

Effect of discretization in thickness direction on eigenfrequency analysis of a circular plate by the finite-difference time-domain method

Masahiro Toyoda^{1,*}, Daiji Takahashi² and Yasuhito Kawai¹

¹Department of Architecture, Faculty of Environmental and Urban Engineering, Kansai University, 3-3-35, Yamate-cho, Suita, 564-8680 Japan

²Department of Architecture and Architectural Engineering, Graduate School of Engineering, Kyoto University, C1-4-385, Kyoto University Katsura, Nishikyo-ku, Kyoto, 615-8540 Japan

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1. Introduction

The authors have proposed a prediction method for floor impact noise by the finite-difference time-domain (FDTD) method [1]. In the paper, solid parts of the target building, i.e., concrete walls and floor slabs are not treated as plate elements but as assembly of small volumes. This treatment yields distribution of shear stress in the thickness direction of the plate. In the case of thin plates such as glass windows and interior panels, their vibrations can be predicted by the FDTD method with the thin-plate theory, where the shear-stress distribution is neglected [2]. However neglect of the shear-stress distribution can cause a serious problem in prediction of vibration when dealing with thick plates such as walls and slabs. Therefore thick walls and slabs should be treated as assembly of small volumes for the accurate analyses. Although the walls and slabs are discretized into three or four cells in the thickness direction in the previous work [1], it is not described how many cells are required to sufficiently reproduce the shear-stress distribution.

Herein the effect of discretization in thickness direction of a plate is investigated. The eigenfrequencies of a clamped circular plate calculated by the FDTD method are compared with those obtained by the thin-/thick-plate theories [3]. Based on the comparisons, the effect of number of cells in thickness direction for plate-vibration analysis is discussed.

2. Thin plate theory

In this section, prior to the investigation using the FDTD method, theoretical estimation of eigenfrequencies for a circular thin plate is introduced. The homogeneous equation of motion for the axisymmetric displacement of a circular thin plate $w(r)$, where r is distance from the origin, can be written as

$$\{\nabla^4 - k_B^4\}w(r) = 0, \quad (1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}, \quad (2)$$

$$k_B^4 = \frac{\rho h \omega^2}{D}, \quad (3)$$

ρ , h , $D = Eh^3/12(1 - \nu^2)$, E , ν , and ω are density, thickness, flexural rigidity, Young's modulus, Poisson's ratio of the plate, and the angular frequency, respectively. The time factor $e^{-i\omega t}$ is suppressed throughout, where i is the imaginary unit. Equation (1) can be rewritten as

$$\{\nabla^2 + k_B^2\}\{\nabla^2 - k_B^2\}w(r) = 0. \quad (4)$$

The complete solution of Eq. (4) is given by

$$w(r) = c_1 J_0(k_B r) + c_2 I_0(k_B r) + c_3 Y_0(k_B r) + c_4 K_0(k_B r), \quad (5)$$

where J_0 , I_0 , Y_0 , and K_0 are the 0th order Bessel function of first kind, modified Bessel function of first kind, Bessel function of second kind (Neumann function), and modified Bessel function of second kind, respectively. Considering that $w(r)$ is finite, the constants c_3 and c_4 must be zero. When the plate is supported at its edge, i.e., $w(a) = 0$ where a is the radius of the plate, Eq. (5) can be rewritten as

$$w(r) = c_1 \left\{ J_0\left(\frac{\beta}{a} r\right) - \frac{J_0(\beta)}{I_0(\beta)} I_0\left(\frac{\beta}{a} r\right) \right\}, \quad (6)$$

where

$$\beta = ak_B. \quad (7)$$

In addition, when the plate is clamped, i.e., $\partial w(r)/\partial r|_{r=a} = 0$, the relation,

$$\frac{J_1(\beta)}{J_0(\beta)} - \frac{I_1(\beta)}{I_0(\beta)} = 0, \quad (8)$$

can be derived. Substituting Eq. (7) into Eq. (3), the eigenfrequencies can be given by,

$$f_m = \frac{1}{2\pi} \sqrt{\frac{D}{\rho h}} \left(\frac{\beta_m}{a} \right)^2, \quad (9)$$

where β_m are the values which satisfy Eq. (8).

