.00	9.08	8.00	-1.18
ftv170	2634.00	2634.00	16.50	16.53	1.00	1.00	4.39	4.39	0.00

Source: the authors

The CS-RCP and MC-RCP procedures gave similar results. In 10 out of the 13 instances, the relative value of the

associated bounds was no greater than 3%. However, the particularities of some instances resulted in big differences in



the results of the two procedures. For the instance, br17 v(AP) = 0, and the bound given by the MC-RCP procedure reduced the AP gap from 100% to 64.10%, whereas the CS-RCP procedure reduced it to 5.13%. The number of valid inequalities that can be identified in the CS-RCP procedure is higher than for the MC-RCP procedure. However, the proportion of cuts generated in relation to possible total is very close in both algorithms. The CS-RCP procedure identified 11 cuts out of 20, a ratio of 0.55, while the MC-RCP identified 3 out of 5, with a ratio of 0.6. It is noteworthy that for the instance ftv170 the dual bound and the number of cuts was the same in both procedures.

*The work of Rocha, Fernandes and Soares [16] features an application of the Volume Algorithm to solve the dual Lagrangean problem associated to the formulation MC-ATSP. They test the procedure on several TSPLIB instances, which includes the ones used in the present work. The number of iterations vary from instance to instance hanging from 1000 to 10000, depending on the size of the instance. They do not report dual bounds for the ftv170 instance and therefore this instance is not included in the comparison. The average relative value of the best bounds obtained with the Volume Algorithm ( $\kappa$ ) in comparison with the MC-RCP procedure ( $\rho$ ) is 17%, and the standard deviation is 28.75%. Considering the number of iterations required to the MC-RCP, the relax and cut procedure proposed in this work provides good dual bounds with a reduced computational effort. The bounds obtained with the MC-RCP and the associated relaxed solution could be used as a starting point to the Volume Algorithm. This might improve the performance of the algorithm in terms of reducing the total number of iterations and execution time.*

#### 4. Concluding Remarks

In this work, we studied two procedures to obtain dual bounds to the Asymmetric traveling salesman problem. The procedures are based on the relax and cut method that starting from the optimal solution to the assignment problem, identifies violated inequalities and dualizes them to build a Lagrangean function. Two classes of valid inequalities are used. The CS-RCP procedure is based on cut set subtour elimination constraints, and the MC-RCP procedure is based on the multi-commodity subtour elimination constraints.

The procedures were tested on a set of instances from the TSPLIB. The computational results obtained with both procedures are encouraging. The quality of the bounds given by the two algorithms is similar. The CPU time required to compute the dual bounds with both the CS-RCP and MC-RCP are small when compared to the time necessary to obtain dual bounds solving the linear relaxation of the MC-ATSP formulation.

The formulation CS-ATSP and MC-ATSP are equivalent and give the same linear relaxation values. Still, a combination of the two types of subtour elimination constraints can be useful in a relax and cut procedure. They could be used sequentially, since the valid inequalities identified are distinct. The matrix of reduced costs resulting from the CS-RCP can be used as starting point for the MC-

RCP. The implementation of the CS-RCP was important for two reasons: it served as a benchmark for the MC-RCP and updated the work of Balas and Christofides [2] since it was tested with instances of the TSPLIB while in the original work it was tested only with random data.

The dual bounds obtained with the relax and cut procedures presented here can be useful to speed up the solution of large instances of the ATSP by the implicit enumeration methods present in commercial and noncommercial solvers.

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#### References

- [1] Álvarez, P.J., Calderón, C.A.G. and Calderón, G.G., Route optimization of urban public transportation. DYNA, 88 (180), pp. 41-49, 2013.
- [2] Balas, E., Christofides, N., A restricted Lagrangean approach to the traveling salesman problem, Mathematical Programming, 21, pp. 19-46, 1981. DOI: 10.1007/BF01584228
- [3] Dantzig G., Fulkerson R. and Johnson S., Solution of a large-scale traveling-salesman problem, Operations Research 2, pp. 393-410, 1954. DOI: 10.1287/opre.2.4.393
- [4] De Souza, C.C. and Cavalcante, V.F., Exact algorithms for the Vertex separator problem in graphs, Networks 57 (3), pp. 212-230, 2011. DOI: 10.1002/net.20420
- [5] Escudero, L.F., Guinard, M. and Malik, K., A Lagrangian relax-and-cut approach for the sequential ordering problem with precedence relationships, Annals of Operations Research 50, pp. 219-237, 1994. DOI: 10.1007/BF02085641
- [6] Fischetti, M., Salvagnin, D., A relax-and-cut framework for gomory mixed-integer cuts, Mathematical Programming Computation 3 (2), pp. 79-102, 2011. DOI: 10.1007/s12532-011-0024-x
- [7] Gavish, B., Augmented Lagrangean based algorithms for centralized network design, IEEE Transactions on Communications 33, pp. 1247-1257, 1985. DOI: 10.1109/TCOM.1985.1096250
- [8] Guinard, M., Lagrangean relaxation, Top 11 (2), pp. 151-228, 2003. DOI: 10.1109/TCOM.1985.1096250
- [9] Kawashima, M.S., Relax and cut: Limitantes duais para o problema do caixeiro viajante, MSc. Thesis, UNESP - Universidade Estadual Paulista, São José do Rio Preto, Brasil, 2014.
- [10] Kawashima, M.S., Rangel, S., Litvinchev, I. and Infante, L., Relax and cut: Dual bounds for the traveling salesman problem. Memorias del CILOG 2014. San Luis Potosí, México: Mexican Logistics & Supply Chain Association, 2014.
- [11] Lucena, A., Steiner problem in graphs: Lagrangean relaxation and cutting planes, COAL Bulletin 21, pp. 2-8, 1993.
- [12] Lucena, A., Non-Delayed relax and cut algorithms, Annals of Operations Research 140, pp. 375-410, 2005. DOI: 10.1007/s10479-005-3977-1
- [13] Maldonado, M., Rangel, S., Ferreira, D., A Study of different subsequence elimination strategies for the soft drink production planning, Journal of Applied Research and Technology 12, 631-641, 2014. DOI: 10.1016/S1665-6423(14)70080-X
- [14] Öncan, T., Altinel, I.K., Laporte G., A comparative analysis of several asymmetric traveling salesman problem formulations, Computers & Operations Research, 36, pp. 637-654, 2009. DOI: 10.1016/j.cor.2007.11.008

- [15] Reinelt, G., TSPLIB - A traveling salesman problem library, ORSA Journal Computing, [on line] 3, pp. 376-384, 1991. Available at: <https://www.iwr.uniheidelberg.de/groups/comopt/software/TSPLIB95/>, last visited in 11/04/2014
- [16] Rocha, A.N., Fernandes, E.M.G.P. and Soares, J., Aplicação do algoritmo volumétrico à resolução aproximada e exacta do problema do caixeiro viajante assimétrico, Investigaçao Operacional 25 (2), pp. 277-294, 2005.
- [17] Serna, M.D.A., Jaimes, W.A. and Cortes, J.A.Z., Commodities distribution using alternative types of transport. A study in the Colombian bread SME's. DYNA, 77 (163), pp. 222-233, 2010.
- [18] Sherali, H.D. and Smith, J.C., Dynamic Lagrangean dual and reduced RLT constructs for solving 0-1 mixed-integer programs, Top 20 (1), pp. 173-189, 2012. DOI: 10.1007/s11750-011-0199-3

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