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Use of velocity potentials in the definition of absorbing boundaries for FDTD analysis of elastic wave fields

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1. Introduction

In a solid material, two types of elastic waves exist: longitudinal waves and shear waves. In numerical analysis, absorbing boundaries are sometimes required to express the unlimited space of the solid material. However, the absorbing boundary must be coded in a sophisticated manner [1–3] to account for the different acoustic characteristics of these two types of waves, and it proves quite difficult to absorb both waves perfectly at the same time.

The present author has been involved in evaluating the effectiveness and potential utility of scalar and vector velocity potentials in elastic wave fields [4,5]. The present report introduces a new usage for the scalar and vector velocity potentials in coding the absorbing boundary in a manner suitable for the finite-difference time-domain (FDTD) analysis of elastic wave fields. If a linear approximation is assumed in a homogenous solid, the longitudinal and shear waves propagate individually without interaction or mode conversion. Thus, absorbing boundaries can be realized for both longitudinal and shear waves simultaneously through the use of two (scalar and vector) separate velocity potentials.

Randall also used potentials (not velocity potentials) in defining the absorbing boundaries of elastic wave fields appropriate for use in an FDTD formulation [2]. However, the inclusion of a Laplacian in the constitutional equations for Randall's formulation at the points of the absorbing boundary, amongst other factors, rendered the method not fully suitable for the FDTD algorithm. Here, velocity potentials are treated as scalar waves in definition of the absorbing boundary. The calculation procedure is as follows: (1) Stress and particle velocities are analyzed, excluding the absorbing boundaries. (2) Velocity potentials are treated individually as scalar waves and formulated by a leap-frog algorithm matched to the FDTD formulation. These wave fields are set near the absorbing boundaries and coded as absorbing boundary conditions. (3) Particle velocities at the absorbing boundaries are then calculated using the velocity potentials. The method proposed here allows for easy and effective formulation of arbitrary scalar wave formulations for absorbing boundary conditions.

The method is applied in this report to formulation of the absorbing boundary condition for a simple model as verification of the method's effectiveness.

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2. Fundamental equations

For simplicity, a two-dimensional surface-strain problem will be assumed. The fundamental constitutional equations for solids in this system are expressed as follows.

$$\frac{\partial}{\partial t}\begin{bmatrix} T_1 \\ T_3 \\ T_5 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & c_{55} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial \dot{u}}{\partial x} \\ \dfrac{\partial \dot{w}}{\partial z} \\ \dfrac{\partial \dot{u}}{\partial z} + \dfrac{\partial \dot{w}}{\partial x} \end{bmatrix} \tag{1}$$

$$\rho \frac{\partial}{\partial t}\begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial T_1}{\partial x} + \dfrac{\partial T_5}{\partial z} \\ \dfrac{\partial T_3}{\partial z} + \dfrac{\partial T_5}{\partial x} \end{bmatrix} \tag{2}$$

Equations (1) and (2) describe Hook's law and Newton's second law, respectively [6]. Isotropic characteristic being assumed, the conditions of $c_{33} = c_{11}$ and $c_{55} = (c_{11} - c_{13})/2$ are imposed on the stiffness constant matrix elements. In the equations, ρ is mass density, $\dot{u}$ and $\dot{w}$ are particle velocities in the x and z directions, T_1 and T_3 are normal stresses, and T_5 is shear stress. The relationships between the stress vector (T_i: $i = 1, 3, 5$) and stress tensor (T_{jk}: $j,\ k = x, z$) are defined as $T_1 = T_{xx}$, $T_3 = T_{zz}$ and $T_5 = T_{xz}$ [6].