3. Thick plate theory

In this section, theoretical estimation of eigenfrequencies for a circular thick plate is discussed. Lee *et al.* [4] suggested the pseudospectral method using Chebyshev polynomial. Some approximation errors due to polynomial expansion however must be included in the method. Herein, to obtain the

*e-mail: toyoda@kansai-u.ac.jp

more precise estimation of eigenfrequencies, the following procedure is considered.

The homogeneous equation of motion for the axisymmetric displacement of a circular thick plate $w(r)$ can be written as [3]

$$\{\nabla^4 + (S + R)\nabla^2 + (SR - k_B^4)\}w(r) = 0, \quad (10)$$

where

$$S = \frac{\rho\omega^2}{\kappa^2 G}, \quad (11)$$

$$R = \frac{\rho h^3 \omega^2}{12D}, \quad (12)$$

$G = E/2(1 + \nu)$ is shear modulus of the plate. κ^2 is a constant relating shear stress to strain and the value herein is set to $\pi^2/12$ [3]. Using the eigenvalues $\delta_{1,2}$, Eq. (10) can be rewritten as

$$\{\nabla^2 + \delta_1^2\}\{\nabla^2 + \delta_2^2\}w(r) = 0, \quad (13)$$

$$\delta_1^2 + \delta_2^2 = S + R, \quad (14)$$

$$\delta_1^2 \delta_2^2 = SR - k_B^4. \quad (15)$$

The complete solution of Eq. (13) is given by

$$w(r) = c_1 J_0(\delta_1 r) + c_2 J_0(\delta_2 r) + c_3 Y_0(\delta_1 r) + c_4 Y_0(\delta_2 r). \quad (16)$$

Considering that $w(r)$ is finite and the plate is supported at its edge, Eq. (16) can be rewritten as

$$w(r) = c_1 \left\{ J_0\left(\frac{\gamma_1}{a} r\right) - \frac{J_0(\gamma_1)}{J_0(\gamma_2)} J_0\left(\frac{\gamma_2}{a} r\right) \right\}, \quad (17)$$

where

$$\gamma_{1,2} = a\delta_{1,2}. \quad (18)$$

When the plate is clamped, the relation,

$$\gamma_1 \frac{J_1(\gamma_1)}{J_0(\gamma_1)} - \gamma_2 \frac{J_1(\gamma_2)}{J_0(\gamma_2)} = 0, \quad (19)$$

can be derived. On the other hand, from Eqs. (14), (15), and (18), $\gamma_{1,2}$ can be given by

$$\gamma_{1,2} = \frac{a^2}{2} \left\{ (S + R) \pm \sqrt{(S - R)^2 + 4k_B^4} \right\}. \quad (20)$$

At an arbitrary frequency, the set of $\gamma_{1,2}$ can be uniquely obtained from Eq. (20). If the set of $\gamma_{1,2}$ satisfies Eq. (19), the corresponding values of $\delta_{1,2}$ are the eigenvalues and the frequency is the eigenfrequency. Scanning a target range of frequency with the examination of Eq. (19), all the eigenfrequencies in the range can be obtained.

4. FDTD analysis

In the FDTD analysis, a circular plate is discretized in the Euclidean system. The z direction is its thickness direction. Although fundamental procedures of the FDTD calculation are the same as those described in the previous work [1], damping is neglected here and the implementation of boundary conditions introduced in the previous work [5] is employed. The center of the plate is excited by a Gaussian pulse and the z -directional velocity at the excitation point is calculated. Driving-point impedance levels are given by

Table 1 Comparison between the eigenfrequencies of a circular plate of 150 mm thickness obtained by the thin-/thick-plate theories and the FDTD method with cubic discretization with the spatial intervals $\Delta x = \Delta y = \Delta z (= \Delta h)$. Relative errors of the eigenfrequencies to those with the spatial intervals $\Delta h = 5$ mm are noted in brackets.

