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HOMOTOPY GROUPS OF THE SPACE OF SELF MAPS OF THE PROJECTIVE 3-SPACE

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Abstract

We compute the homotopy groups of the space of self maps of the 3 dimensional projective space.

1. Introduction

For spaces X and Y with base points, we denote by $\mathrm{map}_*(X, Y)$ the space of maps from X to Y preserving base points. We take the trivial map 0 as the base point of $\mathrm{map}_*(X, Y)$. Homotopical properties of $\mathrm{map}_*(X, Y)$ have long been studied in algebraic topology. In the recent decade, several people have been interested in the case where X is a Lie group and $X = Y$ [**6**, **7**, **8**, **13**, **14**, **15**, **16**]. In the present note, we study the case $X = Y = \mathrm{SO}(3) = \mathbf{P}^3$ and we compute the homotopy groups $\pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3))$ for $n \le 20$, where $\mathbf{P}^3$ is the 3-dimensional projective space. As an application to our computations we know $\pi_n(\mathrm{aut}(\mathbf{P}^3))$ for $n \le 20$, where $\mathrm{aut}(\mathbf{P}^3)$ is the space of self homotopy equivalences of $\mathbf{P}^3$. Results will be given in the section 2, and proofs will be given in sections 3, 4 and 5.

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2. Results

The groups $\pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3))$ for $n = 0, 1, 3$ are well-known (cf. (3.1) below):

(2.1) $\pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3))$

$$\cong \begin{cases} \mathbf{Z} & n = 0 \ \ ([\mathbf{12},\ \text{Theorem IIa}] \text{ or } [\mathbf{8},\ \text{Proposition 4.1}]) \\ (\mathbf{Z}_2)^2 & n = 1 \ \ ([\mathbf{3},\ (9.1.3)] \text{ or } [\mathbf{6},\ \text{Lemma 7.3}]) \\ (\mathbf{Z}_4)^2 \oplus \mathbf{Z}_3 & n = 3 \ \ ([\mathbf{17},\ \text{Corollary 5}] \text{ or } [\mathbf{15},\ \text{Lemma 2.1(6)}]) \end{cases}$$

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Here $\mathbf{Z}_k$ denotes the cyclic group of order k and $(\mathbf{Z}_k)^m$ is the direct sum of m copies of $\mathbf{Z}_k$. We denote by $\mathbf{Z}_k\{x\}$ the cyclic group of order k generated by x. We write $(\mathbf{Z}_k)^m\{x_1,\dots,x_m\} = \mathbf{Z}_k\{x_1\} \oplus \cdots \oplus \mathbf{Z}_k\{x_m\}$.

We work in the category of spaces with base points, unless otherwise stated. The base point is denoted by $*$. We do not distinguish in notation between a map and its homotopy class. We denote the suspension functor by E, that is, $E^nX = X \wedge \mathrm{S}^n$ and $E^nf : E^nX \to E^nY$ are the n-fold suspensions of a space X and a map $f : X \to Y$, respectively. For spaces X and Y, we denote the set of homotopy classes of maps from X to Y by $[X, Y]$, that is, $[X, Y] = \pi_0(\mathrm{map}_*(X, Y))$. We follow notations of [**18**] for elements of homotopy groups of spheres. Given a map $g : \mathrm{S}^m \to \mathrm{S}^n$ such that $2g \simeq 0$, let $\bar{g} : \mathrm{S}^m \cup_{2\iota_m} e^{m+1} \to \mathrm{S}^n$ denote an extension of g, that is, $\bar{g} = g$ on S^m, where $\mathrm{S}^m \cup_{2\iota_m} e^{m+1}$ is the mapping cone of $2\iota_m$. We should be careful not to confuse Toda's elements $\bar{\varepsilon}_n$ (resp. $\bar{\mu}_n$) with extensions $\overline{\varepsilon_n}$ (resp. $\overline{\mu_n}$) of Toda's elements ε_n (resp. μ_n). We set

$$\Gamma_n = [\mathrm{S}^{n+1} \cup_{2\iota_{n+1}} e^{n+2}, \mathrm{S}^3] \quad (n \geq 0).$$

Notice that Γ_n is a finite abelian group and $E^n(\mathrm{S}^1 \cup_{2\iota_1} e^2) = \mathrm{S}^{n+1} \cup_{2\iota_{n+1}} e^{n+2}$. We will prove the following assertion in §3.

Proposition 2.1. $\pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)) \cong \Gamma_n \oplus \pi_{n+3}(\mathrm{S}^3)$.

We refer $\pi_{n+3}(\mathrm{S}^3)$ for $n \leq 20$ to [**18**, **9**] (cf. Lemma 4.1(2) below).

In order to compute Γ_n, we use the following cofibre sequence

$$\mathrm{S}^{n+2} \xleftarrow{-2\iota_{n+2}} \mathrm{S}^{n+2} \xleftarrow{q_n} \mathrm{S}^{n+1} \cup_{2\iota_{n+1}} e^{n+2} \xleftarrow{i_n} \mathrm{S}^{n+1} \xleftarrow{2\iota_{n+1}} \mathrm{S}^{n+1}. \tag{2.2}$$

Our main result is the following theorem which will be proved in §4.