The velocity potentials ϕ and $\boldsymbol{A}$ are defined as follows.

$$\dot{\boldsymbol{u}} = \mathrm{grad}\,\phi + \mathrm{rot}\,\boldsymbol{A} \tag{3}$$

where ϕ is a scalar velocity potential and $\boldsymbol{A} = [\varphi_1, \varphi_2, \varphi_3]$ is a vector velocity potential. In the two-dimensional case mentioned above, $\dot{u}$ and $\dot{w}$ are expressed as follows.

$$\dot{u} = \frac{\partial \phi}{\partial x} - \frac{\partial \varphi_2}{\partial z} \tag{4}$$

$$\dot{w} = \frac{\partial \phi}{\partial z} + \frac{\partial \varphi_2}{\partial x} \tag{5}$$

From Eqs. (4) and (5), we obtain

$$\frac{\partial \dot{U}_1}{\partial x} + \frac{\partial \dot{U}_3}{\partial z} = Z_0\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) = \frac{Z_0}{c_{\mathrm{P}}{}^2}\frac{\partial^2 \phi}{\partial t^2} \tag{6}$$

$$-\frac{\partial \dot{U}_1}{\partial z} + \frac{\partial \dot{U}_3}{\partial x} = Z_0\left(\frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial z^2}\right) = \frac{Z_0}{c_{\mathrm{S}}{}^2}\frac{\partial^2 \varphi_2}{\partial t^2} \tag{7}$$

where $\dot{U}_1 = Z_0\dot{u}_1$ and $\dot{U}_3 = Z_0\dot{u}_3$, $Z_0 = \sqrt{\rho c_{11}}$ is the acoustic characteristic impedance of a longitudinal plane wave, and

c_P, c_S is the phase velocity of a longitudinal and shear wave, respectively. From Eqs. (6) and (7) we derive the following.

$$\frac{\partial \dot{U}_1}{\partial x} + \frac{\partial \dot{U}_3}{\partial z} = \frac{Z_0}{c_P{}^2}\frac{\partial \dot{\phi}}{\partial t} \tag{8}$$

$$-\frac{\partial \dot{U}_1}{\partial z} + \frac{\partial \dot{U}_3}{\partial x} = \frac{Z_0}{c_S{}^2}\frac{\partial \dot{\varphi}_2}{\partial t} \tag{9}$$

$$\dot{\phi} = \frac{\partial \phi}{\partial t}, \quad \dot{\varphi}_2 = \frac{\partial \varphi_2}{\partial t} \tag{10}$$

Here, $\dot{\phi}$ and $\dot{\varphi}_2$ represent the time differential of a scalar velocity potential and a vector velocity potential, respectively.

The wave equations of velocity potentials (6) and (7) will be used to define the absorbing boundary. The ability to relate the calculation results for the velocity potentials to the values of the elastic variables is important for formulating equations suitable for FDTD analysis. Thus, the wave equations (6) and (7) are broken down here into the following six constitutional equations.

$$\begin{aligned} \rho\frac{\partial i_{\phi x}}{\partial t} &= -\frac{\partial \dot{\phi}}{\partial x} \\ \rho\frac{\partial i_{\phi z}}{\partial t} &= -\frac{\partial \dot{\phi}}{\partial z} \\ \frac{1}{c_{11}}\frac{\partial \dot{\phi}}{\partial t} &= -\left(\frac{\partial i_{\phi x}}{\partial x} + \frac{\partial i_{\phi z}}{\partial z}\right) \end{aligned} \tag{11}$$

$$\begin{aligned} \rho\frac{\partial i_{\varphi z}}{\partial t} &= -\frac{\partial \dot{\varphi}_2}{\partial x} \\ \rho\frac{\partial i_{\varphi x}}{\partial t} &= -\frac{\partial \dot{\varphi}_2}{\partial z} \\ \frac{1}{c_{55}}\frac{\partial \dot{\varphi}_2}{\partial t} &= -\left(\frac{\partial i_{\varphi z}}{\partial x} + \frac{\partial i_{\varphi x}}{\partial z}\right) \end{aligned} \tag{12}$$

where $i_{\phi i}$ and $i_{\varphi j}$ $(i, j = x, z)$ are expediently adopted variables.

3. FDTD formulation of elastic and velocity potential equations

Figure 1 shows the lattice configurations of the FDTD method for analyzing the elastic, scalar velocity potential and vector velocity potential wave fields. The lattice regions of the elastic and velocity potentials fields overlap.