Method	1st	2nd	3rd
Thin plate	101.0 Hz (1.3%)	393.2 Hz (6.6%)	880.8 Hz (14.3%)
Thick plate	99.8 Hz (0.1%)	373.2 Hz (1.2%)	786.3 Hz (2.0%)
$\Delta h = 50$ mm	103.0 Hz (3.3%)	378.0 Hz (2.5%)	784.0 Hz (1.7%)
$\Delta h = 30$ mm	102.7 Hz (3.0%)	378.0 Hz (2.5%)	787.7 Hz (2.2%)
$\Delta h = 10$ mm	100.7 Hz (1.0%)	371.3 Hz (0.7%)	776.0 Hz (0.7%)
$\Delta h = 5$ mm	99.7 Hz	368.7 Hz	770.7 Hz

$$Z(f) = 10 \log_{10} \frac{|F(f)/v_z(f)|^2}{Z_0^2}, \quad (21)$$

where $Z_0 (= 1 \text{ Ns/m})$ is a criterion for the impedance level. $F(f)$ and $v_z(f)$ are the frequency characteristics of the Gaussian pulse and z -directional velocity, respectively. From the frequency characteristics of impedance level, the eigenfrequencies can be estimated by searching where the level is locally minimum.

5. Comparison

The plate is assumed to be homogeneous concrete with a density of $2,400 \text{ kg/m}^3$, Young's modulus of $2.4 \times 10^{10} \text{ N/m}^2$, and Poisson's ratio of 0.2. The radius of the plate is 1,500 mm.

Table 1 shows the comparison between the eigenfrequencies of a circular plate of 150 mm thickness obtained by the thin-/thick-plate theories and the FDTD method with cubic discretization. In the FDTD method, the spatial intervals $\Delta x = \Delta y = \Delta z (= \Delta h)$ are set to 50, 30, 10, and 5 mm, which correspond to three, five, fifteen, and thirty cells in thickness direction of the plate, respectively. In the table, relative errors of the eigenfrequencies to those obtained by the FDTD method with 5 mm spatial intervals are also shown. The results with 5 mm intervals are assumed to be the correct values here because the reference points per wavelength are quite many and the eigenfrequencies seems to converge sufficiently. It can be seen from Table 1 that the errors by the thin-plate theory is somewhat small for first eigenfrequency and extremely large for second and third eigenfrequencies. As for the thick-plate theory, although some errors can be seen because of the approximation of shear-stress distribution in thickness direction, the eigenfrequencies are relatively in good agreement with those by the FDTD method with 5 mm intervals. These errors by plate theories tend to increase as the

Table 2 Comparison between the eigenfrequencies of a circular plate of 150 mm thickness obtained by the FDTD method with rectangular discretization ($\Delta x = \Delta y = 50$ mm). Relative errors of the eigenfrequencies to those with cubic discretization ($\Delta h = 5$ mm) are noted in brackets.

Δz	1st	2nd	3rd
30 mm	103.0 Hz (3.3%)	379.3 Hz (2.9%)	789.3 Hz (2.4%)
10 mm	101.3 Hz (1.6%)	374.0 Hz (1.4%)	779.7 Hz (1.2%)
5 mm	100.2 Hz (0.5%)	369.6 Hz (0.2%)	770.3 Hz (0.1%)

order of vibration becomes higher. On the other hand, the errors by the FDTD method decrease as the order of vibration becomes higher. The errors also tend to decrease as the spatial intervals become smaller. However the tendency is not monotonically; the errors for 50 and 30 mm intervals are in the same range. This would be caused by complex cancellations of errors due to numerical dispersion, staircase approximation of the circular shape, and rough distribution of shear stress. The next discussion is devoted to extraction of the most important effect from these causes.

Table 2 shows the comparison between the eigenfrequencies of a circular plate of 150 mm thickness obtained by the FDTD method with rectangular discretization where the spatial intervals $\Delta x = \Delta y$ are set to 50 mm and Δz is set to 30, 10, and 5 mm, which correspond to five, fifteen, and thirty cells in thickness direction of the plate, respectively. In the table, relative errors of the eigenfrequencies to those obtained by the FDTD method with cubic discretization with spatial intervals $\Delta h = 5$ mm are also shown. From Tables 1 and 2, it can not be said that the aspect ratio of the discretized cell seriously affects the errors on eigenfrequency because differences between the errors in Tables 1 and 2 are less than 1% for the same Δz . This means that the errors due to rough distribution of shear stress have more important impact on the eigenfrequencies than those due to numerical dispersion and staircase approximation of the circular shape.

