

A CLASS OF UNIVALENT FUNCTIONS DEFINED BY A DIFFERENTIAL INEQUALITY

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Abstract

Let \mathcal{A} be the class of analytic functions in the unit disk \mathbf{D} with the normalization $f(0) = f'(0) - 1 = 0$. For $\lambda > 0$, denote by $\mathcal{M}(\lambda)$ the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\left| z^2 \left(\frac{z}{f(z)} \right)'' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, \quad z \in \mathbf{D}.$$

We show that functions in $\mathcal{M}(1)$ are univalent in \mathbf{D} and we present one parameter family of functions in $\mathcal{M}(1)$ that are also starlike in \mathbf{D} . In addition to certain inclusion results, we also present characterization formula, necessary and sufficient coefficient conditions for functions in $\mathcal{M}(\lambda)$, and a radius property of $\mathcal{M}(1)$.

1. Introduction and main results

Let \mathcal{H} be the class of analytic functions in the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$, mapping \mathbf{D} into the complex plane \mathbf{C} and \mathcal{A} be the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in \mathcal{H} . Let \mathcal{S} denote the class of functions f in \mathcal{A} such that f is univalent in \mathbf{D} . For $\lambda > 0$, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{U}(\lambda)$ if

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, \quad z \in \mathbf{D}.$$

Denote by $\mathcal{P}(\lambda)$, the subclass of \mathcal{A} , consisting of functions f for which

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2\lambda, \quad z \in \mathbf{D}.$$

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Set $\mathcal{U}(1) := \mathcal{U}$ and $\mathcal{P}(1) := \mathcal{P}$, see [4, 9]. We have the strict inclusion $\mathcal{P} \subsetneq \mathcal{U} \subsetneq \mathcal{S}$ (see [1, 4, 10] for a proof). Many properties of the classes $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ have been studied extensively in [5, 6, 7, 8, 9]. More generally

$$\mathcal{P}(\lambda) \subsetneq \mathcal{U}(\lambda) \subsetneq \mathcal{S} \quad \text{for } 0 < \lambda \leq 1$$

and for a proof of this inclusion, we refer to [5]. Also, it is well-known that there are only nine functions in \mathcal{S} having integral coefficients in the power series expansions of $f \in \mathcal{S}$ (see [3]). That is, if we set $\mathcal{S}_{\mathbf{Z}} = \{f \in \mathcal{S} : a_n \in \mathbf{Z}\}$, then

$$\mathcal{S}_{\mathbf{Z}} = \left\{ z, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2} \right\}.$$

Further, it is easy to see that the corresponding $g \in \mathcal{S}_{\mathbf{Z}}$ have the property that

$$g'(z) \left(\frac{z}{g(z)} \right)^2 - 1 \in \{0, -z^2, 0, \mp z^2, -z^2\} \quad \text{and} \quad z^2 \left(\frac{z}{g(z)} \right)'' \in \{0, 2z^2, 0, \mp 2z^2, 2z^2\},$$

respectively. Consequently, we obtain the interesting fact that each function in $\mathcal{S}_{\mathbf{Z}}$ belongs to $\mathcal{U} \cap \mathcal{P}$. Finally, we observe that

$$z^2 \left(\frac{z}{g(z)} \right)'' + g'(z) \left(\frac{z}{g(z)} \right)^2 - 1 \in \{0, z^2, 0, \mp z^2, z^2\}.$$

In view of this observation, we introduce the following:

DEFINITION 1. For $\lambda > 0$, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{M}(\lambda)$ if $|M_f(z)| \leq \lambda$ for $z \in \mathbf{D}$, where

$$(1) \quad M_f(z) = z^2 \left(\frac{z}{f(z)} \right)'' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1.$$

Also, denote the class $\mathcal{M}(1)$ by \mathcal{M} .

Now, we state our main results and the proofs of these will be given in Section 3.

THEOREM 1 (Inclusion property). For $0 < \lambda \leq 1$, we have the strict inclusion $\mathcal{M}(\lambda) \subsetneq \mathcal{U}(\lambda) \cap \mathcal{P}(\lambda) \subsetneq \mathcal{S}$. In particular, $\mathcal{M} \subsetneq \mathcal{U} \cap \mathcal{P} = \mathcal{P} \subsetneq \mathcal{S}$.