Theorem 2.2.

n	0	1	2	3	4	5
Γ_n	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_4$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$
generators		q_1	$q_2^*\eta_3$	$\overline{\eta_3}$	$q_4^*\nu', \eta_3\overline{\eta_4}$	$q_5^*(\nu'\eta_6), \eta_3^2\overline{\eta_5}$
relations				$2\overline{\eta_3} = q_3^*\eta_3^2$		

6	7	8	9	10	11
$\mathbf{Z}_4$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_4$	$(\mathbf{Z}_2)^2 \oplus \mathbf{Z}_4$
$\nu'\overline{\eta_6}$	$\nu'\eta_6\overline{\eta_7}$		$q_9^*\varepsilon_3$	$q_{10}^*\mu_3, \overline{\varepsilon_3}$	$q_{11}^*\varepsilon', \varepsilon_3\overline{\eta_{11}}, \overline{\mu_3}$
$2(\nu'\overline{\eta_6}) = q_6^*(\nu'\eta_6^2)$				$2\overline{\varepsilon_3} = q_{10}^*(\eta_3\varepsilon_4)$	$2\overline{\mu_3} = q_{11}^*(\eta_3\mu_4)$

12	13
$(\mathbf{Z}_2)^5$	$(\mathbf{Z}_2)^3 \oplus \mathbf{Z}_4$
$q_{12}^*\mu', q_{12}^*(\varepsilon_3\nu_{11}), q_{12}^*(\nu'\varepsilon_6), \varepsilon_3\eta_{11}\overline{\eta_{12}}, \mu_3\overline{\eta_{12}}$	$q_{13}^*(\nu'\mu_6), \mu_3\eta_{12}\overline{\eta_{13}}, \overline{\varepsilon_3\nu_{11}}, \varepsilon'\overline{\eta_{13}}$
	$2(\varepsilon'\overline{\eta_{13}}) = q_{13}^*(\nu'\eta_6\varepsilon_7)$

14	15	16	17
$\mathbf{Z}_2 \oplus \mathbf{Z}_4$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_4$
$\nu'\eta_6\overline{\varepsilon_7}, \nu'\overline{\mu_6}$	$q_{15}^*(\varepsilon_3\nu_{11}^2), \nu'\eta_6\overline{\mu_7}$	$q_{16}^*\bar{\varepsilon}_3, \varepsilon_3\overline{\nu_{11}^2}$	$q_{17}^*(\mu_3\sigma_{12}), \overline{\overline{\varepsilon}}_3$
$2(\nu'\overline{\mu_6}) = q_{14}^*(\nu'\eta_6\mu_7)$			$2\overline{\overline{\varepsilon}}_3 = q_{17}^*(\eta_3\bar{\varepsilon}_4)$

18	19
$(\mathbf{Z}_2)^3 \oplus \mathbf{Z}_4$	$(\mathbf{Z}_2)^4 \oplus \mathbf{Z}_4$
$q_{18}^*\bar{\varepsilon}', q_{18}^*\bar{\mu}_3, \bar{\varepsilon}_3\overline{\eta_{18}}, \overline{\mu_3\sigma_{12}}$	$q_{19}^*(\mu'\sigma_{14}), q_{19}^*(\nu'\bar{\varepsilon}_6), \bar{\varepsilon}_3\eta_{18}\overline{\eta_{19}}, \mu_3\sigma_{12}\overline{\eta_{19}}, \overline{\overline{\mu}}_3$
$2\overline{\mu_3\sigma_{12}} = q_{18}^*(\eta_3\mu_4\sigma_{13})$	$2\overline{\overline{\mu}}_3 = q_{19}^*(\eta_3\bar{\mu}_4)$

20	21
$(\mathbf{Z}_2)^5$	$(\mathbf{Z}_2)^2 \oplus \mathbf{Z}_4$
$q_{20}^*\bar{\mu}', q_{20}^*(\nu'\mu_6\sigma_{15}), \mu_3\sigma_{12}\eta_{19}\overline{\eta_{20}}, \nu'\overline{\overline{\varepsilon}}_6, \eta_3\overline{\overline{\mu}}_4$	$q_{21}^*(\nu'\bar{\mu}_6), \eta_3^2\overline{\overline{\mu}}_5, \nu'\overline{\mu_6\sigma_{15}}$
	$2(\nu'\overline{\mu_6\sigma_{15}}) = q_{21}^*(\nu'\eta_6\mu_7\sigma_{16})$

Here we have used the following notations: $\overline{\eta_n} = E^{n-3}\overline{\eta_3}$, $\overline{\varepsilon_n} = E^{n-3}\overline{\varepsilon_3}$, $\overline{\mu_n} = E^{n-3}\overline{\mu_3}$, $\overline{\overline{\varepsilon}}_n = E^{n-3}\overline{\overline{\varepsilon}}_3$, $\overline{\overline{\mu}}_n = E^{n-3}\overline{\overline{\mu}}_3$ *and* $\overline{\mu_n\sigma_{n+12}} = E^{n-3}\overline{\mu_3\sigma_{12}}$ *for* $n \geq 4$.

