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PAPER

Theoretical analysis of two-dimensional vibration of single piano string using equivalent mechanical circuit models

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Abstract: When a piano string is struck by a hammer, it begins to vibrate vertically, i.e., in a plane perpendicular to the soundboard. After the vertical vibration has begun, the string then begins to rotate owing to horizontal vibration. The rotation direction changes several seconds later, which suggests that the frequencies of the vertical and horizontal components of the vibration are slightly different. In this article, we describe the modeling and theoretical analysis of the two-dimensional motion of a piano string. To this end, the string and soundboard are represented by an equivalent mechanical circuit. The string with two-dimensional movement is divided into two independent strings, each with one-dimensional movement. The vertical and horizontal motions are initialized to have the same frequency and are connected by a bridge that is represented as an ideal transformer. A soundboard is attached to the vertically vibrating string. Once the circuit is excited, the two strings vibrate at slightly different frequencies.

Keywords: Piano, String, Soundboard, Equivalent mechanical circuit, Two-dimensional vibration

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1. INTRODUCTION

In this article, we describe the interaction between the vibration of a piano string and the soundboard. Weinreich used electrodes to observe the low-frequency motion of a single piano string and showed that the horizontal motion of the string has a slow decay rate while the vertical motion has a fast decay rate [1]. In order to explain this finding, Iwaoka and Nakamura presented a physical model that describes how the horizontal vibration of the string is generated by its vertical vibration [2]. Their model assumes that the transversal force of the strings is transmitted through the bridge pin that is mounted on the soundboard. They investigated two-dimensional motions of the string by numerical simulation. However, they did not pay attention to the frequency of the motion. Later, Tanaka *et al.* used an optical method to observe the two-dimensional motion of a single string, and they found that the vertical and horizontal vibrations have slightly different frequencies [3]. However, they did not determine the reason why the decay rates and frequencies are different.

Incidentally, in an earlier paper, one of the present authors introduced a method of using an equivalent circuit to provide theoretical solutions for the generation of two-dimensional motion in a string [4]. In the present article, we use an equivalent circuit analysis to theoretically examine the generation of two-dimensional motion in a single piano string and thus support the experimental results obtained by Tanaka *et al.*, mainly by showing why the string vibrations of the vertical and horizontal directions provide different frequencies.

This article is organized as follows. First, we present a physical model of the vibration mechanism of piano strings and a soundboard using an equivalent circuit approach. Second, the obtained circuit is then simplified while retaining its essential characteristics. Third, we use Laplace transforms to analytically solve several differential equations of the simplified model. We then use the analytical solutions to clarify the mechanism of two-dimensional vibration, and end with some concluding remarks.

2. PHYSICAL MODEL OF A STRING AND SOUNDBOARD

2.1. Equivalent Circuit of a Single String and Soundboard

As soon as a piano string is struck by a hammer, almost all the vertical vibration energy is transmitted to the soundboard via a bridge. In addition, a small part of the vibration energy is used to generate a horizontal vibration, because the bridge is not located at the modal center of the

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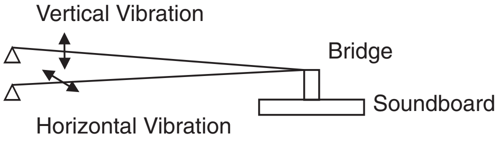

Fig. 1 Vibration model for a single string and a soundboard.

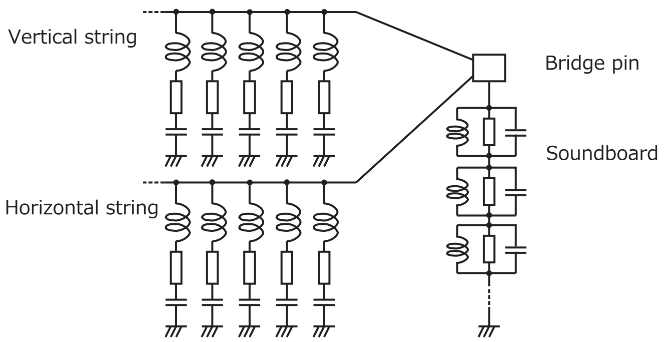

Fig. 2 Equivalent mechanical circuit in the lumped element model for a single string and a soundboard.

soundboard [5]. The piano string can then move horizontally as well as vertically relative to the soundboard, even if the string is initially excited in the vertical direction [1,3].

Since the movement of a string that is vibrating in two dimensions can be decomposed into vertical and horizontal components, the string can be considered to be equivalent to two independent strings that move in orthogonal directions. A model of a vertically vibrating soundboard, a bridge, and vertically and horizontally vibrating strings is shown in Fig. 1.

Equivalent mechanical circuits in the lumped element model will be represented using the mobility analogy: the vibration of the string is represented by a series resonant circuit with mechanical elements, and the vibration of the soundboard is represented by a parallel resonant circuit [6]. In the mobility analogy, the force acts as the through variable and the velocity acts as the across variable for the mechanical circuit. We will use f to denote force and u to denote vibration velocity. The string and the soundboard are connected by the bridge; the equivalent mechanical circuit of this is shown in Fig. 2. For simplicity, we will consider only the first vibration modes of the string and soundboard.