Reformulating first equation of Eq. (1) by the FDTD method gives

$$\begin{aligned} &\frac{2\Delta d}{2\Delta t}[\{-T_1{}^{n+1}(i+1,k)\} - \{-T_1{}^{n-1}(i+1,k)\}] \\ &\quad = -c_{11}\{\dot{u}^n(i+2,k) - \dot{u}^n(i,k)\} \\ &\qquad - c_{13}\{\dot{w}^n(i+1,k+1) - \dot{w}^n(i+1,k-1)\} \end{aligned} \tag{13}$$

where i and k indicate the number of numerical points in the x- and z-directions, n is a time step, and Δd and Δt are the spatial and temporal discrete lengths. The spatial lattice interval is arranged as $\Delta x = \Delta z = \Delta d$.

Similarly, first equation of Eq. (2) becomes

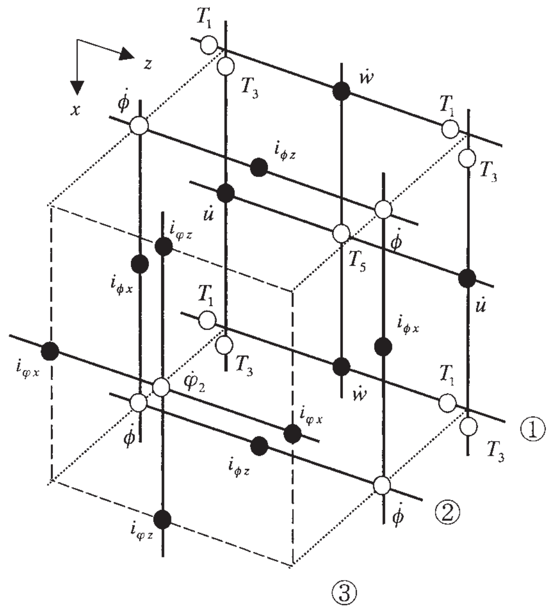

Fig. 1 Variables at the crossing point of three lattice networks. (1) Elastic stress and particle velocity field, (2) scalar velocity potential field ϕ, and (3) vector velocity potential field φ_2. Thin dotted lines indicate discrete overlapping points.

$$\begin{aligned} &\rho\frac{2\Delta d}{2\Delta t}\{\dot{u}^n(i,k) - \dot{u}^{n-2}(i,k)\} \\ &\quad = -[\{-T_1{}^{n-1}(i+1,k)\} - \{-T_1{}^{n-1}(i-1,k)\}] \\ &\qquad - [\{-T_5{}^{n-1}(i,k+1)\} - \{-T_5{}^{n-1}(i,k-1)\}] \end{aligned} \tag{14}$$

The other differential equations of Eqs. (11) and (12) are formulated in a similar manner. Using the FDTD formulation, the stress and particle velocities can be calculated alternately according to the time step Δt.

Equations (8) and (9) are used to obtain velocity potentials from particle velocities, with approximation by the center finite difference. The equations in FDTD formulation are then derived as follows.

$$\begin{aligned} \rho\dot{\phi}^{n+1}(i+1,k) = \rho\dot{\phi}^{n-1}(i+1,k) &\\ + VPL\cdot\{\dot{U}_1{}^n(i+2,k) - \dot{U}_1{}^n(i,k) &\\ + \dot{U}_3{}^n(i+1,k+1) - \dot{U}_3{}^n(i+1,k-1)\} & \end{aligned} \tag{15}$$

$$\begin{aligned} \rho\dot{\varphi}_2{}^{n+1}(i,k+1) = \rho\dot{\varphi}_2{}^{n-1}(i,k+1) &\\ + VSL\cdot\{\dot{U}_1{}^n(i,k) - \dot{U}_1{}^n(i,k+2) &\\ + \dot{U}_3{}^n(i+1,k) - \dot{U}_3{}^n(i-1,k)\} & \end{aligned} \tag{16}$$

$$VPL = \frac{c_P}{\Delta d/\Delta t}, \quad VSL = VPL\cdot\frac{c_S}{c_P} \tag{17}$$

