

**Technical Report**

## **Resonance frequencies and loss factors of various single-degree-of-freedom systems**

Hideo Suzuki

*Ono Sokki Co., Ltd.,*

*1-16-1, Hakusan, Midori-ku, Yokohama, 226-8507 Japan*

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A single-degree-of-freedom (SDOF) system with a mass ( $m$ ), a spring ( $k$ ), and a damping ( $c$ ) is a basic mechanical system. It is well known that a complex mechanical system is represented by a combination of infinite number of SDOF systems. The modal analysis theory is based on this principle, but only one type among various types of SDOF systems is presently employed for a modeling of complex systems. When one needs to estimate  $m$ ,  $k$ , and  $c$  of a SDOF system from the measurement of the resonance frequency and the loss factor, the relationships between them must be exactly known. Those relationships are known for commonly used SDOF systems, but those for rather unfamiliar types are not known. In this technical report, various types of SDOF systems and their equivalent electrical circuits are listed, and equations that relate the resonance frequencies to the undamped resonance (natural) frequencies and the loss factors are given. It is also shown that, for some type of SDOF systems, some cares must be taken how to interpret the loss factor.

**Keywords:** SDOF systems, Resonance frequency, Loss factor, Model analysis

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### **1. INTRODUCTION**

A single-degree-of-freedom (SDOF) system with a mass ( $m$ ), a spring ( $k$ ), and a damping ( $c$ ) is a basic mechanical system. It is well known that a complex mechanical system is represented by a combination of infinite number of SDOF systems. The modal analysis theory is based on this principle, but only one type among various types of SDOF systems is presently employed for a modeling of complex systems.

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quencies to the undamped resonance (natural) frequencies and the loss factors are given. It is also shown that, for some type of SDOF systems, some cares must be taken how to interpret the loss factor.

### **2. DEFINITION OF RESONANCE AND ANTI-RESONANCE FREQUENCIES**

In this technical report, the resonance frequency of a SDOF system is defined as the frequency at which the driving point admittance (velocity/force) gives the maximum value. In a similar way, the anti-resonance frequency is defined as the frequency at which the driving point impedance (force/velocity) gives the maximum value. The resonance (also means "anti-resonance" in some cases) can be defined by use of the displacement or acceleration instead of velocity.

The first column (except the column for the model type) of Fig. 1 shows various types of SDOF systems with the mass ( $m$ ), the spring ( $k$ ) and the damping ( $c$ ). Type A is the most common SDOF system

that is used for the modeling of a complex mechanical system. It has the (angular) resonance frequency given by

$$\omega_0 = \omega_A = \sqrt{k/m} \quad (1)$$

The “displacement” or “acceleration” based resonance frequencies for type A SDOF system are given, respectively, by<sup>1)</sup>

$$\omega_{\text{dis}} = \omega_0 \sqrt{1 - 2\zeta^2} = \omega_0 \sqrt{1 - \eta^2/2} \quad (2)$$

and

$$\omega_{\text{ncc}} = \omega_0 / \sqrt{1 - 2\zeta^2} = \omega_0 / \sqrt{1 - \eta^2/2} \quad (3)$$

where  $\eta$  is the loss factor ( $\zeta = \eta/2$  is the damping ratio) which is given by

$$\eta = 2\zeta = \frac{c}{\sqrt{mk}} = \omega_0 \frac{c}{k} = \frac{m\dot{c}}{\omega_0} \quad (4)$$

The displacement or acceleration resonance frequency is dependent on the loss factor and always lower or higher than  $\omega_0$ . A benefit of using the velocity resonance frequency is that the resonance frequency is independent of the loss factor. In general, the resonance frequency is independent of the loss factor if the admittance or impedance characteristic of a SDOF system is symmetric with respect to  $\omega_0$  when the frequency axis is expressed in the logarithmic scale.