The mechanical elements for the first mode of the string, i.e., the equivalent stiffness K, the equivalent mass M, and the equivalent resistance R, are given by $K = \pi T/l$, $M = \mu l/\pi$, and $R = Q\sqrt{MK} = Q\sqrt{\mu T}$, respectively, where l is the string length, T is the tension, μ is the linear mass density, and Q is the quality factor. (See Appendix A.1 for the mechanical elements derived from physical parameters.) Likewise, the mechanical elements of the soundboard for the corresponding vibrating mode are

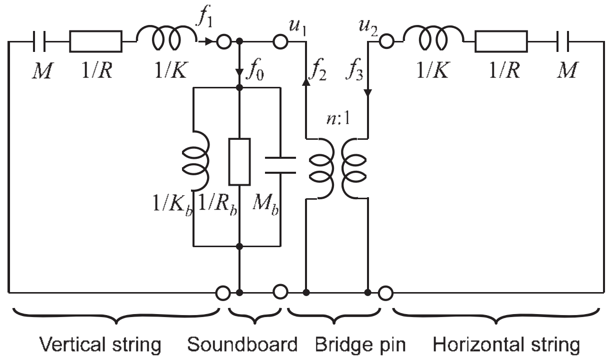

Fig. 3 Equivalent mechanical circuit in the lumped element model for the first mode of vibration of a single string and a soundboard.

given at the bridge point as follows: equivalent stiffness K_b, equivalent mass M_b, and quality factor Q_b. The equivalent resistance R_b of the soundboard is determined by M_b, K_b, and Q_b as $R_\mathrm{b} = \sqrt{M_\mathrm{b}K_\mathrm{b}}/Q_\mathrm{b}$.

When the soundboard moves up and down, the bridge tilts, so that the point of the bridge pin moves parallel as well as perpendicular to the soundboard [5]. It is reasonable to model the bridge as a lever, because the bridge rotates like a one-armed straight lever whose fulcrum is located on the edge of the soundboard. The lever is represented as an ideal transformer in the equivalent mechanical circuit.

The circuit of a single string and a soundboard is presented in Fig. 3. In this circuit, the vertical vibration satisfies the following equation:

$$\frac{1}{K}\frac{df_1}{dt} + \frac{1}{R}f_1 + \frac{1}{M}\int f_1 dt + u_1 = 0. \tag{1}$$

Since the mechanical elements of the string for the horizontal vibration are the same as those for the vertical vibration, we obtain

$$\frac{1}{K}\frac{df_3}{dt} + \frac{1}{R}f_3 + \frac{1}{M}\int f_3 dt + u_2 = 0. \tag{2}$$

The vertical and horizontal vibrations are coupled at the bridge pin, which is represented as an ideal transformer with a turn ratio of $n:1$. These vibrations are related through

$$u_2 = \frac{1}{n}u_1, \tag{3}$$

$$f_3 = nf_2. \tag{4}$$

Using Eqs. (3) and (4), Eq. (2) at the primary terminals of the transformer can be rewritten as

$$\frac{n^2}{K}\frac{df_2}{dt} + \frac{n^2}{R}f_2 + \frac{n^2}{M}\int f_2 dt + u_1 = 0. \tag{5}$$

From the viewpoint of the vertically vibrating string and the soundboard, the removal of the transformer changes the

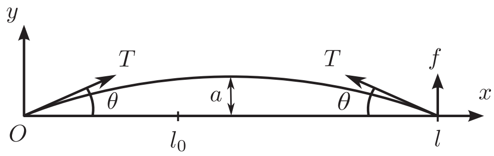

Fig. 4 First mode of a string.

horizontally vibrating string into another vertically vibrating string with an admittance of n^2 times.

If the soundboard vibration is represented by a single parallel circuit with K_b, R_b, and M_b, then it follows that

$$M_b \frac{du_1}{dt} + R_b u_1 + K_b \int u_1 dt = f_0, \tag{6}$$

where

$$f_0 = f_1 + f_2. \tag{7}$$

Now we examine the relationship between the transversal force and displacement of the string. If the string moves in the first mode with the displacement $y = a \sin(\pi x/l)$, as shown in the Fig. 4, the angle θ is

$$\theta = \frac{d}{dx}\left(a \sin\frac{\pi x}{l}\right)\bigg|_{x=0} = \frac{\pi a}{l}, \tag{8}$$

where l is the speaking length of the string. The agraffe and the bridge are assumed to be located at O and l, respectively. The transversal force f at the end of the string is expressed as $f = T \sin\theta$. With the string displacement $y_0 = a \sin(\pi l_0/l)$ at observation point $x = l_0$, f is described as

$$f \approx \begin{cases} T\dfrac{a}{l_0}\sin\dfrac{\pi l_0}{l} & \left(l_0 \leq \dfrac{l}{2}\right) \\ T\dfrac{a}{l-l_0}\sin\dfrac{\pi(l-l_0)}{l} & \left(l_0 > \dfrac{l}{2}\right). \end{cases} \tag{9}$$

Since the transversal force is proportional to the string displacement, the forces f_1 and f_3 are identical to the vertical and horizontal displacements of the string.

2.2. Simplification of Soundboard Circuit

In order to obtain analytical solutions, we simplify the parallel circuit of the soundboard and represent it as a series circuit.

The mechanical admittance $Y_b(\omega)$ of the parallel circuit of the soundboard is given by

$$Y_b(\omega) = \frac{1}{R_b + K_b/j\omega + j\omega M_b} = g_0(\omega) + jb_0(\omega), \tag{10}$$

where

$$g_0(\omega) = \frac{R_b}{R_b^2 + (K_b/\omega - \omega M_b)^2}, \tag{11}$$

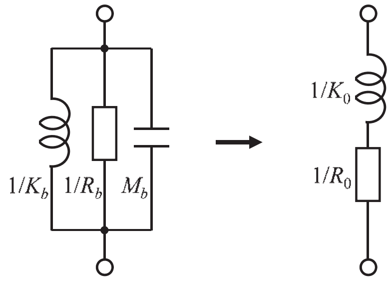

Fig. 5 Simplification of the soundboard.

$$b_0(\omega) = \frac{K_b/\omega - \omega M_b}{R_b^2 + (K_b/\omega - \omega M_b)^2}. \tag{12}$$

As will be shown later, when the resonance frequency $\omega_s = \sqrt{K/M}$ of the string is lower than that of the soundboard, $\omega_b = \sqrt{K_b/M_b}$, and the quality factor of the soundboard Q_b is lower than that of the string as $Q_b < Q$, the vibration of the soundboard is stiffness-dominated.