Rees [**17**, Corollary 5] determined Γ_3 by methods different with ours. By Proposition 2.1, Theorem 2.2 and [**18**, **9**], we readily have

COROLLARY 2.3.

n	0	1	2	3	4	5	6
$\pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3))$	$\mathbf{Z}$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_4)^2 \oplus \mathbf{Z}_3$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^3$	$\mathbf{Z}_4 \oplus \mathbf{Z}_3$

7	8	9	10	11	12
$\mathbf{Z}_2 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_5$	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^2 \oplus (\mathbf{Z}_4)^2 \oplus \mathbf{Z}_3$	$(\mathbf{Z}_2)^4 \oplus (\mathbf{Z}_4)^2 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_7$	$(\mathbf{Z}_2)^7$

13	14	15	16
$(\mathbf{Z}_2)^4 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_3$	$(\mathbf{Z}_2)^2 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_5$	$(\mathbf{Z}_2)^3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_5$	$(\mathbf{Z}_2)^4 \oplus \mathbf{Z}_3$

17	18	19	20
$(\mathbf{Z}_2)^3 \oplus (\mathbf{Z}_4)^2 \oplus \mathbf{Z}_3$	$(\mathbf{Z}_2)^5 \oplus (\mathbf{Z}_4)^2 \oplus \mathbf{Z}_3$	$(\mathbf{Z}_2)^5 \oplus (\mathbf{Z}_4)^2 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_{11}$	$(\mathbf{Z}_2)^7$

Let $\mathrm{aut}(X)$ denote the space of self homotopy equivalences of X which are not necessarily preserving the base point, and $\mathrm{aut}_*(X)$ the space of based self homotopy equivalences. Then $\mathrm{aut}_*(X)$ is a submonoid of the monoid $\mathrm{aut}(X)$ whose operation is the composition. By [**12**, Theorem IIa, Theorem IIb] or [**1**, Corollary 6], we have $\pi_0(\mathrm{aut}_*(\mathbf{P}^3)) \cong \pi_0(\mathrm{aut}(\mathbf{P}^3)) \cong \mathbf{Z}_2$. The following assertion will be proved in §5.

Proposition 2.4. *If* $n \geq 1$, *then* $\pi_n(\mathrm{aut}_*(\mathbf{P}^3)) \cong \Gamma_n \oplus \pi_{n+3}(\mathrm{S}^3)$ *and* $\pi_n(\mathrm{aut}(\mathbf{P}^3)) \cong \Gamma_n \oplus \pi_{n+3}(\mathrm{S}^3) \oplus \pi_n(\mathbf{P}^3)$.

3. Proof of Proposition 2.1

As is well-known, we have $\mathbf{P}^3 = \mathrm{S}^1 \cup_{2\iota_1} e^2 \cup e^3$ and

$$\pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)) \cong [\mathbf{P}^3 \wedge \mathrm{S}^n, \mathbf{P}^3]. \tag{3.1}$$

It follows from (2.1) that the assertion of Proposition 2.1 is true for $n = 0, 1$. By [**4**, (3.1)] (or [**2**]), we have $\mathbf{P}^3 \wedge \mathrm{S}^n \simeq (\mathrm{S}^{n+1} \cup_{2\iota_{n+1}} e^{n+2}) \vee \mathrm{S}^{n+3}$ for $n \geq 2$. Therefore

$$\pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)) \cong [\mathrm{S}^{n+1} \cup_{2\iota_{n+1}} e^{n+2}, \mathbf{P}^3] \oplus \pi_{n+3}(\mathbf{P}^3) \quad \text{if} \ \ n \geq 2.$$

The covering map $p : \mathrm{S}^3 \to \mathbf{P}^3$ induces isomorphisms $\Gamma_n \cong [\mathrm{S}^{n+1} \cup_{2\iota_{n+1}} e^{n+2}, \mathbf{P}^3]$ for $n \geq 1$ and $\pi_{n+3}(\mathrm{S}^3) \cong \pi_{n+3}(\mathbf{P}^3)$ for $n \geq 0$. Hence we have Proposition 2.1 for $n \geq 2$. This completes the proof of Proposition 2.1.

4. Proof of Theorem 2.2

Let $\pi_k(\mathrm{S}^3; 2)$ be $\pi_3(\mathrm{S}^3)$ if $k = 3$ and the 2-primary subgroup of $\pi_k(\mathrm{S}^3)$ if $k \neq 3$.

By applying the cohomotopy functor $[\ \ , \mathrm{S}^3]$ to the cofibre sequence (2.2), we have the following exact sequence of abelian groups.

$$\pi_{n+2}(\mathrm{S}^3) \xrightarrow{(-2\iota_{n+2})^*} \pi_{n+2}(\mathrm{S}^3) \xrightarrow{q_n^*} \Gamma_n \xrightarrow{i_n^*} \pi_{n+1}(\mathrm{S}^3) \xrightarrow{(2\iota_{n+1})^*} \pi_{n+1}(\mathrm{S}^3).$$

Since $(2\iota_k)^* : \pi_k(\mathrm{S}^3) \to \pi_k(\mathrm{S}^3)$ is the multiplication by 2, $(-2\iota_k)^* = (2\iota_k)^*(-\iota_k)^* = -(2\iota_k)^*$ and $(-2)\pi_k(\mathrm{S}^3) = 2\pi_k(\mathrm{S}^3)$, it follows that the above exact sequence induces the following two exact sequences.

