

New unconditionally stable scheme for solving two-dimensional acoustic wave problems

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1. Introduction

In this paper, we consider the following two-dimensional acoustic wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = v^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + f(x, y, t) & (x, y) \in \Omega \times [0, T] \\ u(x, y, 0) = 0, \quad \frac{\partial u(x, y, 0)}{\partial t} = 0. \end{cases} \quad (1)$$

where u is the wave displacement, v is the sound velocity, and f is an external source. A Mur's first-order absorbing boundary condition (ABC) is considered.

$$\left[\frac{\partial u}{\partial x} - \frac{1}{c} \frac{\partial u}{\partial t} \right]_{x \in \partial \Omega} = \left[\frac{\partial u}{\partial y} - \frac{1}{c} \frac{\partial u}{\partial t} \right]_{y \in \partial \Omega} = 0 \quad (2)$$

The conventional finite difference scheme (FDS) works efficiently for solving the acoustic wave problems [1–3]. However, the FDS is an explicit time-marching algorithm, which means its time step should be limited by the Courant-Friedrichs-Levy (CFL) condition. Since time step size strongly depends on the smallest cell in a computational area, the FDS costs much time to solve acoustic wave problems with fine structures. Hence, to overcome the conflict in the calculative stability of the FDS, some unconditionally stable schemes [4–6] containing the alternating direction implicit (ADI) difference scheme have been proposed. Huang *et al.* [7] and Fu *et al.* [8] proposed a new orthogonal decomposition scheme using the associated Hermite polynomials to eliminate the CFL stability condition (AH-FDS). However, the AH-FDS has the drawback of a need for large internal storage and heavy computation.

In this work, we proposed a new unconditionally stable scheme for the acoustic wave equation using the weighted Laguerre polynomials and finite difference scheme (WLP-FDS). First, the time derivatives in the wave equation are expanded by the Laguerre polynomials and weighting functions. Since these orthogonal polynomials converge to zero with time, the sound field expanded by the weighted Laguerre polynomials converges to zero simultaneously. Then, by applying Galerkin's method and using the orthogonal property of weighted Laguerre basis functions, we can eliminate the time variables and thus obtain an uncondition-

ally stable scheme from the computations. Finally, we can solve the implicit equation recursively and reconstruct numerical results using the expansion coefficients.

2. Formulations and WLP-FDS

Consider the Laguerre polynomials defined by

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}), \quad n \geq 0; \quad t \geq 0. \quad (3)$$

The Laguerre polynomials have a recursive and simple relationship given by

$$\begin{aligned} L_0(t) &= 1 \\ L_1(t) &= 1 - t \\ nL_n(t) &= (2n - 1 - t)L_{n-1}(t) - (n - 1)L_{n-2}(t) \\ n &\geq 2; \quad t \geq 0. \end{aligned} \quad (4)$$

These polynomials are orthogonal with respect to the weighting function e^{-t} .

$$\int_0^\infty L_n(t) L_p(t) e^{-t} dt = \begin{cases} 1, & n = p \\ 0, & n \neq p \end{cases} \quad (5)$$

Then, a set of orthogonal functions $\{\phi_0, \phi_1, \phi_2, \dots\}$ can be obtained by

$$\phi_n(s \cdot t) = e^{-s \cdot t/2} L_n(s \cdot t), \quad (6)$$

where $s > 0$ is a time scale factor. Note that the functions are convergent to zero absolutely as $t \rightarrow \infty$. Then, the arbitrary functions spanned by the above functions are convergent to zero absolutely as $t \rightarrow \infty$. The above basis functions are orthogonal with respect to the scaled time \bar{t} as

$$\int_0^\infty \phi_n(\bar{t}) \cdot \phi_p(\bar{t}) d\bar{t} = \begin{cases} 1, & n = p \\ 0, & n \neq p, \end{cases} \quad (7)$$

where $\bar{t} = st$ is a scaled time variable. We introduce an appropriate scale factor to use the basis functions properly. The partial differential with respect to x and y can be obtained by using the above basis functions.

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^\infty \frac{\partial}{\partial x^2} u_n(x, y) \phi_n(\bar{t}), \quad \frac{\partial^2 u}{\partial y^2} = \sum_{n=0}^\infty \frac{\partial}{\partial y^2} u_n(x, y) \phi_n(\bar{t}) \quad (8)$$

From [9], we obtain the first derivative of $u(x, y, t)$ versus time t as

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$$\frac{\partial u(x, y, t)}{\partial t} = s \sum_{n=0}^{\infty} \left[0.5u_n(x, y) + \sum_{k=0, n>0}^{n-1} u_k(x, y) \right] \phi_n(\bar{t}). \quad (9)$$