The parallel circuit of the soundboard can be reduced to a series circuit, as shown in Fig. 5, with the resistance $R_0(\omega)$ and the stiffness $K_0(\omega)$ obtained as

$$R_0(\omega) = \frac{1}{g_0(\omega)} = R_b + \frac{(K_b/\omega - \omega M_b)^2}{R_b}, \tag{13}$$

$$K_0(\omega) = \frac{\omega}{b_0(\omega)} = K_b\{1 - (\omega/\omega_b)^2\} + \frac{\omega R_b^2}{K_b/\omega - \omega M_b}. \tag{14}$$

By setting $K_0 = K_0(\omega_s)$ and $R_0 = R_0(\omega_s)$, the series resonant circuit of the simplified soundboard vibration can be written as

$$u_1 = \frac{1}{R_0} f_0 + \frac{1}{K_0}\frac{df_0}{dt} \tag{15}$$

instead of Eq. (6). Applying the simplification of the soundboard and removing the transformer as in Eq. (5) produces the circuit represented in Fig. 6. The mechanical admittance $Y_b(\omega)$ is approximated by the simplified series circuit as

$$Y_b(\omega) \approx \frac{1}{R_0} + \frac{j\omega}{K_0}. \tag{16}$$

3. ANALYSIS BY LAPLACE TRANSFORM

The initial conditions of the vertical vibration induced by a hammer strike are such that the equivalent mass M for the vertical string has momentum $-q$ where $q = MU$. This means that M is moved by an equivalent initial velocity U. No force is applied to the string or the soundboard, i.e., $f_0(0) = f_1(0) = f_2(0) = f_3(0) = 0$.

The Laplace transforms of the differential equations are the following polynomial equations. The Laplace transform

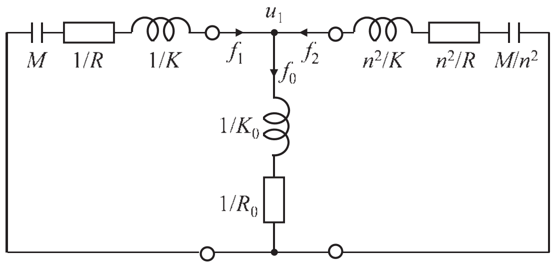

Fig. 6 Simplified equivalent circuit of a single string and a soundboard.

of Eq. (1) for a vertically vibrating string with a given initial speed is

$$\frac{sF_1(s)}{K} + \frac{F_1(s)}{R} + \left(\frac{F_1(s)}{Ms} + \frac{f_1^{(-1)}(0)}{Ms}\right) + U_1(s) = 0, \quad (17)$$

where $F_1(s)$ and $U_1(s)$ are the Laplace transforms of $f_1(t)$ and $u_1(t)$, respectively, and $f_1^{(-1)}(0)$ is the initial value of the antiderivative of $f_1(t)$, which is equal to the momentum q. Then we have

$$\left(\frac{1}{K}s + \frac{1}{R} + \frac{1}{Ms}\right)F_1(s) + U_1(s) = \frac{U}{s}. \quad (18)$$

Similarly, Eqs. (5), (15), and (7) are transformed into

$$n^2\left(\frac{1}{K}s + \frac{1}{R} + \frac{1}{Ms}\right)F_2(s) + U_1(s) = 0, \quad (19)$$

$$U_1(s) = \left(\frac{1}{K_0}s + \frac{1}{R_0}\right)F_0(s), \quad (20)$$

$$F_0(s) = F_1(s) + F_2(s), \quad (21)$$

where $F_2(s)$ and $F_0(s)$ are the Laplace transforms of $f_2(t)$ and $f_0(t)$, respectively. From Eqs. (18) to (21), $F_0(s)$ and $U_1(s)$ are obtained as

$$F_0(s) = \frac{U}{\{1/K + (1+n^{-2})/K_0\}s^2 + \{1/R + (1+n^{-2})/R_0\}s + 1/M}, \quad (22)$$

$$U_1(s) = \frac{(s/K_0 + 1/R_0)U}{\{1/K + (1+n^{-2})/K_0\}s^2 + \{1/R + (1+n^{-2})/R_0\}s + 1/M}. \quad (23)$$

From Eq. (18), the force $F_1(s)$ is obtained as

$$\begin{aligned} F_1(s) &= \frac{U/s - U_1(s)}{s/K + 1/R + 1/Ms} \\ &= \frac{U}{s^2/K + s/R + 1/M} \cdot \frac{(1/K + n^{-2}/K_0)s^2 + (1/R + n^{-2}/R_0)s + 1/M}{\{1/K + (1+n^{-2})/K_0\}s^2 + \{1/R + (1+n^{-2})/R_0\}s + 1/M}. \end{aligned}$$

Decomposing the right-hand side into partial fractions yields

$$\begin{aligned} F_1(s) = \frac{1}{n^2+1}\Bigg[&\frac{1}{s^2/K + s/R + 1/M} \\ &+ \frac{n^2}{\{1/K + (1+n^{-2})/K_0\}s^2 + \{1/R + (1+n^{-2})/R_0\}s + 1/M}\Bigg]U. \end{aligned} \quad (24)$$

Likewise, $F_2(s)$ is calculated as

$$\begin{aligned} F_2(s) &= F_0(s) - F_1(s) \\ &= \left[-\frac{1/(n^2+1)}{s^2/K + s/R + 1/M} + \frac{1/(n^2+1)}{\{1/K + (1+n^{-2})/K_0\}s^2 + \{1/R + (1+n^{-2})/R_0\}s + 1/M}\right]U. \end{aligned} \quad (25)$$