However, it is also necessary to have a formulation that determines the particle velocities from the velocity potentials. Appropriate relationships can be derived from Eqs. (4) and (5) by applying the center finite difference approximation, as follows.

$$\dot{U}_1{}^n(i,k) = \dot{U}_1{}^{n-2}(i,k) + VPL \cdot \{\rho\dot{\phi}^{n-1}(i+1,k) - \rho\dot{\phi}^{n-1}(i-1,k) - \rho\dot{\varphi}_2{}^{n-1}(i,k+1) + \rho\dot{\varphi}_2{}^{n-1}(i,k-1)\} \quad (18)$$

$$\dot{U}_3{}^n(i+1,k+1) = \dot{U}_3{}^{n-2}(i+1,k+1) + VSL \cdot \{\rho\dot{\phi}^{n-1}(i,k+2) - \rho\dot{\phi}^{n-1}(i,k) + \rho\dot{\varphi}_2{}^{n-1}(i+2,k) - \rho\dot{\varphi}_2{}^{n-1}(i,k)\} \quad (19)$$

$$VPL = \frac{c_P}{\Delta d/\Delta t}, \quad VSL = VPL \cdot \frac{c_S}{c_P} \quad (20)$$

4. Calculation procedure, model and result

Figure 2 shows a calculation model for verification of the proposed definition of an absorbing boundary in the FDTD analysis of elastic fields. The elastic variables were calculated for the entire the region except for particle velocity points at the absorbing boundaries. The scalar and vector velocity potentials were calculated in the shaded region of Fig. 2. The scalar and vector velocity potentials on the dotted lines at x_1 and z_1 in Fig. 2 were calculated by Eqs. (15) and (16), and $\dot{U}_1$ and $\dot{U}_3$ on the absorbing boundaries were determined from Eqs. (18) and (19).

A flow chart of the computational procedure is shown in Fig. 3. The lattice figures beside the chart are discrete points of the elastic, scalar velocity potential and vector velocity potential fields around the point of $z = z_1$. The solid black and open circles are calculation points.

A first-order approximation is applied on the absorbing boundaries of the velocity potential fields $\dot{\phi}$ and $\dot{\varphi}_2$. The boundaries a-o and b-o are symmetrical, and a-c and c-b are the absorbing boundaries. Stresses T_1 and T_3 at the center point (0,0) and T_5 at (1,1) are the input points in Fig. 3. The waveform of the input signal is a single-period sinusoid.

Figure 4 shows the stress $(T_1 + T_3)/2$ at three time steps. Wave propagation associated with the stress $(T_1 + T_3)/2$ is know to be almost entirely longitudinal [4]. As shown in Fig. 4, the longitudinal wave is well attenuated at the absorbing boundaries. Figure 5 shows the case for propagation of the T_5 wave, which will be almost entirely shear. Despite the distribution of stress T_5 being more complicated than the shear wave [4], as well as the low amplitude of T_5 compared to $(T_1 + T_3)/2$ and the many little waves that appear behind the headmost T_5 wave (S), the absorbing effect appears very good. It is expected that use of a technique such as the pseudo-maximum likelihood method [7] will allow the absorbing effect to be further improved.