From the earlier discussion and Theorem 1, we easily see that

$$\mathcal{S}_{\mathbf{Z}} \subsetneq \mathcal{M} \subsetneq \mathcal{P} \subsetneq \mathcal{U} \subsetneq \mathcal{S}$$

and it is worth recalling that the Koebe function belongs to \mathcal{M} .

Example 1. Consider the function f defined by

$$\frac{z}{f(z)} = 1 + \frac{1}{2}z + \frac{\lambda}{2}z^3$$

where $0 < \lambda \leq 1$. Then

$$\left| \frac{z}{f(z)} \right| \geq 1 - \frac{1}{2}|z| - (\lambda/2)|z|^3 > \frac{1-\lambda}{2} \geq 0$$

and so $z/f(z) \neq 0$ in \mathbf{D} whenever $0 < \lambda \leq 1$. Further

$$f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 = -\lambda z^3 \quad \text{and} \quad M_f(z) = 2\lambda z^3$$

so that there exists a function $f \in \mathcal{U}(\lambda)$ such that $f \notin \mathcal{M}(\lambda)$. Also, for each μ with $|\mu| \leq 1/2$, it is easy to see that the function f defined by

$$f(z) = \frac{z}{1 + \mu z + \frac{1}{2}z^3}$$

belongs $\mathcal{U} \setminus \mathcal{M}$.

THEOREM 2 (Sufficiency coefficient condition). *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in \mathbf{D} that satisfy the coefficient condition*

$$(2) \quad \sum_{n=2}^{\infty} (n-1)^2 |b_n| \leq \lambda.$$

Then the function f defined by $f(z) = z/\phi(z)$ is in $\mathcal{M}(\lambda)$.

For example, according to (2) with $\lambda = 1$, each function in $\mathcal{S}_{\mathbf{Z}}$ belongs to \mathcal{M} .

Let \mathcal{S}^* denote the class of univalent functions in $f \in \mathcal{S}$ such that the range $f(\mathbf{D})$ is a starlike domain (with respect to the origin). Analytically, $f \in \mathcal{S}^*$ if and only if $\operatorname{Re}(zf'(z)/f(z)) > 0$ in \mathbf{D} .

It is easy to see that each $g \in \mathcal{S}_{\mathbf{Z}}$ is starlike in \mathbf{D} . Also, it has been shown that for arbitrarily small values of λ we have $\mathcal{U}(\lambda) \not\subset \mathcal{S}^*$. Indeed, Fournier and Ponnusamy [2, Theorem 3] obtained that every function $f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n \in \mathcal{A}$ satisfying

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| < \frac{-|a_2(f)| + \sqrt{2 - |a_2(f)|^2}}{2}, \quad |z| < 1,$$

belongs to \mathcal{S}^* . Moreover, there exists a non-starlike function $f \in \mathcal{U}$ such that

$$0 < \frac{-|a_2(f)| + \sqrt{2 - |a_2(f)|^2}}{2} < \sup_{|z| < 1} \left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq 1 - |a_2(f)|.$$

In particular, $\mathcal{U} \not\subset \mathcal{S}^*$. Moreover, Theorem 1 shows that $\mathcal{M} \subsetneq \mathcal{U}$ and therefore, it is natural to ask whether the class \mathcal{M} is included in \mathcal{S}^* . This remains an open question.

If f and g are analytic functions on \mathbf{D} with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the convolution (Hadamard product) of f and g , denoted by $f * g$, is an analytic function on \mathbf{D} given by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathbf{D}.$$

Although \mathcal{U} is not included in \mathcal{M} , in the following result, we show that the class \mathcal{U} can be used to construct functions belonging to \mathcal{M} .

THEOREM 3 (Multiplier theorem). *Let $f \in \mathcal{U}(\lambda_1)$ and $g \in \mathcal{U}(\lambda_2)$ have the form*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \quad \text{and} \quad \frac{z}{g(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

*and such that $\frac{z}{f(z)} * \frac{z}{g(z)} \neq 0$ on \mathbf{D} . Then the function H defined by*

$$H(z) = \frac{z}{(z/f(z)) * (z/g(z))}$$

is in the class $\mathcal{M}(\lambda)$, where $\lambda = \lambda_1 \lambda_2$. In particular, if $f, g \in \mathcal{U}$ then $H \in \mathcal{M}$.