The inverse Laplace transform of $F_0(s)$ is

$$f_0 = \frac{Ue^{-\frac{t}{\tau_0}}}{\sqrt{\frac{1/K + (1+n^{-2})/K_0}{M} - \frac{\{1/R + (1+n^{-2})/R_0\}^2}{4}}} \sin\sqrt{\frac{1}{M\{1/K + (1+n^{-2})/K_0\}} - \frac{\{1/R + (1+n^{-2})/R_0\}^2}{4\{1/K + (1+n^{-2})/K_0\}^2}}\, t, \quad (26)$$

where

$$\tau_0 = \frac{2\{1/K + (1+n^{-2})/K_0\}}{1/R + (1+n^{-2})/R_0}. \quad (27)$$

Likewise,

$$f_1 = \frac{1}{n^2+1}\frac{U}{\sqrt{\frac{1}{MK} - \frac{1}{4R^2}}}e^{-\frac{K}{2R}t}\sin\left(\sqrt{\frac{K}{M} - \frac{K^2}{4R^2}}t\right) + \frac{n^2}{n^2+1}f_0, \quad (28)$$

$$f_2 = -\frac{1}{n^2+1}\frac{U}{\sqrt{\frac{1}{MK}-\frac{1}{4R^2}}}e^{-\frac{K}{2R}t}\sin\left(\sqrt{\frac{K}{M}-\frac{K^2}{4R^2}}t\right) + \frac{1}{n^2+1}f_0. \tag{29}$$

Now, if we assume that the conductances $1/R$ of the string and $1/R_0$ of the soundboard are sufficiently small, i.e.,

$$\frac{1/K+(1+n^{-2})/K_0}{M} \gg \frac{\{1/R+(1+n^{-2})/R_0\}^2}{4},$$
$$\frac{1}{MK} \gg \frac{1}{4R^2}, \tag{30}$$

then we can neglect some terms that include R and R_0 in Eq. (26) and Eq. (28). The vibrating force of the soundboard is then approximated as

$$f_0 \approx U\sqrt{\frac{M}{1/K+(1+n^{-2})/K_0}}e^{-\frac{t}{\tau_0}}\sin\omega_0 t, \tag{31}$$

where

$$\omega_0 = \sqrt{\frac{1}{M\{1/K+(1+n^{-2})/K_0\}}}, \tag{32}$$

and the vibrating force of the vertical vibration of the string is approximated as

$$f_1 \approx \frac{1}{n^2+1}U\sqrt{MK}e^{-\frac{K}{2R}t}\sin\omega t + \frac{n^2}{n^2+1}f_0, \tag{33}$$

where

$$\omega = \sqrt{\frac{K}{M}}. \tag{34}$$

Let f_s be the vibrating force for the string without the soundboard; it is equal to f_1 when $n=0$ in Eq. (33). Then,

$$f_s = U\sqrt{MK}e^{-\frac{K}{2R}t}\sin\omega t, \tag{35}$$

and f_1 can be written using f_s and f_0:

$$f_1 = \frac{1}{n^2+1}f_s + \frac{n^2}{n^2+1}f_0. \tag{36}$$

The part of the vertical component of the vibrating force that is contributed by the force of horizontal string is

$$f_2 = -\frac{1}{n^2+1}f_s + \frac{1}{n^2+1}f_0. \tag{37}$$

The relationships between the forces are shown in Fig. 7. Force f_0 denotes the force of the string vibration including the influence of the soundboard, and f_s denotes the force induced by the string-only vibration. As will be shown in Sect. 5.1, f_0 decays faster than f_s because the time constant of f_0 is smaller than that of f_s.

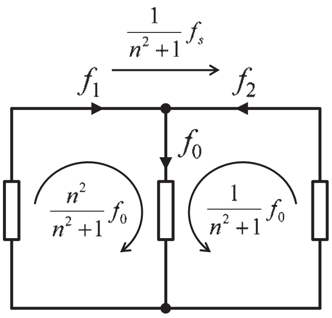

Fig. 7 Relationships of the forces for two loops.

The vibrating force for the horizontal vibration is obtained from Eq. (4) as

$$f_3 = -\frac{n}{n^2+1}f_s + \frac{n}{n^2+1}f_0. \tag{38}$$

The velocity of the soundboard u_1 is calculated by Eqs. (15) and (31). Suppose that $1/K \gg 1/K_0$, that is, the quality factor of the string is considerably higher than that of the soundboard. This assumption enables to neglect small-amplitude terms in Eq. (15) and leads to the following equation:

$$u_1 \approx U\sqrt{\frac{M(1/R_0^2+\omega_0^2/K_0^2)}{1/K+(1+n^{-2})/K_0}} \cdot e^{-\frac{1/R+(1+n^{-2})/R_0}{2\{1/K+(1+n^{-2})/K_0\}}t}\sin(\omega_0 t+\varphi), \tag{39}$$

where ω_0 is the same as the frequency of the soundboard vibrating force given by Eq. (32) and φ is

$$\varphi = \tan^{-1}\left(\frac{\omega_0/K_0}{1/R_0}\right). \tag{40}$$

We thus note that the vibration velocity of the soundboard has the exponential decay rate, but not a beat or double decay rate.

4. NUMERICAL CALCULATION

4.1. String Parameters

We used the data obtained by Tanaka *et al.* for a grand piano to determine the parameters for the note E1 [3]. The string of a short grand piano (YAMAHA C3) has a speaking length of $l = 1.27$ m. Unfortunately, no other physical parameters have yet been reported. We thus assigned the following parameters for the wound E1 string: the averaged linear mass density $\mu = 0.13$ kg/m and the fundamental frequency ν of the string is 41.20 Hz. From these values, the string tension T is calculated to be $T = 4l^2\mu\nu^2$ when the string is driven at its fundamental frequency. The quality factor is assumed to be $Q = 5{,}000$. The corresponding mechanical elements are $K = 3.52 \times 10^3$ N/m, $M = 5.25 \times 10^{-2}$ kg, and $R = 6.80 \times 10^4$ N·s/m.