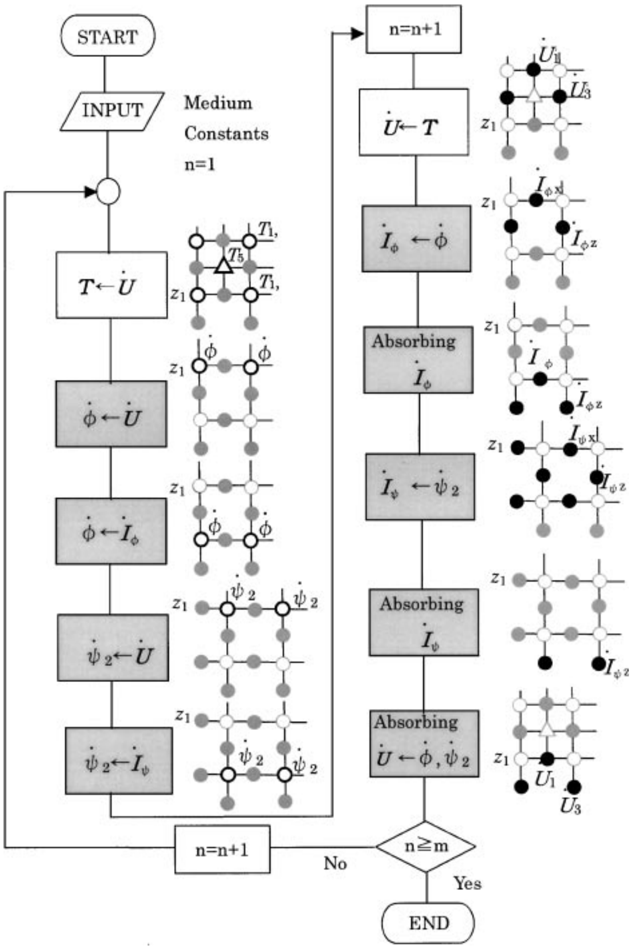

Fig. 3 Flow chart of FDTD analysis for elastic wave fields in solids with absorbing boundaries defined using velocity potentials. Hatching parts of the chart show calculations of the new technique.

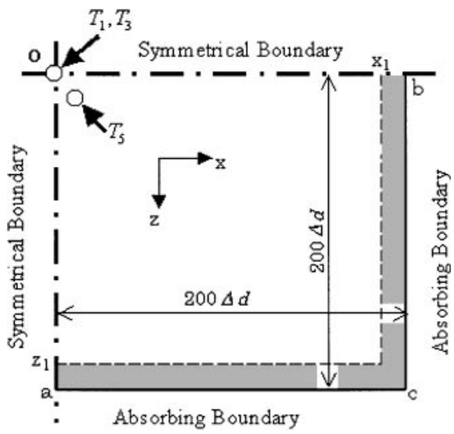

Fig. 2 Simple analysis model for confirming the effectiveness of the proposed definition of absorbing boundaries.

5. Conclusion

Velocity potentials were introduced for defining the absorbing boundary in the numerical FDTD analysis of two-dimensional elastic wave fields. In this method, velocity potentials are treated as scalar waves individually and formulated numerically by a leap-flog algorithm appropriate to the FDTD method, and the absorbing boundary condition is coded at the boundary of two scalar waves representing longitudinal and shear waves. Thus, the method allows for easy and effective formulation of arbitrary scalar wave formulations for absorbing boundary conditions, and is readily applicable to systems including both longitudinal and shear waves.