COROLLARY 1 (Necessary coefficient condition). *Let $f \in \mathcal{M}$ of the form*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots.$$

Then we have

$$\sum_{n=2}^{\infty} (n-1)^4 |b_n|^2 \leq 1.$$

At this place it is appropriate to remind the reader of the fact that the inequality $\sum_{n=2}^{\infty} (n-1) |b_n|^2 \leq 1$ follows merely from the condition $f \in \mathcal{S}$ for the expansion $\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$. This result is known as the Prawitz theorem which is indeed an immediate consequence of Gronwall's area theorem. Thus, the necessary condition given in Corollary 1 is much stronger than this result.

THEOREM 4 (Characterization theorem). *Every $f \in \mathcal{M}(\lambda)$ has the representation*

$$\frac{z}{f(z)} = 1 - \frac{f''(0)}{2} z + \lambda \int_0^1 \frac{w(tz)}{t^2} \log(1/t) dt,$$

for some $w : \mathbf{D} \rightarrow \mathbf{D}$ with $w(0) = w'(0) = 0$.

Let \mathcal{F} and \mathcal{G} be two subclasses of \mathcal{A} . If for every $f \in \mathcal{F}$, $r^{-1}f(rz) \in \mathcal{G}$ for $r \leq r_0$, and r_0 is the maximum value for which this holds, then we say that r_0 is the \mathcal{G} -radius in \mathcal{F} . There are many results of this type in the theory of univalent functions, see [8] and the references therein.

Because $\mathcal{M} \subsetneq \mathcal{U}$, it is natural to investigate the \mathcal{M} -radius in \mathcal{U} .

THEOREM 5 (Radii property). *If $f \in \mathcal{U}$ and $g(z) = \frac{1}{r}f(rz)$, then $g \in \mathcal{M}$ for $0 < r \leq r_0$, where $r_0 \approx 0.62977$ is the unique positive root of the equation $2r^6 - 2r^4 + 3r^2 - 1 = 0$.*

2. Preliminary lemmas

Let \mathcal{P}_n denote the class of functions p in \mathcal{H} such that $p^{(k)}(0) = 0$ for $k = 0, 1, 2, \dots, n$, where $p^{(0)}(0) = p(0)$. We set

$$\mathcal{B}_n = \{w \in \mathcal{H} : |w(z)| \leq 1, w^{(k)}(0) = 0 \text{ for } k = 0, 1, \dots, n\}.$$

LEMMA 1. *Suppose that $p \in \mathcal{P}_n$, $\lambda > 0$ and α is a complex number such that $\operatorname{Re}(1/(1-\alpha)) > -n$. If p satisfies the condition*

$$(3) \quad |(1-\alpha)zp'(z) + \alpha p(z)| \leq \lambda, \quad z \in \mathbf{D}$$

then

$$|p(z)| \leq \frac{\lambda|z|^{n+1}}{|1-\alpha|(n+\operatorname{Re}(1/(1-\alpha)))}, \quad z \in \mathbf{D}.$$

Proof. First, we rewrite (3) as

$$(1-\alpha)zp'(z) + \alpha p(z) = \lambda w(z),$$

where $w \in \mathcal{B}_n$. Now, by integration, we get

$$p(z) = \frac{\lambda}{1-\alpha} \int_0^1 t^{(\alpha/(1-\alpha))-1} w(tz) dt.$$

Because $|w(z)| \leq |z|^{n+1}$ for $z \in \mathbf{D}$ by Schwarz' lemma, we obtain that

$$|p(z)| \leq \frac{\lambda}{|1-\alpha|} \left(\frac{|z|^{n+1}}{n+1+\operatorname{Re}(\alpha/(1-\alpha))} \right), \quad z \in \mathbf{D}$$

and the desired conclusion follows. \square

COROLLARY 2. *Suppose that $p \in \mathcal{P}_n$, $\lambda > 0$ and $\alpha \neq 1$ is a real number such that $n+1/(1-\alpha) > 0$. Then*

$$(4) \quad |(1-\alpha)zp'(z) + \alpha p(z)| \leq \lambda \Rightarrow |p(z)| \leq \frac{\lambda(1-\alpha)}{|1-\alpha|(n(1-\alpha)+1)}$$

for $z \in \mathbf{D}$.