Another definition of the resonance frequency based on a time waveform exists. If an impulse is given to type A SDOF system with a zero initial displacement, the displacement response is given by ( $\zeta < 1$  is assumed)<sup>1)</sup>

$$x(t) = A e^{-\omega_0 \zeta t} \sin \omega_d t, \quad t \geq 0 \quad (5)$$

where  $A$  is a coefficient determined by the initial velocity and  $\omega_d$  given by

$$\omega_d = \omega_0 (1 - \zeta^2) \quad (6)$$

is called the damped resonance frequency. This is smaller than  $\omega_0$  but larger than  $\omega_{\text{dis}}$  of type A SDOF system.

**NOTE 1:** Only the resonance type SDOF systems (types A, C, E, and G) show the vibratory motion when an impulse is applied. If a frequency response function is measured by the impulse method, the anti-resonance of type B, D, F and H SDOF systems are not observed. If a continuous excitation by a shaker is applied to those systems, the anti-resonance can be observed.

### 3. RESONANCE FREQUENCIES OF VARIOUS SDOF SYSTEMS

The second and third columns of Fig. 1 show equivalent electrical circuits of the mechanical systems shown in the first column. The second and third columns are obtained by the mobility and impedance analogies, respectively.<sup>2)</sup> The fourth column shows the admittance curves of each SDOF systems.

#### 3.1 Type A and B SDOF Systems

Type A and type B SDOF systems are dual with each other, that is, the characteristic of the force of one type has the same property of the velocity characteristic of the other type, and vice versa. The anti-resonance frequency of type B SDOF system is given by Eq.(1).

$$\omega_B = \omega_0 = \sqrt{k/m} \quad (7)$$

The loss factor  $\eta'$  of type B SDOF system can be defined from the analogy to type A SDOF system and using the mobility analogy circuit such that

$$\eta' (= 2\zeta') = \frac{1/c}{\sqrt{(1/k)(1/m)}} = \sqrt{\frac{mk}{c}} \quad (8)$$

Equation (8) indicates that the loss factor of type B SDOF system is larger with a smaller damping.

#### 3.2 Type C and D SDOF Systems

Type C and D SDOF systems have the duality relationship with each other. The resonance frequency of type C SDOF system is obtained as follows.

The impedance of type C SDOF system is given from the equivalent electrical circuit (impedance analogy) such that

$$\begin{aligned} Z(\omega) &= j\omega m + \frac{c(k/j\omega)}{c + (k/j\omega)} \\ &= m \frac{j\eta' \omega_0 \omega + (\omega_0^2 - \omega^2)}{j\omega + \eta' \omega_0} \end{aligned} \quad (9)$$

The squared absolute value is given by

$$|Z(\omega)|^2 = \frac{\omega^4 + \omega_0^2 [(\eta')^2 - 2] \omega^2 + \omega_0^4}{\omega^2 + (\eta' \omega_0)^2} \quad (10)$$

By differentiating Eq.(10) with respect to  $\omega$  and letting the result equal to zero, the resonance frequency is obtained as

$$\omega_C = \omega_0 \sqrt{1 + 2(\eta')^2 - (\eta')^2} \quad (11)$$

From the duality, the resonance frequency of type D

type	mechanical model	equivalent electrical circuit		admittance
		mobility analogy	impedance analogy	
A				
B				
C				
D				
E				
F				
G				
H				

**Fig. 1** Tables of various mechanical SDOF systems and their equivalent electrical circuits and admittance curves.

SDOF system is given by

$$\omega_b = \omega_0 \sqrt{1 + 2(\eta)^2 - (\eta)^2} \quad (12)$$

given by

$$Y(\omega) = \frac{1}{j\omega m} + \frac{1}{c + (k/j\omega)} = \frac{1}{c} \frac{[(\omega^2 - \omega_0^2) - j\{\eta\omega_0\}\omega]}{[\omega^2 - j(\omega_0/\eta)\omega]} \quad (13)$$

### 3.3 Type E and F SDOF Systems

A method to obtain the anti-resonance frequency of type F SDOF system is the same as the method described above. The admittance of this system is

Following the same procedure as in 3.2, the resonance frequency is given by

$$\omega_F = \omega_0 \sqrt{\frac{\sqrt{1 + 2\eta^2 + \eta^2}}{1 + 2\eta^2 - \eta^4}} \quad (14)$$