Table 3 shows the comparison between the eigenfrequencies of a circular plate of 150, 200, and 250 mm thickness obtained by the thin-/thick-plate theories and the FDTD method with cubic discretization. In the FDTD method, the spatial intervals Δh are set to 50 mm. Herein considered number of cells in thickness direction is three, four, and five. From the results shown in Table 2, the eigenfrequencies obtained by the FDTD method with rectangular discretization of $\Delta x = \Delta y = 50$ mm and $\Delta z = 5$ mm are assumed as the criteria and the relative errors of the eigenfrequencies to them are shown in Table 3. As a matter of course, the thin plate theory gives relatively large errors in the case of 250 mm. While the thick plate theory gives good estimations for all configurations of thickness, the errors tend to increase as the order of vibration becomes higher. On the other hand, the errors by the FDTD method tend to decrease as the order of

Table 3 Comparison between the eigenfrequencies of a circular plate obtained by the thin-/thick-plate theories and the FDTD method with cubic discretization ($\Delta h = 50$ mm). Relative errors of the eigenfrequencies to those with rectangular discretization ($\Delta x = \Delta y = 50$ mm, $\Delta z = 5$ mm) are noted in brackets.

Method	Thickness	1st	2nd	3rd
Thin plate	150 mm	101.0 Hz (0.8%)	393.2 Hz (6.4%)	880.8 Hz (14.3%)
	200 mm	134.7 Hz (2.3%)	524.2 Hz (11.5%)	1174.5 Hz (24.0%)
	250 mm	168.3 Hz (4.5%)	655.3 Hz (18.2%)	1468.1 Hz (35.8%)
Thick plate	150 mm	99.8 Hz (0.4%)	373.2 Hz (1.0%)	786.3 Hz (2.1%)
	200 mm	132.0 Hz (0.2%)	480.3 Hz (2.1%)	978.8 Hz (3.4%)
	250 mm	163.2 Hz (1.4%)	576.3 Hz (4.0%)	1137.1 Hz (5.2%)
$\Delta z = 50$ mm	150 mm	103.0 Hz (2.8%)	378.0 Hz (2.3%)	784.0 Hz (1.8%)
	200 mm	135.0 Hz (2.5%)	480.0 Hz (2.1%)	962.0 Hz (1.6%)
	250 mm	165.0 Hz (2.5%)	565.3 Hz (2.0%)	1098.3 Hz (1.6%)
$\Delta z = 5$ mm	150 mm	100.2 Hz	369.6 Hz	770.3 Hz
	200 mm	131.7 Hz	470.3 Hz	947.0 Hz
	250 mm	161.0 Hz	554.3 Hz	1081.3 Hz

vibration becomes higher. It can also be seen that the errors by the FDTD method slightly decrease as the plate becomes thicker. This tendency would be caused by the effect of rough distribution of shear stress. However it can be said that the effect is insignificant in this range from three to five cells in thickness direction.

6. Conclusion

Herein, in the aim of investigating the effect of number of cells in thickness direction for plate-vibration analysis by the FDTD method, the numerical eigenfrequencies by the FDTD method with various spatial intervals are compared with theoretical ones by the thin-/thick-plate theories. The findings obtained from these comparisons are summarized as follows:

- (1) The eigenfrequencies obtained by the FDTD method are affected by numerical dispersion, staircase approximation of the circular shape, and rough distribution of shear stress. Among them, the errors due to rough distribution of shear stress have the most important impact on the eigenfrequencies.
- (2) While the errors by the thin-/thick-plate theories tend to increase as the order of vibration becomes higher and the plate becomes thicker, those by the FDTD method show the opposite tendency.
- (3) In the range from three to five cells in the thickness direction, although the errors tend to decrease as the number of cells increase, the difference is not significant.

- (4) The errors of first, second, and third eigenfrequencies for the plate considered here can be less than 3% if the number of cells in thickness direction is equal to or more than three.

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