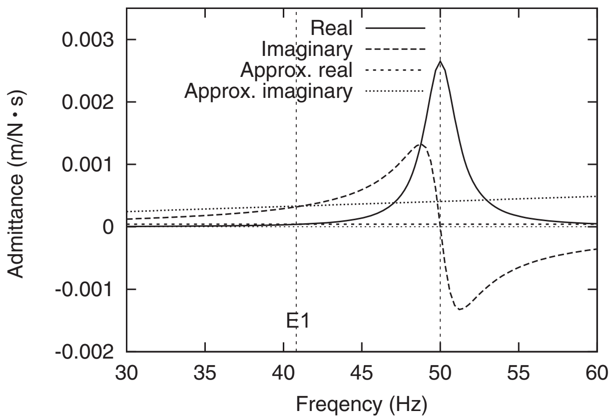

Fig. 8 Characteristics of the soundboard and its approximation.

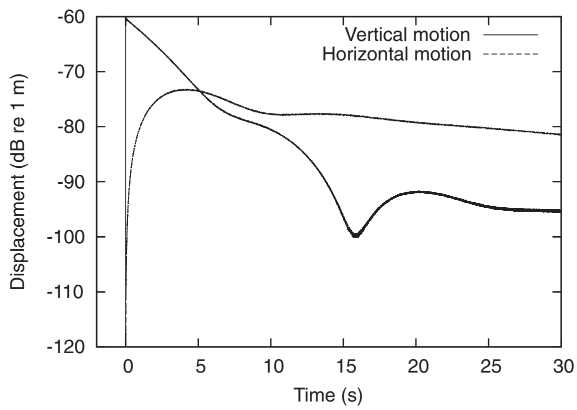

Fig. 9 Displacement of the string (soundboard quality factor is $Q_b = 20$).

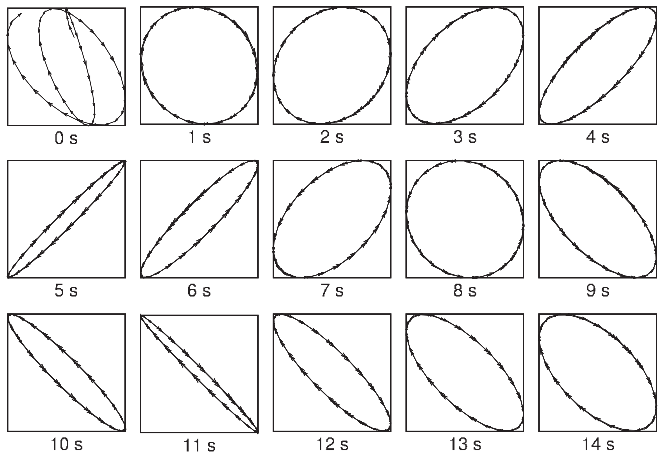

Fig. 10 Time variations of the displacement of the string (soundboard quality factor is $Q_b = 20$). Directions of rotation are indicated by arrows.

4.2. Soundboard Parameters

Suzuki reported that the frequency for the first mode of a soundboard without cast iron or strings for a 6 ft grand piano is 49.7 Hz [7]. Mamou-Mani *et al.* [8] used the finite-element method to show that the tension of the strings leads to a modification in the soundboard's mode frequencies. They also showed that the mode frequencies depend on the initial conditions of the soundboard. We will assume that the soundboard has an equivalent mass of $M_b = 24$ kg and a resonance frequency of $\nu_b = 50$ Hz with a quality factor of $Q_b = 20$. (See Appendix A.2.)

The stiffness of the soundboard K_b can be calculated from its equivalent mass M_b and its mode frequency $\nu_b = \sqrt{K_b/M_b}/2\pi$. Hence, the mechanical elements of the equivalent circuit are $K_b = 2.37 \times 10^6$ N/m, $M_b = 24.0$ kg, and $R_b = 3.77 \times 10^2$ N·s/m.

Since the parameters of the string and the soundboard satisfy $\nu < \nu_b$ and $Q \gg Q_b$, the mechanical parallel circuit of these elements can be simplified to a series circuit, which was described in Sect. 2.2. The stiffness of the series circuit K_0 for the string frequency is calculated, using Eq. (14), with the string frequency $\nu = \omega/2\pi = 41.20$ Hz, the mode frequency of the soundboard $\nu_b = \omega_b/2\pi = 50$ Hz, and the stiffness K_0 of the parallel circuit as

$$K_0 = 7.73 \times 10^5 \text{ N/m},$$

and R_0 is calculated as

$$R_0 = 2.33 \times 10^4 \text{ N·s/m}$$

using Eq. (13). The parameters of the string and soundboard satisfy the inequality condition (30), which allows the approximation. The real and imaginary parts of the admittance of the soundboard are given by Eqs. (11) and (12). Furthermore, their approximations are given by $g_0(\omega) \approx 1/R_0$ and $b_0(\omega) \approx \omega/K_0$, which are obtained from Eq. (16). The results are shown in Fig. 8. From this figure, we note that the approximation has the same value as the admittance at the string frequency.

The transformation ratio $n : 1$ of the vertical to horizontal string velocities at the bridge, which determines the coupling degree, is assumed to be $n = 5$, the value for a grand piano measured by Mori *et al.* [9].

4.3. Experimental Results

The time-varying amplitudes of the vertical and horizontal string displacements are shown in Fig. 9. The displacements are calculated by forces f_1 and f_3 using Eq. (9), assuming tension $T = 1{,}400$ N at the observation point of $l_0 = 0.46$ m ($< l/2$) under the same condition as described by Tanaka *et al.* [3]. In the first 16 seconds, the vertical vibration of the string rapidly decays, with the exception of a temporary increase at about 7 s. The horizontal vibration, however, increases in amplitude and decays with beats that have a period of about 8 s. The beats of both strings gradually disappear and have almost vanished by 25 s.