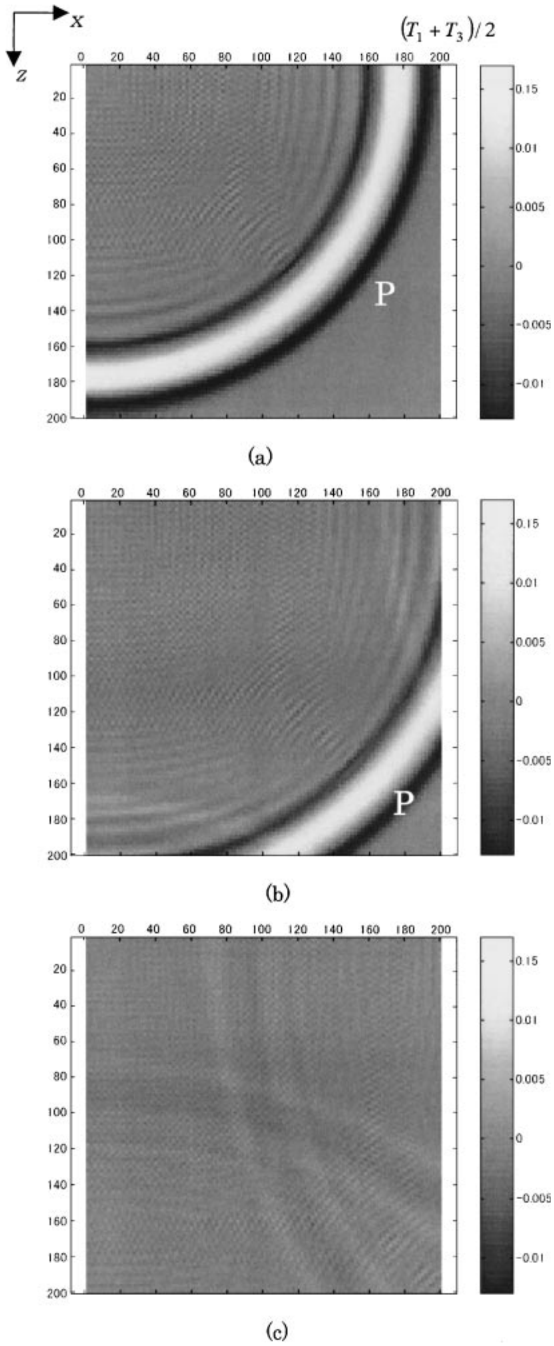

Fig. 4 Propagation of stress $(T_1 + T_3)/2$ analyzed using the calculation model in Fig. 2 at times steps of (a) 200, (b) 250, and (c) 350.

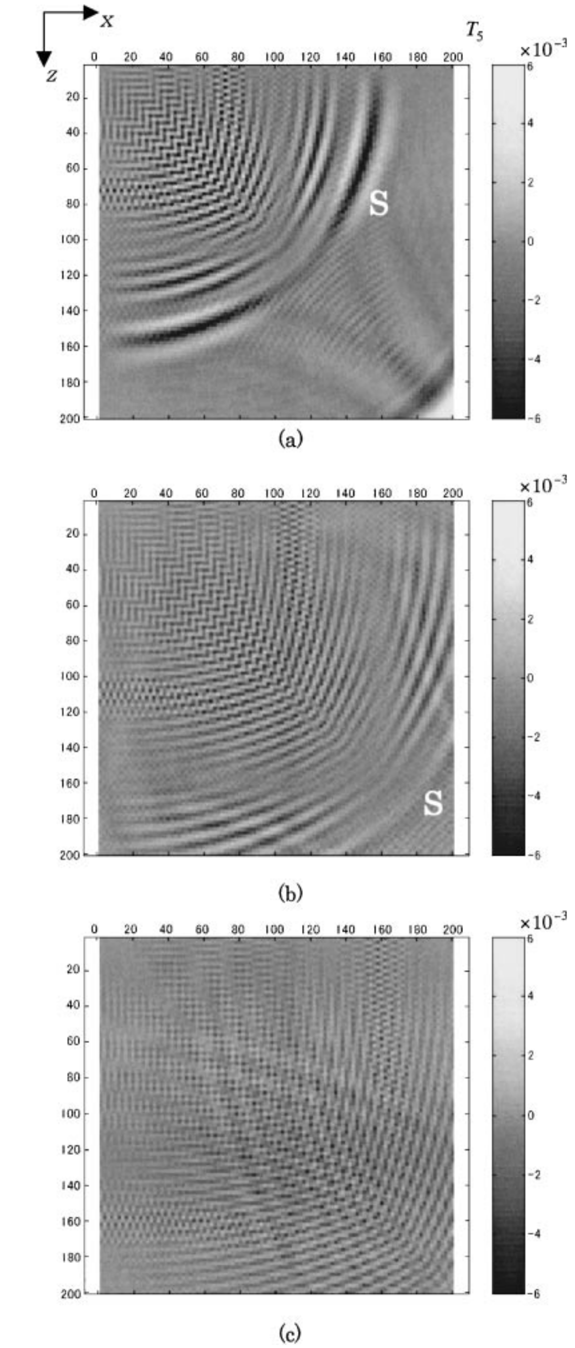

Fig. 5 Propagation of stress T_5 analyzed using the calculation model in Fig. 2 at time steps of (a) 300, (b) 450, and (c) 650.

Through application to a simple model, the method was confirmed to be effective for implementing an absorbing boundary for elastic wave fields in FDTD analysis. The method is currently being extended to three-dimensional cases.

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