(4.1) $\pi_{n+2}(S^3) \xrightarrow{2} \pi_{n+2}(S^3) \xrightarrow{q_n^*} \Gamma_n \xrightarrow{i_n^*} \{\beta \in \pi_{n+1}(S^3) \mid 2\beta = 0\} \to 0,$

(4.2) $\pi_{n+2}(S^3;2) \xrightarrow{2} \pi_{n+2}(S^3;2) \xrightarrow{q_n^*} \Gamma_n \xrightarrow{i_n^*} \{\beta \in \pi_{n+1}(S^3;2) \mid 2\beta = 0\} \to 0.$

We will use the following known results.

LEMMA 4.1. (1) ([**10**, (2.3)]). *For every n, the suspension $E : \pi_n(S^3) \to \pi_{n+1}(S^4)$ is an isomorphism onto a direct summand, that is, there is a homomorphism $\varphi_n : \pi_{n+1}(S^4) \to \pi_n(S^3)$ such that $\varphi_n \circ E$ is the identity.*
(2) ([**18**, **9**]). *We have the following table*:

n	1, 2	3	4	5	6	7	8	9, 10	11	12	13
$\pi_n(S^3;2)$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_4$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_4$
generator		ι_3	η_3	η_3^2	ν'	$\nu'\eta_6$	$\nu'\eta_6^2$		ε_3	$\mu_3, \eta_3\varepsilon_4$	$\eta_3\mu_4, \varepsilon'$

n	14	15	16	17	18	19	20
$\pi_n(S^3;2)$	$(\mathbf{Z}_2)^2 \oplus \mathbf{Z}_4$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2 \oplus \mathbf{Z}_4$
generator	$\varepsilon_3\nu_{11}, \nu'\varepsilon_6, \mu'$	$\nu'\mu_6, \nu'\eta_6\varepsilon_7$	$\nu'\eta_6\mu_7$	$\varepsilon_3\nu_{11}^2$	$\bar{\varepsilon}_3$	$\mu_3\sigma_{12}, \eta_3\bar{\varepsilon}_4$	$\bar{\mu}_3, \eta_3\mu_4\sigma_{13}, \bar{\varepsilon}'$

n	21	22	23
$\pi_n(S^3;2)$	$(\mathbf{Z}_2)^2 \oplus \mathbf{Z}_4$	$\mathbf{Z}_2 \oplus \mathbf{Z}_4$	$(\mathbf{Z}_2)^2$
generator	$\nu'\bar{\varepsilon}_6, \eta_3\bar{\mu}_4, \mu'\sigma_{14}$	$\nu'\mu_6\sigma_{15}, \bar{\mu}'$	$\nu'\bar{\mu}_6, \nu'\eta_6\mu_7\sigma_{16}$

For each n, we can write $\{\beta \in \pi_{n+1}(S^3;2) \mid 2\beta = 0\} = (\mathbf{Z}_2)^m\{y_1, \dots, y_m\}$ with $m \geq 0$ and $\pi_{n+2}(S^3;2) = \mathbf{Z}_{2^{k_1}}\{x_1\} \oplus \cdots \oplus \mathbf{Z}_{2^{k_l}}\{x_l\}$ with $l \geq 0$ and $k_i \geq 1$ for every $i \leq l$. Hence (4.2) induces the following exact sequence:

(4.3) $0 \to (\mathbf{Z}_2)^l\{q_n^*(x_1), \dots, q_n^*(x_l)\} \xrightarrow{\subset} \Gamma_n \xrightarrow{i_n^*} (\mathbf{Z}_2)^m\{y_1, \dots, y_m\} \to 0.$

The following result can be proved easily. So we omit its proof.

LEMMA 4.2. (1) *If $2\overline{y_j} = 0$ for all j in* (4.3), *then*

$$\Gamma_n = (\mathbf{Z}_2)^{l+m}\{q_n^*(x_1), \dots, q_n^*(x_l), \overline{y_1}, \dots, \overline{y_m}\}.$$

(2) *If $2\overline{y_j} = 0$ for all $j < m$ and $2\overline{y_m} = q_n^*(x_l)$ in* (4.3), *then*

$$\Gamma_n = (\mathbf{Z}_2)^{l+m-2}\{q_n^*(x_1), \dots, q_n^*(x_{l-1}), \overline{y_1}, \dots, \overline{y_{m-1}}\} \oplus \mathbf{Z}_4\{\overline{y_m}\}.$$

In order to determine the group extension of (4.3), we will compute $2\overline{y_j}$ $(1 \leq j \leq m)$ by using the following two lemmas.

Lemma 4.3. (1) ([**18**]). $\pi_{n+4}(S^n;2)=0$ *for* $n\geq 6$, $\pi_{n+5}(S^n;2)=0$ *for* $n\geq 7$, $\pi_{n+6}(S^n;2)=\mathbf{Z}_2\{\nu_n^2\}$ *for* $n\geq 5$, $2\nu'=\eta_3^3$, $4\nu_5=\eta_5^3$, $\eta_{10}\sigma_{11}=\sigma_{10}\eta_{17}$, $\eta_3\varepsilon_4=\varepsilon_3\eta_{11}$, $2\varepsilon'=\eta_3^2\varepsilon_5$, $2\mu'=\eta_3^2\mu_5$, $\nu'\varepsilon_6=\varepsilon'\eta_{13}$, $\eta_3\bar{\varepsilon}_4=\bar{\varepsilon}_3\eta_{18}=\varepsilon_3^2=\varepsilon_3\bar{\nu}_{11}$, $2\bar{\varepsilon}'=\eta_3^2\bar{\varepsilon}_5$, $2\bar{\mu}'=\eta_3^2\bar{\mu}_5$.