Suppose that $0 \neq \alpha < 1$. Then (4) becomes

$$\left| \left(\frac{1}{\alpha} - 1 \right) zp'(z) + p(z) \right| \leq \frac{\lambda}{|\alpha|} \Rightarrow |p(z)| \leq \frac{\lambda}{n(1-\alpha)+1}, \quad z \in \mathbf{D}.$$

Now, if we allow $\alpha \rightarrow -\infty$, then the last relation gives that

$$|-zp'(z) + p(z)| \leq 0 \Rightarrow |p(z)| \leq 0, \quad z \in \mathbf{D}$$

so that $p(z) = 0$ is the only solution which satisfies the above implication. Now, we state an improved version of it.

LEMMA 2. Suppose that $p \in \mathcal{P}_n$ ($n \geq 1$) satisfies the condition

$$(5) \quad |-zp'(z) + p(z)| \leq \lambda, \quad z \in \mathbf{D}$$

for some $\lambda > 0$. Then we have

$$|p(z)| \leq \frac{\lambda |z|^{n+1}}{n} \quad \text{and} \quad |zp'(z)| \leq \lambda |z|^{n+1} \left(1 + \frac{1}{n} \right), \quad z \in \mathbf{D}.$$

Proof. The condition (5) implies that

$$-zp'(z) + p(z) = \lambda w(z)$$

where $w \in \mathcal{B}_n$. It follows easily that

$$p(z) = -\lambda \int_0^1 t^{-2} w(tz) dt \quad \text{and} \quad -zp'(z) = \lambda w(z) + \lambda \int_0^1 t^{-2} w(tz) dt.$$

Because $|w(z)| \leq |z|^{n+1}$ for $z \in \mathbf{D}$, by Schwarz' lemma, the desired conclusion follows from the last two formulas. \square

We state the above lemmas in a general form in order to apply them for functions with missing coefficients. However, for our application the case $n = 1$ suffices. Setting $n = 1$ in Lemma 2, we have

COROLLARY 3. Suppose that p is analytic in \mathbf{D} , $p(0) = p'(0) = 0$ and satisfies the condition

$$(6) \quad |-zp'(z) + p(z)| \leq \lambda, \quad z \in \mathbf{D}$$

for some $\lambda > 0$. Then we have

$$|p(z)| \leq \lambda |z|^2 \quad \text{and} \quad |zp'(z)| \leq 2\lambda |z|^2, \quad z \in \mathbf{D}.$$

3. Proofs

Proof of Theorem 1. Set

$$(7) \quad p(z) = \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 = -z \left(\frac{z}{f(z)} \right)' + \frac{z}{f(z)} - 1.$$

Then p is analytic in \mathbf{D} , $p(0) = p'(0) = 0$,

$$(8) \quad -zp'(z) = z^2 \left(\frac{z}{f(z)} \right)'' \quad \text{and} \quad -zp'(z) + p(z) = M_f(z),$$

where M_f is defined by (1). Now, suppose that $f \in \mathcal{M}(\lambda)$. Then, we obtain that

$$|-zp'(z) + p(z)| \leq \lambda, \quad z \in \mathbf{D}.$$

By Corollary 3, it follows that

$$|p(z)| = \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| \leq \lambda |z|^2 \quad \text{and} \quad |zp'(z)| = \left| z^2 \left(\frac{z}{f(z)} \right)'' \right| \leq 2\lambda |z|^2, \quad z \in \mathbf{D}$$

and therefore, $f \in \mathcal{U}(\lambda) \cap \mathcal{P}(\lambda)$. \square

Proof of Theorem 2. Let f be given by $f(z) = z/\phi(z)$, where $\phi(z) \neq 0$ in \mathbf{D} . Then the power series representation of ϕ gives

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

By (7) and (8), it follows easily that

$$M_f(z) = \sum_{n=2}^{\infty} (n-1)^2 b_n z^n.$$

Thus, using the coefficient condition (2), we deduce that

$$|M_f(z)| \leq \sum_{n=2}^{\infty} (n-1)^2 |b_n| |z|^n \leq \sum_{n=2}^{\infty} (n-1)^2 |b_n| \leq \lambda$$

and therefore, $f \in \mathcal{M}(\lambda)$. \square

Proof of Theorem 3. Suppose that $f \in \mathcal{U}(\lambda_1)$ and $g \in \mathcal{U}(\lambda_2)$. By hypotheses, $\frac{z}{H(z)} \neq 0$ for $z \in \mathbf{D}$, and f and g have the power series representation of the form

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \quad \text{and} \quad \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

respectively. As $f \in \mathcal{U}(\lambda_1)$, we have

$$\left| -z \left(\frac{z}{f(z)} \right)' + \frac{z}{f(z)} - 1 \right| = \left| \sum_{n=2}^{\infty} (n-1) b_n z^n \right| \leq \lambda_1.$$