Figure 10 shows the two-dimensional movement of the string, where the vertical axis is the amplitude of the

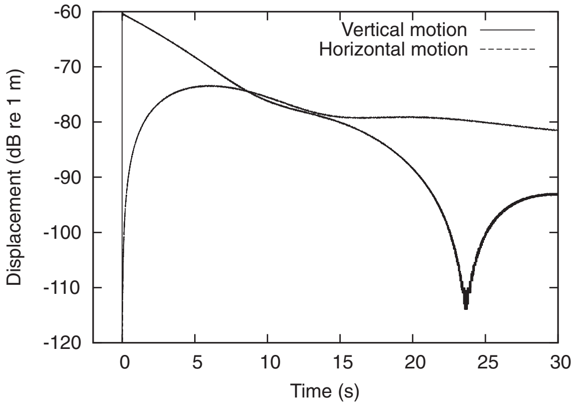

Fig. 11 Displacement of the string (soundboard quality factor is $Q_b = 30$).

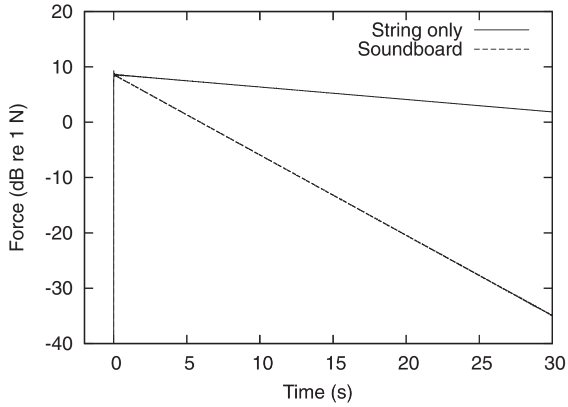

Fig. 13 Decay characteristics of the force of the string f_s and the soundboard f_0 at the bridge.

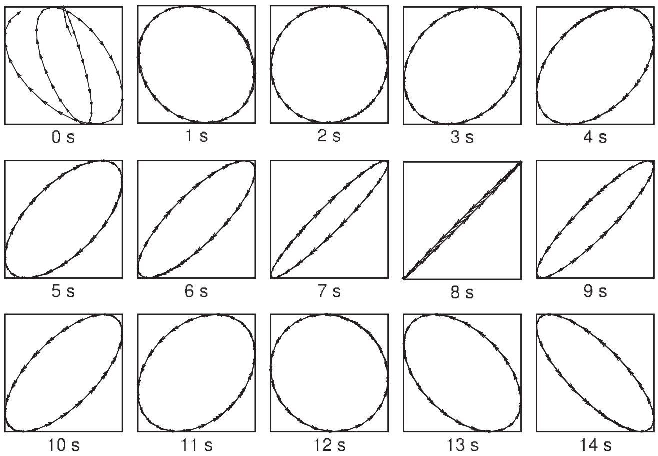

Fig. 12 Time variations of the displacement of the string (soundboard quality factor is $Q_b = 30$).

displacement calculated from force f_1 and the horizontal axis is calculated from force f_3. The scales of both axes are arbitrary, and the arrows indicate the direction in which the string rotates. The string starts vibrating vertically immediately after the hammer strikes, and then rotates clockwise. Interestingly, after 6 s, the rotation changes to the counter-clockwise direction. This indicates that the frequencies of the vertical and horizontal vibrations are slightly different. Tanaka *et al.* obtained similar tendencies in a simulation [3]. Their simulation, however, assigned different initial frequencies to the two directions. In the present model, the two strings generate different frequencies even when the parameters, such as l, μ, and T, are the same.

Figures 11 and 12 show the amplitudes of the displacement and the two-dimensional movements of the string when the initial quality factor of the soundboard is $Q_b = 30$. Since the bandwidth of the soundboard with $Q_b = 30$ is narrower than that with $Q_b = 20$, the compliance $1/K_b$ is smaller. The resultant compliance $1/K_0$ does not have a large effect on the vibrations that have a 16 s period of beats or an 8 s rotation switching period.

5. DISCUSSION

5.1. Characteristics of Strings and the Soundboard

The vibrating frequency of the soundboard is $\nu_0 = \omega_0/2\pi = \sqrt{1/[M\{1/K + (1+n^{-2})/K_0\}]}/2\pi = 41.10\,\text{Hz}$, which is lower than the frequency of the string-only vibration $\nu = \omega/2\pi = \sqrt{K/M}/2\pi = 41.20\,\text{Hz}$. This is because, owing to the influence of the horizontal component of the string vibration, the string compliance $1/K = 2.84 \times 10^{-4}\,\text{m/N}$ is added in series to the soundboard compliance $1/K_0 = 1.29 \times 10^{-6}\,\text{m/N}$ multiplied by $(1 + n^{-2})$. The characteristic impedances of the soundboard and the string are given by $\sqrt{\{1/K + (1+n^{-2})/K_0\}/M}$ and $\sqrt{1/MK}$, respectively. The reciprocal of the characteristic impedance affects the initial amplitude of the vibrations of the soundboard and string forces. The decay characteristic of the vibrating force is $\exp[-\{1/R + (1+n^{-2})/R_0\}t/2\{1/K + (1+n^{-2})/K_0\}]$ for the soundboard, while that for the string without the soundboard is $\exp(-Kt/2R)$. The decay characteristics of vibration for the string-only f_s and the soundboard f_0 are shown in Fig. 13. As can be seen, the vibrating force of the soundboard f_0 decays faster than that of the string f_s. The relationship between f_0 and f_s can also be expressed using the concept of a time constant. The time constant of the series circuit that consists of the equivalent stiffness and resistance of the soundboard given by R_0/K_0 is smaller than that of the series circuit of the string given by $2R/K$. Thus, in both directions, the string vibrations approach the string-only vibration.

5.2. Characteristics of Vertical and Horizontal Vibrating Strings

The forces of both vertical and horizontal vibrations have two components f_s and f_0 with different amplitudes.