(2) ([**11**]). $\eta_3\mu_4=\mu_3\eta_{12}$, $\nu'\mu_6=\mu'\eta_{14}$, $\nu'\bar{\varepsilon}_6=\bar{\varepsilon}_3\nu_{18}$, $\eta_3\bar{\mu}_4=\bar{\mu}_3\eta_{20}$.

Proof. By Propositions 5.8 and 5.9, (5.3), (5.5), (7.5), (7.7), (7.12), Lemmas 6.4, 6.6, 12.3, 12.4 and 12.10 of [**18**], we have (1). We have (2) from Proposition (2.2)(2),(4) and Proposition (2.17)(4),(10) of [**11**] which were proved by standard methods of [**18**]. □

Lemma 4.4. *If* $\beta\in\pi_{n+1}(S^3;2)$ *is of order* 2, *then every* $\bar{\beta}\in[S^{n+1}\cup_{2\iota_{n+1}} e^{n+2},S^3]$ *satisfies* $2\bar{\beta}=q_n^*(\beta\circ\eta_{n+1})$.

Proof. Take $x\in\{2\iota_3,\beta,2\iota_{n+1}\}_0$ arbitrarily, where $\{\gamma,E^k\delta,E^k\varepsilon\}_k$ is the Toda bracket [**18**]. Then $\{2\iota_3,\beta,2\iota_{n+1}\}_0=x+2\pi_{n+2}(S^3)$ and

$$\begin{aligned}2\bar{\beta}&=2\iota_3\circ\bar{\beta}\in\{2\iota_3,\beta,2\iota_{n+1}\}_0\circ q_n \quad (\text{by [18, Proposition 1.9]})\\&=\{q_n^*(x)\}\quad(\text{by (4.1)}),\end{aligned}$$

that is, $2\bar{\beta}=q_n^*(x)$. We have

$$\begin{aligned}Ex&\in E\{2\iota_3,\beta,2\iota_{n+1}\}_0\subset-\{2\iota_4,E\beta,2\iota_{n+2}\}_1\quad(\text{by [18, Proposition 1.3]})\\&=E(\beta\eta_{n+1})+2\pi_{n+3}(S^4)\quad(\text{by [18, Corollary 3.7]}).\end{aligned}$$

Hence there exists $y\in\pi_{n+3}(S^4)$ such that $Ex=E(\beta\eta_{n+1})+2y$, that is, $E(x-\beta\eta_{n+1})=2y$. We have $x-\beta\eta_{n+1}=\varphi_{n+2}E(x-\beta\eta_{n+1})=2\varphi_{n+2}(y)\in 2\pi_{n+2}(S^3)$ by Lemma 4.1(1) and so $q_n^*(x-\beta\eta_{n+1})=0$ by (4.1). Therefore $q_n^*(x)=q_n^*(\beta\eta_{n+1})$. Thus $2\bar{\beta}=q_n^*(\beta\eta_{n+1})$. This completes the proof. □

4.1. Γ_n **for** $n=0,1,2,7,8,9$**.** By (4.3) and Lemma 4.1(2), we obtain the results.

4.2. Γ_3**.** By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0\to\mathbf{Z}_2\{q_3^*(\eta_3^2)\}\xrightarrow{\subset}\Gamma_3\xrightarrow{i_3^*}\mathbf{Z}_2\{\eta_3\}\to 0.$$

By setting $\beta=\eta_3$ in Lemma 4.4, we have

$$2\overline{\eta_3}=q_3^*(\eta_3^2). \tag{4.4}$$

Hence we have the result by Lemma 4.2(2). From now on, we will denote $E^n\overline{\eta_3}$ by $\overline{\eta_{n+3}}$.

4.3. Γ_n **for** $n = 4, 15, 16$. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to \mathbf{Z}_2\{q_4^* \nu'\} \xrightarrow{\subset} \Gamma_4 \xrightarrow{i_4^*} \mathbf{Z}_2\{\eta_3^2\} \to 0.$$

Since $i_4^*(\eta_3\overline{\eta_4}) = \eta_3^2$ and $2(\eta_3\overline{\eta_4}) = (2\eta_3)\overline{\eta_4} = 0$, we obtain the result for $n = 4$ by Lemma 4.2(1). By similar reason, we obtain the results for $n = 15, 16$.

4.4. Γ_5. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to \mathbf{Z}_2\{q_5^*(\nu'\eta_6)\} \xrightarrow{\subset} \Gamma_5 \xrightarrow{i_5^*} \mathbf{Z}_2\{2\nu'\} \to 0.$$

Since $i_5^*(\eta_3^2\overline{\eta_5}) = \eta_3^3 = 2\nu'$ by Lemma 4.3(1) and $2(\eta_3^2\overline{\eta_5}) = (2\eta_3^2)\overline{\eta_5} = 0$, we have the result by Lemma 4.2(1).