Therefore, with $z = re^{i\theta}$ for $r \in (0, 1)$ and $0 \leq \theta \leq 2\pi$, the last inequality gives

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} (n-1) b_n z^n \right|^2 d\theta \leq \lambda_1^2.$$

Allowing $r \rightarrow 1^-$, we obtain the inequality

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 \leq \lambda_1^2.$$

Similarly, as $g \in \mathcal{U}(\lambda_2)$, we have

$$\sum_{n=2}^{\infty} (n-1)^2 |c_n|^2 \leq \lambda_2^2.$$

Now, since

$$\frac{z}{f(z)} * \frac{z}{g(z)} = 1 + b_1 c_1 z + b_2 c_2 z^2 + \dots$$

and

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n| |c_n| \leq \left(\sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 \right)^{1/2} \left(\sum_{n=2}^{\infty} (n-1)^2 |c_n|^2 \right)^{1/2} \leq \lambda_1 \lambda_2,$$

by (2), we conclude that $H \in \mathcal{M}(\lambda)$, $\lambda = \lambda_1 \lambda_2$. □

Proof of Corollary 1. As in the proof of Theorems 2 and 3, we see that

$$M_f(z) = \sum_{n=2}^{\infty} (n-1)^2 b_n z^n$$

and therefore, we easily have the desired necessary condition. □

Proof of Theorem 4. Let $f \in \mathcal{M}(\lambda)$. By assumption,

$$M_f(z) = \lambda w(z)$$

for some $w \in \mathcal{B}_1$. Let $\phi(z) = 1 + b_1 z + b_2 z^2 + \dots$ denote $z/f(z)$. Then

$$M_f(z) = \sum_{n=2}^{\infty} (n-1)^2 b_n z^n = \lambda w(z)$$

which leads to

$$\phi(z) - 1 - b_1 z = \sum_{n=2}^{\infty} b_n z^n = \lambda z \operatorname{Li}_2(z) * w(z),$$

where

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is the dilogarithm. By using the well-known representation

$$\text{Li}_2(z) = z \int_0^1 \frac{\log(1/t)}{1-tz} dt,$$

we obtain

$$\begin{aligned} \phi(z) &= 1 + b_1 z + \lambda w(z) * z^2 \int_0^1 \frac{\log(1/t)}{1-tz} dt \\ &= 1 + b_1 z + \lambda \int_0^1 w(z) * \frac{z^2}{1-tz} \log(1/t) dt \\ &= 1 + b_1 z + \lambda \int_0^1 \frac{w(tz)}{t^2} \log(1/t) dt. \end{aligned}$$

Since $b_1 = -f''(0)/2$, the desired representation follows. \square

Proof of Theorem 5. Let $f \in \mathcal{U}$. Then, because f is univalent, f has the form

$$(9) \quad f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}, \quad z \in \mathbf{D}.$$

Since $f \in \mathcal{U}$, we have (see the proof of Theorem 3)

$$(10) \quad \sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 \leq 1.$$

We need to show that $\frac{1}{r} f(rz) \in \mathcal{M}$ for $0 < r \leq r_0$ where $r_0 \approx 0.62977$ is the root of the equation $r^4(1+r^2) = (1-r^2)^3$ lying in the interval $(0, 1)$.

Using (9), for $0 < r \leq 1$, we can write

$$\frac{z}{\frac{1}{r} f(rz)} = 1 + \sum_{n=1}^{\infty} (b_n r^n) z^n.$$

According to Theorem 2, it suffices to show that

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n r^n| \leq 1$$

for $0 < r \leq r_0$.

Now, by the Cauchy-Schwarz inequality and (10),

$$\begin{aligned}\sum_{n=2}^{\infty} (n-1)^2 |b_n| r^n &\leq \left(\sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 \right)^{1/2} \left(\sum_{n=2}^{\infty} (n-1)^2 r^{2n} \right)^{1/2} \\ &\leq \left(\sum_{n=2}^{\infty} (n-1)^2 r^{2n} \right)^{1/2} = \left(\frac{r^4(1+r^2)}{(1-r^2)^3} \right)^{1/2}.\end{aligned}$$

In particular, for $0 < r \leq r_0$, the last expression is less than or equal to 1. The proof is complete. \square

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