From Eq. (9), we can conclude that the second derivative of $u(x, y, t)$ with respect to time t is

$$\frac{\partial^2 u}{\partial t^2} = s^2 \sum_{n=0}^{\infty} \left[0.25u_n(x, y) + \sum_{k=0, n>0}^{n-1} (n-k)u_k(x, y) \right] \phi_n(\bar{t}) \quad (10)$$

Inserting Eqs. (8) and (10) into the acoustic wave equation, we can obtain

$$\begin{aligned} & s^2 \sum_{n=0}^{\infty} \left[0.25u_n(x, y) + \sum_{k=0, n>0}^{n-1} (n-k)u_k(x, y) \right] \phi_n(\bar{t}) \\ &= v^2 \sum_{n=0}^{\infty} \frac{\partial}{\partial x^2} u_n(x, y) \phi_n(\bar{t}) + v^2 \sum_{n=0}^{\infty} \frac{\partial}{\partial y^2} u_n(x, y) \phi_n(\bar{t}) \quad (11) \\ &+ \sum_{n=0}^{\infty} f_n(x, y) \phi_n(\bar{t}). \end{aligned}$$

To eliminate the time terms $\phi_n(\bar{t})$, Galerkin's method is applied to the above equation using the orthogonal property of weighted Laguerre basis functions. We multiply both sides of Eq. (11) by $\phi_p(\bar{t})$, and then, by integrating over time from 0 to ∞ , we can obtain the following equation using Eq. (7):

$$\begin{aligned} & s^2 \left[0.25u_p(x, y) + \sum_{k=0, p>0}^{p-1} (p-k)u_k(x, y) \right] \\ &= v^2 \frac{\partial^2 u_p(x, y)}{\partial x^2} + v^2 \frac{\partial^2 u_p(x, y)}{\partial y^2} + f_p(x, y) \quad (12) \end{aligned}$$

where

$$f_p(x, y) = \int_0^{T_f} f(x, y, t) \phi_p(\bar{t}) d\bar{t}. \quad (13)$$

In Eq. (13), T_f is a finite time interval. Rewriting Eq. (12) using the finite difference scheme in space, we obtain

$$\begin{aligned} u_p(x, y)|_{i,j} &= \frac{4}{s^2} f_p(x, y)|_{i,j} - 4 \sum_{k=0, p>0}^{p-1} (p-k)u_k(x, y)|_{i,j} \\ &+ \frac{4v^2}{s^2 \cdot \Delta y_j^2} [u_p(x, y)|_{i,j+1} - 2u_p(x, y)|_{i,j} + u_p(x, y)|_{i,j-1}] \quad (14) \\ &+ \frac{4v^2}{s^2 \cdot \Delta x_i^2} [u_p(x, y)|_{i+1,j} - 2u_p(x, y)|_{i,j} + u_p(x, y)|_{i-1,j}], \end{aligned}$$

where Δx_i and Δy_j denote the spatial size in the x - and y -directions, respectively. From Eq. (14), we can see that each order of a field variable is related to the adjacent four field variables. Rewriting Eq. (14), we have

$$\begin{aligned} & -\frac{4v^2}{s^2 \cdot \Delta x_i^2} u_p(x, y)|_{i-1,j} - \frac{4v^2}{s^2 \cdot \Delta y_j^2} u_p(x, y)|_{i,j-1} \\ &+ \left(1 + \frac{8v^2}{s^2 \cdot \Delta x_i^2} + \frac{8v^2}{s^2 \cdot \Delta y_j^2} \right) u_p(x, y)|_{i,j} \quad (15) \\ &- \frac{4v^2}{s^2 \cdot \Delta y_j^2} u_p(x, y)|_{i,j+1} - \frac{4v^2}{s^2 \cdot \Delta x_i^2} u_p(x, y)|_{i+1,j} \end{aligned}$$

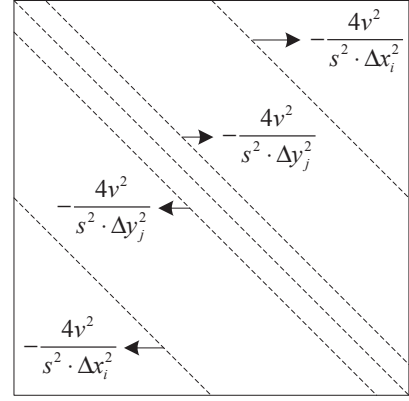


Fig. 1 Five-diagonal sparse matrix $[A]$.

$$= \frac{4}{s^2} f_p(x, y)|_{i,j} - 4 \sum_{k=0, p>0}^{p-1} (p-k)u_k(x, y)|_{i,j}.$$

Rewriting Eq. (15) in a matrix form, we can obtain

$$[A]\{u^p\} = \{f^p\} - \{(p-k)\beta^{p-1}\}. \quad (16)$$

In Eq. (16), $[A]$ is a five-diagonal matrix and its shape is shown in Fig. 1. In $[A]$, the values in central line are the coefficients of $u_p(x, y)|_{i,j}$ in Eq. (15). $\{f^p\}$ is the term of the external source due to Eq. (13). $\{\beta^{p-1}\}$ is the accumulation term from order zero to order $p-1$.