4.5. Γ_6. By (4.3) and Lemma 4.1(2), we have the following short exact sequence:

$$0 \to \mathbf{Z}_2\{q_6^*(\nu'\eta_6^2)\} \xrightarrow{\subset} \Gamma_6 \xrightarrow{i_6^*} \mathbf{Z}_2\{\nu'\eta_6\} \to 0.$$

Since $i_6^*(\nu'\overline{\eta_6}) = \nu'\eta_6$ and $2(\nu'\overline{\eta_6}) = \nu'(2\overline{\eta_6}) = \nu'(\eta_6^2 q_6) = q_6^*(\nu'\eta_6^2)$ by (4.4), we have the result by Lemma 4.2(2).

4.6. Γ_{10}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^2\{q_{10}^*\mu_3, q_{10}^*(\eta_3\varepsilon_4)\} \xrightarrow{\subset} \Gamma_{10} \xrightarrow{i_{10}^*} \mathbf{Z}_2\{\varepsilon_3\} \to 0.$$

We have $2\overline{\varepsilon_3} = q_{10}^*(\varepsilon_3\eta_{11}) = q_{10}^*(\eta_3\varepsilon_4)$ by Lemma 4.4 and Lemma 4.3(1). Hence we obtain the result by Lemma 4.2(2). From now on, we will denote $E^n\overline{\varepsilon_3}$ by $\overline{\varepsilon_{n+3}}$.

4.7. Γ_{11}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^2\{q_{11}^*\varepsilon', q_{11}^*(\eta_3\mu_4)\} \xrightarrow{\subset} \Gamma_{11} \xrightarrow{i_{11}^*} (\mathbf{Z}_2)^2\{\mu_3, \eta_3\varepsilon_4\} \to 0.$$

We have

$$2\overline{\mu_3} = q_{11}^*(\mu_3\eta_{12}) = q_{11}^*(\eta_3\mu_4) \tag{4.5}$$

by Lemma 4.4 and Lemma 4.3(2). On the other hand, $i_{11}^*(\eta_3\overline{\varepsilon_4}) = \eta_3\varepsilon_4$ and $2(\eta_3\bar{\varepsilon}_4) = (2\eta_3)\bar{\varepsilon}_4 = 0$. Hence we obtain the result by Lemma 4.2(2). From now on, we will denote $E^n\overline{\mu_3}$ by $\overline{\mu_{n+3}}$.

4.8. Γ_{12}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^3\{q_{12}^*\mu', q_{12}^*(\varepsilon_3\nu_{11}), q_{12}^*(\nu'\varepsilon_6)\} \xrightarrow{\subset} \Gamma_{12} \xrightarrow{i_{12}^*} (\mathbf{Z}_2)^2\{2\varepsilon', \mu_3\eta_{12}\} \to 0.$$

We have $i_{12}^*(\eta_3^2\overline{\varepsilon_5}) = \eta_3^2\varepsilon_5 = 2\varepsilon'$ by Lemma 4.3(1) and $2(\eta_3^2\overline{\varepsilon_5}) = (2\eta_3^2)\overline{\varepsilon_5} = 0$. We have $i_{12}^*(\mu_3\overline{\eta_{12}}) = \mu_3\eta_{12}$ and $2(\mu_3\overline{\eta_{12}}) = (2\mu_3)\overline{\eta_{12}} = 0$. Hence we obtain the result by Lemma 4.2(1).

4.9. Γ_{13}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^2\{q_{13}^*(\nu'\mu_6), q_{13}^*(\nu'\eta_6\varepsilon_7)\} \xrightarrow{\subset} \Gamma_{13} \xrightarrow{i_{13}^*} (\mathbf{Z}_2)^3\{2\mu', \varepsilon_3\nu_{11}, \nu'\varepsilon_6\} \to 0.$$

We have $i_{13}^*(\mu_3\eta_{12}\overline{\eta_{13}}) = \mu_3\eta_{12}^2 = 2\mu'$ by Lemma 4.3, and $2(\mu_3\eta_{12}\overline{\eta_{13}}) = (2\mu_3)\eta_{12}\overline{\eta_{13}} = 0$. By setting $\beta = \varepsilon_3\nu_{11}$, $\nu'\varepsilon_6$ in Lemma 4.4, we have $2\overline{\varepsilon_3\nu_{11}} = q_{13}^*(\varepsilon_3\nu_{11}\eta_{14}) = 0$ and $2(\nu'\overline{\varepsilon_6}) = q_{13}^*(\nu'\varepsilon_6\eta_{14}) = q_{13}^*(\nu'\eta_6\varepsilon_7)$ from Lemma 4.3(1). Hence we obtain the result by Lemma 4.2(2).

4.10. Γ_{14}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to \mathbf{Z}_2\{q_{14}^*(\nu'\eta_6\mu_7)\} \xrightarrow{\subset} \Gamma_{14} \xrightarrow{i_{14}^*} (\mathbf{Z}_2)^2\{\nu'\mu_6, \nu'\eta_6\varepsilon_7\} \to 0.$$

We have $i_{14}^*(\nu'\overline{\mu_6}) = \nu'\mu_6$ and $2(\nu'\overline{\mu_6}) = \nu'(2\overline{\mu_6}) = \nu'(\eta_6\mu_7 q_{14}) = q_{14}^*(\nu'\eta_6\mu_7)$ by (4.5). We have $i_{14}^*(\nu'\eta_6\overline{\varepsilon_7}) = \nu'\eta_6\varepsilon_7$ and $2(\nu'\eta_6\overline{\varepsilon_7}) = \nu'(2\eta_6)\overline{\varepsilon_7} = 0$. Hence we obtain the result by Lemma 4.2(2).