From the duality,

$$\omega_E = \omega_0 \sqrt{\frac{\sqrt{1+2\eta'^2+\eta'^2}}{1+2\eta'^2-\eta'^4}} \quad (15)$$

**NOTE 2:** Applications of definitions of loss factor given by Eqs.(4) and (8) were extended to types C-F SDOF systems. Strictly speaking, however, except for types A and B SDOF systems, values given by Eqs.(4) and (8) do not agree with the measured loss factors by the half-power method. This is related to the asymmetry of admittance or impedance curves shown in Fig. 1. When the definitions of the loss factor given by Eqs.(4) and (8) are applied to types C-F SDOF systems, this fact should be reminded.

### 3.4 Type G and H SDOF Systems

Since these types have the symmetric impedance and admittance curves, respectively, the resonance frequencies do not depend on the loss factor, and are given by

$$\omega_G = \omega_H = \sqrt{\frac{k}{m}} \quad (16)$$

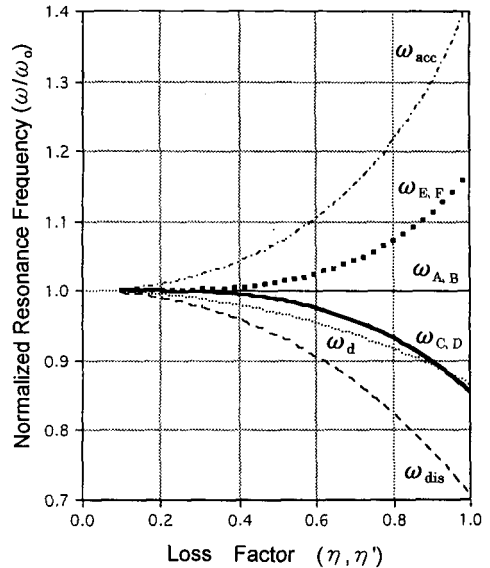
**NOTE 3:** A significant difference of these two types from the previous types is that the admittance or the impedance has an infinitely large or small values regardless of the magnitude of the damping. In this sense, the definition of the loss factor given by Eq.(4) or (8) are not applicable at all to these SDOF systems.

### 3.5 Comparison of the Resonance Frequencies

The dependence of resonance frequencies of SDOF systems from type A to F on the loss factor ( $\eta$  or  $\eta'$ ) are shown in Fig. 2. The validity of the formulae has been checked by numerical calculations, that is, "true" resonance frequencies were obtained from the peak of the admittance or impedance curves and compared with those given by the formulae shown above. Frequencies,  $\omega_{dis}$ ,  $\omega_{acc}$ , and  $\omega_d$  of type A SDOF system are also shown in Fig. (2).

Following remarks can be made from the result.

- i) If Eq.(1) is used to estimate mass  $m$  and spring  $k$  for a system other than type A, B, G, or H, an error may be induced.
- ii) A situation may be worse if measured values of the displacement or the acceleration resonance or anti-resonance frequencies are used.
- iii) Types C/D and E/F SDOF systems have the same order of loss factor dependence but in the opposite direction. For a relatively



**Fig. 2** Dependence of resonance frequencies on the loss factor for various SDOF systems.

large loss factor,  $\eta=0.6$ , the deviation of the resonance frequencies from the undamped resonance frequency are approximately 2%, which may be usually a negligible order.

## 4. CONCLUSIONS

Various definitions of the resonance frequencies were introduced first and then the dependence of resonance frequencies on the loss factors of various SDOF systems were discussed. The interpretation of the loss factor for different types of SDOF systems were also discussed. In most cases, the effect of the loss factor on the resonance frequencies are not very serious. However, in cases where an accurate system identification or evaluation of the loss factor is necessary, a care should be taken as to which type of SDOF system is used as a model and what formula is applied to the measured values. In the modal analysis, when the curve fitting method based on type A SDOF system does not give a satisfactory result, another type of SDOF model may be tried.

## REFERENCES

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