For the boundary conditions, inserting Eq. (9) into Eq. (2), the time derivative can be eliminated using Eq. (7). At $x = X$, we can obtain

$$\frac{\partial}{\partial x} u_p(x, y) - \frac{s}{c} \left[\frac{u_p(x, y)}{2} + \sum_{k=0}^{p-1} u_k(x, y) \right] = 0. \quad (17)$$

Using the central difference scheme and the averaging technique, we can obtain

$$\begin{aligned} u_p(x, y)|_{N_x-1/2,j} &= \frac{u_p(x, y)|_{N_x,j} + u_p(x, y)|_{N_x-1,j}}{2} \\ \frac{\partial}{\partial x} u_p(x, y)|_{N_x-1/2,j} &= \frac{u_p(x, y)|_{N_x,j} - u_p(x, y)|_{N_x-1,j}}{\Delta x}. \end{aligned} \quad (18)$$

Combining Eqs. (17) and (18), we have

$$\begin{aligned} & \left(-\frac{s}{4c} + \frac{1}{\Delta x} \right) u_p(x, y)|_{N_x,j} - \left(\frac{s}{4c} + \frac{1}{\Delta x} \right) u_p(x, y)|_{N_x-1,j} \\ &= \frac{s}{2c} \sum_{k=0}^{p-1} [u_p(x, y)|_{N_x,j} + u_p(x, y)|_{N_x-1,j}]. \end{aligned} \quad (19)$$

Similarly, we can obtain the ABC difference equation at $x = 0$, $y = 0$, and $y = Y$ as follows:

$$\begin{aligned} x = 0: & \left(-\frac{s}{4c} + \frac{1}{\Delta x} \right) u_p(x, y)|_{1,j} - \left(\frac{s}{4c} + \frac{1}{\Delta x} \right) u_p(x, y)|_{2,j} \\ &= \frac{s}{2c} \sum_{k=0}^{p-1} [u_p(x, y)|_{1,j} + u_p(x, y)|_{2,j}] \quad (20) \end{aligned}$$

$$y = 0: \left(-\frac{s}{4c} + \frac{1}{\Delta y} \right) u_p(x, y)|_{i,1} - \left(\frac{s}{4c} + \frac{1}{\Delta y} \right) u_p(x, y)|_{i,2}$$

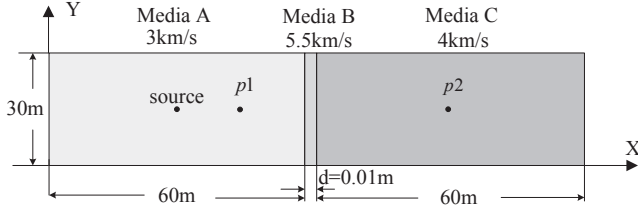


Fig. 2 Computational domain of a 2-D acoustic wave problem.

$$= \frac{s}{2c} \sum_{k=0}^{p-1} [u_p(x, y)|_{i,1} + u_p(x, y)|_{i,2}] \quad (21)$$

$$\begin{aligned} y = Y: & \left(-\frac{s}{4c} + \frac{1}{\Delta y} \right) u_p(x, y)|_{i, N_y} - \left(\frac{s}{4c} + \frac{1}{\Delta y} \right) u_p(x, y)|_{i, N_y-1} \\ & = \frac{s}{2c} \sum_{k=0}^{p-1} [u_p(x, y)|_{i, N_y} + u_p(x, y)|_{i, N_y-1}]. \end{aligned} \quad (22)$$

Inserting Eqs. (19)–(22) into Eq. (16), a modified matrix could be formed:

$$[\tilde{A}]\{u^p\} = \{\tilde{f}^p\} - \{(p-k)\tilde{\beta}^{p-1}\}. \quad (23)$$

Compared with the conventional FDS, this presented scheme has an implicit relationship between the sound fields, which results in the sparse matrix $[\tilde{A}]$. Note that $[\tilde{A}]$ is independent of the order p , and the time step no longer affects the stability of this scheme. In the WLP-FDS, the time step size is only applied to the calculation of Laguerre coefficients using Eq. (13), which has been carried out at the beginning of computations.