4.11. Γ_{17}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^2\{q_{17}^*(\mu_3\sigma_{12}), q_{17}^*(\eta_3\bar{\varepsilon}_4)\} \xrightarrow{\subset} \Gamma_{17} \xrightarrow{i_{17}^*} \mathbf{Z}_2\{\bar{\varepsilon}_3\} \to 0.$$

Since $2\overline{\bar{\varepsilon}_3} = q_{17}^*(\bar{\varepsilon}_3\eta_{18}) = q_{17}^*(\eta_3\bar{\varepsilon}_4)$ by Lemma 4.4 and Lemma 4.3(1), we obtain the result by Lemma 4.2(2). From now on, we will denote $E^n\overline{\bar{\varepsilon}_3}$ by $\overline{\bar{\varepsilon}_{n+3}}$.

4.12. Γ_{18}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^3\{q_{18}^*\bar{\varepsilon}', q_{18}^*\bar{\mu}_3, q_{18}^*(\eta_3\mu_4\sigma_{13})\} \xrightarrow{\subset} \Gamma_{18} \xrightarrow{i_{18}^*} (\mathbf{Z}_2)^2\{\mu_3\sigma_{12}, \eta_3\bar{\varepsilon}_4\} \to 0.$$

We have $2\overline{\mu_3\sigma_{12}} = q_{18}^*(\mu_3\sigma_{12}\eta_{19}) = q_{18}^*(\eta_3\mu_4\sigma_{13})$ by Lemma 4.4 and Lemma 4.3. We have $i_{18}^*(\bar{\varepsilon}_3\overline{\eta_{18}}) = \bar{\varepsilon}_3\eta_{18} = \eta_3\bar{\varepsilon}_4$ by Lemma 4.3(1) and $2(\bar{\varepsilon}_3\overline{\eta_{18}}) = (2\bar{\varepsilon}_3)\overline{\eta_{18}} = 0$. Hence we obtain the result by Lemma 4.2(2). From now on, we will denote $E^n\overline{\mu_3\sigma_{12}}$ by $\overline{\mu_{n+3}\sigma_{n+12}}$.

4.13. Γ_{19}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^3\{q_{19}^*(\mu'\sigma_{14}), q_{19}^*(\nu'\bar{\varepsilon}_6), q_{19}^*(\eta_3\bar{\mu}_4)\} \xrightarrow{\subset} \Gamma_{19} \xrightarrow{i_{19}^*} (\mathbf{Z}_2)^3\{2\bar{\varepsilon}', \bar{\mu}_3, \eta_3\mu_4\sigma_{13}\} \to 0.$$

We have $i_{19}^*(\bar{\varepsilon}_3\eta_{18}\overline{\eta_{19}}) = \bar{\varepsilon}_3\eta_{18}^2 = 2\bar{\varepsilon}'$ by Lemma 4.3 and $2(\bar{\varepsilon}_3\eta_{18}\overline{\eta_{19}}) = (2\bar{\varepsilon}_3)\eta_{18}\overline{\eta_{19}} = 0$. We have $2\overline{\bar{\mu}_3} = q_{19}^*(\bar{\mu}_3\eta_{20}) = q_{19}^*(\eta_3\bar{\mu}_4)$ by Lemma 4.4 and Lemma 4.3(2). We have $i_{19}^*(\mu_3\sigma_{12}\overline{\eta_{19}}) = \mu_3\sigma_{12}\eta_{19} = \eta_3\mu_4\sigma_{13}$ by Lemma 4.3

and $2(\mu_3\sigma_{12}\overline{\eta_{19}}) = (2\mu_3)\sigma_{12}\overline{\eta_{19}} = 0$. Hence we obtain the result by Lemma 4.2(2). From now on, we will denote $E^n\overline{\mu}_3$ by $\overline{\mu_{n+3}}$.

4.14. Γ_{20}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^2\{q_{20}^*(\bar{\mu}'), q_{20}^*(\nu'\mu_6\sigma_{15})\} \xrightarrow{\subset} \Gamma_{20} \xrightarrow{i_{20}^*} (\mathbf{Z}_2)^3\{2(\mu'\sigma_{14}), \nu'\bar{\varepsilon}_6, \eta_3\bar{\mu}_4\} \to 0.$$

We have $i_{20}^*(\mu_3\sigma_{12}\eta_{19}\overline{\eta_{20}}) = \mu_3\sigma_{12}\eta_{19}^2 = 2(\mu'\sigma_{14})$ by Lemma 4.3 and $2(\mu_3\sigma_{12}\eta_{19}\overline{\eta_{20}}) = (2\mu_3)\sigma_{12}\eta_{19}\overline{\eta_{20}} = 0$. We have $i_{20}^*(\nu'\overline{\bar{\varepsilon}_6}) = \nu'\bar{\varepsilon}_6$ and $i_{20}^*(\eta_3\overline{\bar{\mu}_4}) = \eta_3\bar{\mu}_4$. It follows from Lemma 4.3 and Lemma 4.4 that $2(\nu'\overline{\bar{\varepsilon}_6}) = q_{20}^*(\nu'\bar{\varepsilon}_6\eta_{21}) = q_{20}^*(\bar{\varepsilon}_3\nu_{18}\eta_{21}) = 0$ and $2(\eta_3\overline{\bar{\mu}_4}) = q_{20}^*(\eta_3\bar{\mu}_4\eta_{21}) = q_{20}^*(\eta_3^2\bar{\mu}_5) = q_{20}^*(2\bar{\mu}') = 0$. Therefore we obtain the result by Lemma 4.2(1).