The vertical vibration has amplitude f_s, which contributes $1/n^2$ to the soundboard amplitude, as shown in Eq. (36). Initially, the string vibrates with fast decay that mainly depends on the soundboard characteristics. Several seconds later, the vertical vibrations begin to decay more slowly owing to the increasing amplitude of the horizontal string. Then, because the soundboard vibration decays faster than the string vibration (Fig. 13), the amplitudes of the vertical string vibrations and those of the soundboard are slightly different, causing beats, as shown in Fig. 9. Later, the beats gradually vanish because the soundboard vibration continues to decay faster than that of the string alone. For the horizontal vibration, the ratio of the amplitude of the string-only and soundboard vibrations is 1:1, as seen in Eq. (38). As a result, the horizontal vibration has beats from the start, but they gradually vanish. The reason why the string vibration has two components is because the string is excited at the frequency of the soundboard.

The difference in the frequencies of the string-only force and the vibrating force of the soundboard is 0.10 Hz, which can also be calculated from the time at which the rotation direction changes (6 s, as shown in Fig. 10) and the period of the beats (11 s, as shown in Fig. 9). The calculated string frequencies are in agreement with the frequencies measured by Tanaka *et al.* [3]: a vertical vibration frequency of 40.70 Hz and a horizontal vibration frequency of 40.77 Hz; the difference is 0.07 Hz.

6. CONCLUSION

In this article, we considered the vibration of a single string coupled to a soundboard. The motion of the string in two dimensions was decomposed into those of two individual strings, each moving in one dimension. The strings coupled to the soundboard were modeled by the equivalent mechanical circuit, using the mobility analogy. The parallel circuits representing the soundboard were simplified to series circuits. The circuits, which consisted of two strings, one with vertical and the other with horizontal motion, and a simplified soundboard, were analyzed using Laplace transforms. The analysis clarified that the motion of the vertically and horizontally vibrating strings comprises two components: string-only and soundboard vibrations.

When the string frequency is lower than the soundboard mode frequency, the string frequency decreases because the soundboard is stiffness-dominated. The frequency of the vertically vibrating string is lower than that of the horizontally vibrating string. This is because the soundboard contributes primarily to the vertical vibration owing to the higher impedance of the vertically vibrating string. To be more precise, the vertical and horizontal string vibrations each have two components with different frequencies and amplitudes; these are the string-only and the soundboard vibrations, as described in the section on Laplace analysis. Since the vertical string has a stronger soundboard vibrating force than the horizontal string vibration does, the frequency of the vertical vibration is lower than that of the horizontal one. The contributions of these two components determine the direction in which the string rotates and the characteristic motion of the strings. The results of numerical calculations using the model were in agreement with the measured values.

The relationship between the frequencies of the string and the soundboard suggests that if the string vibrates at a frequency higher than the resonant frequency of the soundboard, then it would be expected that the vibrating frequency of the string would increase because the soundboard would be mass-dominated.

In this paper, we clarified why the vertical and horizontal strings with the same physical parameters vibrate at different frequencies. We expect that the present physical models will be helpful in designing soundboards using the results of how the string and soundboard vibrations interact at the bridge, and, in particular, how the soundboard vibration affects the string vibration.

A limitation of the simplified model is that it implies that the string vibrations decay with beats but the soundboard vibrations decay exponentially. If the velocity of the soundboard vibration is proportional to the sound pressure, the vibration of a single string does not undergo the double decay sound. While this is true for the simplified model, the parallel circuit soundboard model still has the potential to represent a double decay.

Further research will include evaluating the model by measuring the string motion and the driving point admittance on the same piano.

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APPENDIX

A.1. Correspondence of String Parameters

A vibration mode of the transversal wave of the string can be represented by a series circuit of mechanical elements, because the string can be considered to be a chain of very small masses connected in series by springs. Hundley *et al.* [10] showed that a single mode of string vibration can be represented as a parallel circuit, using the impedance analogy, but they did not show how to obtain the parameters of the elements. In this appendix, we show how the parameters of a mechanical system for the string can be obtained by using the mobility analogy described in Sect. 2.

First, we consider the restoring force. Figure 14 shows a small segment ds that started from x with the interval dx on the x axis. It is assumed that a string that is vibrating in its fundamental mode has a sinusoidal shape $y(x)$:

$$y(x) = a \sin \frac{\pi x}{l}, \tag{A·1}$$

where a is the displacement of the center of the string and l is the length of the string. The following approximations are valid if the amplitude is significantly small:

$$\sin\theta_x \approx \tan\theta_x \approx \frac{dy}{dx}, \tag{A·2}$$

$$\sin\theta_{x+dx} \approx \tan\theta_{x+dx} \approx \frac{dy}{dx} + \frac{d^2y}{dx^2}dx. \tag{A·3}$$

If the tension is invariant throughout the string segment, the restoring force $F_{\mathrm{r}}(x)$ of the segment in the vertical direction can be approximated as

$$\begin{aligned} F_{\mathrm{r}}(x)dx &= -T(x)\sin\theta_x + T(x+dx)\sin\theta_{x+dx} \\ &\approx -T\frac{dy}{dx} + T\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}dx\right) \\ &= T\frac{d^2y}{dx^2}dx. \end{aligned} \tag{A·4}$$

The restoring force over the entire string is obtained by integration over the segments, and it is related to the stiffness of the string. Since the restoring force acts in the direction opposite to the displacement, it has a negative sign,

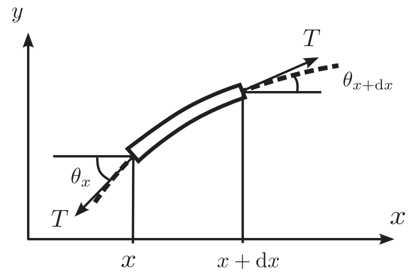

Fig. 14 Segment of a string with tension T.

$$\begin{aligned} K_{\mathrm{a}} &= -\frac{1}{2}\int_0^l T\frac{d^2y}{dx^2}dx \\ &= a\pi T/l, \end{aligned} \tag{A·5}$$

where the string is supported at both ends and the force at one end is half the total restoring force.