3. Numerical example

In this part, a 2-D case is calculated to compare the numerical results obtained using the FDS, AH-FDS and WLP-FDS. A 2-D parallel plate constituted of two isotropic media separated by a fine structure ($d = 0.01$ m, $v = 5.5$ km/s) is considered, as shown in Fig. 2. The sound velocities in three media are 3 km/s, 5.5 km/s and 4 km/s. The source is located at 30 m in the x -direction and is expressed as Eq. (24). Displacement $u(x, y, t)$ is observed at two points $p1(45, 15)$ and $p2(90, 15)$.

$$f(t) = 100 \exp[-(2t - 3)^2] \cdot \sin[2\pi(2t - 1)] \quad (24)$$

In the three schemes, the time support for simulation is set as 1 s. In the conventional FDS, the time step size is set as 1.6×10^{-7} s to satisfy the CFL condition, whereas in the unconditionally stable AH-FDS and WLP-FDS, $\Delta t = 0.001$ s. This value is small enough to calculate Eq. (13). We set the

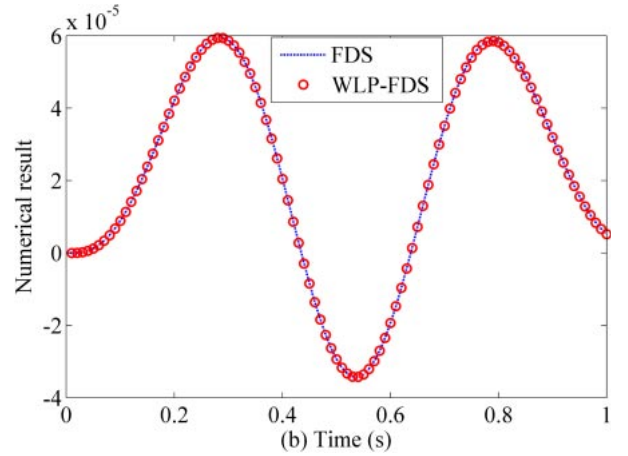
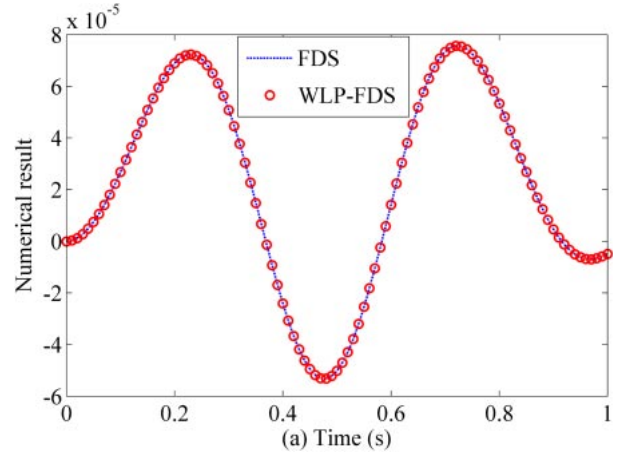


Fig. 3 Numerical results at (a) p_1 and (b) p_2 .

number of iterations of the AH-FDS and WLP-FDS as 60. Figure 3 shows the numerical results of sound field at p_1 and p_2 , respectively.

From Fig. 3, we can observe that the agreement between the proposed WLP-FDS and conventional FDS is quite good. To validate the accuracy of the WLP-FDS quantitatively, the numerical results obtain by the AH-FDS and WLP-FDS are compared with those results of the FDS. The L_2 error, L_∞ error, and root-mean-square (RMS) error at p_2 are tabulated in Table 1. Table 1 also shows the computing information and calculation resources of the three schemes.

From Table 1, we can conclude that the CPU time for the WLP-FDS is much shorter than that for conventional FDS, and can be reduced to about 3.45% that of the AH-FDS. However, the accuracy of the WLP-FDS is almost the same as that of the AH-FDS. On the other hand, the total memory storage for the WLP-FDS is about 12.1% compared with that of the AH-FDS. This is because $[A]$ in Eq. (16) is a single

Table 1 Computational information of the three methods.

Method	No. of iterations	Memory	CPU time	RMS error	L_2 error	L_∞ error
FDS	6,250,000	0.44 Mb	3,287.5 s	0	0	0
AH-FDS	60	34.8 Mb	4.992 s	5.7632(−08)	5.8195(−07)	1.1154(−07)
WLP-FDS	60	4.2 Mb	0.1721 s	5.2834(−08)	5.4590(−07)	6.6841(−08)

five-diagonal matrix; However, matrix $[A]$ in the AH-FDS is a five-diagonal banded nested matrix. At each iteration, the AH-FDS requires more time and computer memory to obtain the final numerical results.

4. Conclusion

In this work, a new unconditionally stable scheme has been presented for solving the 2-D acoustic wave propagation problems. Compared with the conventional FDS, this presented WLP-FDS has high efficiency in solving an acoustic wave problem with fine structures while maintaining a very high accuracy. Moreover, this unconditionally stable scheme has advantages in its shorter calculation time and less required computer memory compared with AH-FDS.

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