4.15. Γ_{21}. By (4.3) and Lemma 4.1(2), we have the following exact sequence:

$$0 \to (\mathbf{Z}_2)^2\{q_{21}^*(\nu'\bar{\mu}_6), q_{21}^*(\nu'\eta_6\mu_7\sigma_{16})\} \xrightarrow{\subset} \Gamma_{21} \xrightarrow{i_{21}^*} (\mathbf{Z}_2)^2\{2\bar{\mu}', \nu'\mu_6\sigma_{15}\} \to 0.$$

We have $i_{21}^*(\bar{\mu}_3\eta_{20}\overline{\eta_{21}}) = \bar{\mu}_3\eta_{20}^2 = 2\bar{\mu}'$ by Lemma 4.3 and $2(\bar{\mu}_3\eta_{20}\overline{\eta_{21}}) = (2\bar{\mu}_3)\eta_{20}\overline{\eta_{21}} = 0$. We have $i_{21}^*(\nu'\overline{\mu_6\sigma_{15}}) = \nu'\mu_6\sigma_{15}$ and $2(\nu'\overline{\mu_6\sigma_{15}}) = q_{21}^*(\nu'\mu_6\sigma_{15}\eta_{22}) = q_{21}^*(\nu'\eta_6\mu_7\sigma_{16})$ by Lemmas 4.4 and 4.3. Hence we obtain the result by Lemma 4.2(2). This completes the proof of Theorem 2.2.

5. Proof of Proposition 2.4

Let $\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3)$ denote the space of self maps of $\mathbf{P}^3$ not necessarily preserving base point. This is a monoid with respect to the composition operation. Borsuk's fibre theorem [**5**, Proposition (6.34)] says that the evaluation map $ev : \mathrm{map}(\mathbf{P}^3, \mathbf{P}^3) \to \mathbf{P}^3$, $f \mapsto f(*)$, is a fibration whose fibre is $\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)$, where the base point $*$ is the unit of the group $\mathbf{P}^3$. By [**12**, Theorems IIa, IIb], we have

$$\begin{aligned}\pi_0(\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3)) &= \pi_0(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)) \\ &= [\mathbf{P}^3, \mathbf{P}^3] \underset{\cong}{\xrightarrow{\xi}} \mathrm{Hom}(H^3(\mathbf{P}^3; \mathbf{Z}), H^3(\mathbf{P}^3; \mathbf{Z})) \cong \mathbf{Z}\end{aligned}$$

where ξ assigns f^* to the homotopy class of $f \in \mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)$. Hence $\mathrm{aut}(\mathbf{P}^3)$ and $\mathrm{aut}_*(\mathbf{P}^3)$ consist of two path components of $\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3)$ and $\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)$, respectively. Therefore $\pi_0(\mathrm{aut}(\mathbf{P}^3)) \cong \pi_0(\mathrm{aut}_*(\mathbf{P}^3)) \cong \mathbf{Z}_2$ and $\pi_n(\mathrm{aut}(\mathbf{P}^3), 1_{\mathbf{P}^3}) \cong \pi_n(\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3), 1_{\mathbf{P}^3})$ and $\pi_n(\mathrm{aut}_*(\mathbf{P}^3), 1_{\mathbf{P}^3}) \cong \pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3), 1_{\mathbf{P}^3})$ for $n \geq 1$. For any $x \in \mathbf{P}^3$, let $L_x : \mathbf{P}^3 \to \mathbf{P}^3$ denote the map $y \mapsto xy$. Then the map $\mathbf{P}^3 \to \mathrm{map}(\mathbf{P}^3, \mathbf{P}^3)$, $x \mapsto L_x$, is a cross section of ev. Hence the homotopy exact sequence of the fibration ev splits so that $\pi_n(\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3), 1_{\mathbf{P}^3}) \cong \pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3), 1_{\mathbf{P}^3}) \oplus \pi_n(\mathbf{P}^3)$. Since all path-components of $\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3)$ have the same homotopy type, $\pi_n(\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3), f) \cong \pi_n(\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3), 0)$ for every

$f \in \mathrm{map}(\mathbf{P}^3, \mathbf{P}^3)$. Similarly all path components of the following spaces have the same homotopy type, respectively: $\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)$, $\mathrm{aut}(\mathbf{P}^3)$ and $\mathrm{aut}_*(\mathbf{P}^3)$. Therefore if $n \geq 1$, then we have $\pi_n(\mathrm{aut}_*(\mathbf{P}^3)) \cong \pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3))$ and $\pi_n(\mathrm{aut}(\mathbf{P}^3)) \cong \pi_n(\mathrm{map}(\mathbf{P}^3, \mathbf{P}^3)) \cong \pi_n(\mathrm{map}_*(\mathbf{P}^3, \mathbf{P}^3)) \oplus \pi_n(\mathbf{P}^3)$. Hence we obtain Proposition 2.4 by Proposition 2.1.

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