The mass-related quantity M_{a} of the deformed string at one end is half the integrated mass of segments:

$$\begin{aligned} M_{\mathrm{a}} &= \frac{1}{2}\int_0^l y(x)\mu dx \\ &= al\mu/\pi. \end{aligned} \tag{A·6}$$

The impedance-related quantity R_{a} of the series resonant circuit is determined using K_{a}, M_{a}, and the quality factor Q as

$$\begin{aligned} R_{\mathrm{a}} &= aQ\sqrt{M_{\mathrm{a}}K_{\mathrm{a}}} \\ &= aQ\sqrt{\mu T}. \end{aligned} \tag{A·7}$$

The transversal velocity-related quantity u_{a} at the bridge is obtained as the sum of each element,

$$u_{\mathrm{a}} = u_{K_{\mathrm{a}}} + u_{R_{\mathrm{a}}} + u_{M_{\mathrm{a}}}, \tag{A·8}$$

where

$$u_{K_{\mathrm{a}}} = \frac{1}{K_{\mathrm{a}}}\frac{df}{dt}, \tag{A·9}$$

$$u_{R_{\mathrm{a}}} = \frac{1}{R_{\mathrm{a}}}f, \tag{A·10}$$

$$u_{M_{\mathrm{a}}} = \frac{1}{M_{\mathrm{a}}}\int f dt. \tag{A·11}$$

Equation (A·8) can be rewritten as

$$u_{\mathrm{a}} = \frac{1}{K_{\mathrm{a}}}\frac{df}{dt} + \frac{1}{R_{\mathrm{a}}}f + \frac{1}{M_{\mathrm{a}}}\int f dt. \tag{A·12}$$

Since a is arbitrary, letting $K = K_{\mathrm{a}}/a$, $R = R_{\mathrm{a}}/a$, $M = M_{\mathrm{a}}/a$, and $u = u_{\mathrm{a}}/a$ yields

$$u = \frac{1}{K}\frac{df}{dt} + \frac{1}{R}f + \frac{1}{M}\int f dt, \tag{A·13}$$

as we see in the series circuit in Ref. [11]. The corre-

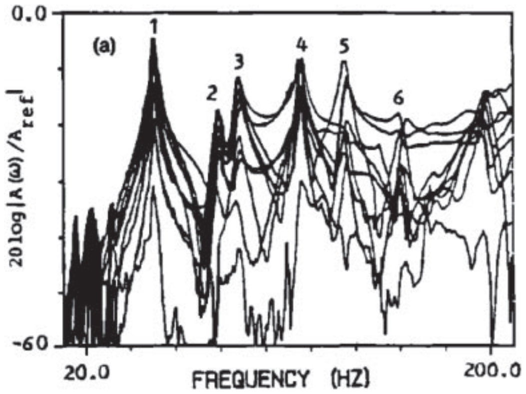

Fig. 15 Measurements of the soundboard acceleration frequency response function at ten measurement points [7].

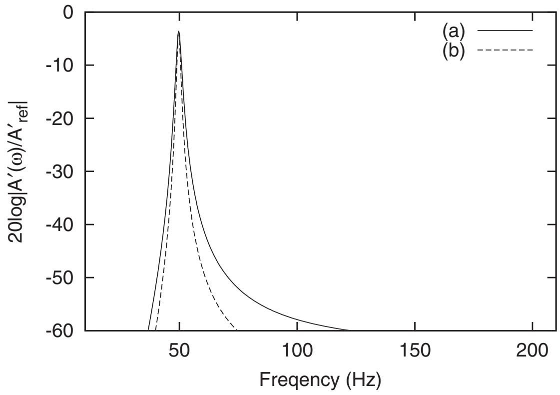

Fig. 16 Approximations of the soundboard acceleration frequency response function. The parameters are (a) $M_b = 24$ kg and $Q_b = 20$ and (b) $M_b = 36$ kg and $Q_b = 30$.

spondence of mechanical elements and physical parameters is $K = \pi T/l$, $M = \mu l/\pi$, and $R = Q\sqrt{\mu T}$. The angular frequency ω is

$$\begin{aligned} \omega &= \sqrt{K/M} \\ &= \frac{\pi}{l}\sqrt{\frac{T}{\mu}}. \end{aligned} \tag{A·14}$$

A.2. Soundboard Parameters

Figure 15 presents measurements of the acceleration frequency response function at ten points on a grand piano soundboard. Although the measured points might not include the bridge of E1, we assume that the driving point accelerance at the bridge of E1 is not too different from those values.

The acceleration frequency response function $A(\omega)$ is given by

$$A(\omega) = \alpha(\omega)/F(\omega), \tag{A·15}$$

where $\alpha(\omega)$ is the acceleration detected by an accelerometer and $F(\omega)$ is the excitation force. The reference A_{ref} is defined as $A_{\mathrm{ref}} = 1\,\mathrm{kg}^{-1}$.

When the soundboard has a single resonance frequency, the mechanical circuit is represented by a parallel resonant circuit of mechanical elements. The mechanical admittance $Y_b(\omega)$ of the soundboard is given by

$$Y_b(\omega) = \frac{1}{R_b + K_b/j\omega + j\omega M_b}, \tag{A·16}$$

where M_b is the effective mass at the driving point of the soundboard; K_b is the stiffness $K_b = \omega_b^2 M_b$; and R_b is the resistance $R_b = \omega_b M_b/Q_b$, where Q_b is the quality factor. Equation (A·16) corresponds to Eq. (10). The absolute value of the acceleration frequency response function $A'(\omega)$ of the circuit is given by

$$\begin{aligned} |A'(\omega)| &= |\omega Y_b(\omega)| \\ &= \frac{\omega}{\sqrt{R_b^2 + (K_b/\omega - \omega M_b)^2}}. \end{aligned} \tag{A·17}$$

Figure 16 shows two approximation curves of $|A'(\omega)/A'_{\mathrm{ref}}|$ with the parameters of $M_b = 24$ kg and $Q_b = 20$, and $M_b = 36$ kg and $Q_b = 30$, and $A'_{\mathrm{ref}} = 1\,\mathrm{kg}^{-1}$. The resonance frequency is 49.7 Hz.