A. K. MISHRA AND P. GOCHHAYAT
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FEKETE-SZEGÖ PROBLEM FOR A CLASS DEFINED BY AN INTEGRAL OPERATOR

Akshaya Kumar Mishra and Priyabrat Gochhayat[1]

Abstract

By making use of an integral operator due to Noor, a new subclass of analytic functions, denoted by $k - \mathscr{UCV}_n$ $(n \in \mathbf{N}_0 := \{0, 1, 2, \ldots\}; 0 \le k < \infty)$; is introduced. For this class the Fekete-Szegö problem is completely settled. The results obtained here also give the Fekete-Szegö inequalities for the classes of k-uniformly convex functions and k-parabolic starlike functions.

1. Introduction and definitions

Let $\mathscr{A}$ denote the family of functions analytic in the *open* unit disk

$$\mathscr{U} := \{z : z \in \mathbf{C}, |z| < 1\}$$

and let $\mathscr{A}_0$ be the class of functions f in $\mathscr{A}$ satisfying the normalization condition $f(0) = f'(0) - 1 = 0$. Thus, the functions in $\mathscr{A}_0$ are given by the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathscr{U}). \tag{1.1}$$

Let $\mathscr{D}^n : \mathscr{A}_0 \to \mathscr{A}_0$ be the operator defined by

$$\begin{aligned} \mathscr{D}^n f(z) &= \frac{z}{(1-z)^{n+1}} * f(z) \quad n > -1 \\ &= z + \sum_{k=2}^{\infty} \binom{k+n-1}{k-1} a_k z^k, \quad (f \in \mathscr{A}_0, n \in \mathbf{N}_0 := \{0, 1, 2, \ldots\}), \end{aligned}$$

where $'*'$ denotes the convolution or Hadamard product. We note that $\mathscr{D}^0 f = f$, $\mathscr{D}^1 f = zf'$ and $\mathscr{D}^n f = \dfrac{z(z^{n-1}f)^{(n)}}{n!}$. The operator $\mathscr{D}^n f$ is called the

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Ruscheweyh derivative of n^{th} order of f. Analogous to $\mathscr{D}^n f$, Noor [13] defined an integral operator $\mathscr{I}_n : \mathscr{A}_0 \to \mathscr{A}_0$ as follows:

Write $f_n(z) = z/(1-z)^{n+1}$ and let $f_n^\dagger$ be defined by the relation

$$f_n(z) * f_n^\dagger(z) = \frac{z}{(1-z)^2}. \tag{1.2}$$

For $f \in \mathscr{A}_0$ let

$$\mathscr{I}_n f(z) = f_n^\dagger(z) * f(z). \tag{1.3}$$

Note that $\mathscr{I}_0 f(z) = zf'(z)$ and $\mathscr{I}_1 f(z) = f(z)$. The operator $\mathscr{I}_n f$ defined by (1.3) is called n^{th} order Noor integral operator of f. For details see (cf. [13], [14], also see [15]).

For any real or complex numbers a, b, c other than $0, -1, -2, \dots$, the Gauss hypergeometric series is defined by

$$_2F_1(a, b; c; z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots. \tag{1.4}$$

We can write (1.3) as

$$\begin{aligned} \mathscr{I}_n f(z) &= [z\,{}_2F_1(2, 1; n+1; z)] * f(z) \\ &= \left[z + \sum_{k=2}^{\infty} \frac{k!}{(n+1)\cdots(n+k-1)} z^k \right] * f(z). \end{aligned} \tag{1.5}$$

Let $\mathscr{S}$, $\mathscr{S}^*$, $\mathscr{CV}$, $\mathscr{UCV}$ and $\mathscr{SP}$ denote, respectively the subclasses of $\mathscr{A}_0$ consisting of functions which are *univalent*, *starlike*, *convex* (cf. [4]), *uniformly convex* (cf. [6]) and *parabolic starlike* (cf. [16])) in $\mathscr{U}$. For fixed k $(0 \le k < \infty)$, the function $f \in \mathscr{A}_0$ is said to be in $k - \mathscr{UCV}$; the class of *k-uniformly convex functions* in $\mathscr{U}$, if the image of every circular arc γ contained in $\mathscr{U}$, with centre ξ where $|\xi| \le k$, is a convex arc. This interesting unification of the concepts of *convex functions* and *uniformly convex functions* is due to [8]. The class $k - \mathscr{SP}$, consisting of *k-parabolic starlike functions*, is defined from $k - \mathscr{UCV}$ via the *Alexander's transform* (see [9]) i.e.

$$f \in k - \mathscr{UCV} \Leftrightarrow g \in k - \mathscr{SP}, \quad \text{where } g(z) = zf'(z) \quad (z \in \mathscr{U}). \tag{1.6}$$

The one variable characterization theorem (cf. [8]) of the the class $k - \mathscr{UCV}$ gives that $f \in k - \mathscr{UCV}$ (respectively $f \in k - \mathscr{SP}$) if only if the values of

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \left(\text{respectively } \frac{zf'(z)}{f(z)} \right) \quad (z \in \mathscr{U})$$

lie in the conic region Ω_k in the w-plane, where

$$\Omega_k := \{ w = u + iv \in \mathbf{C} : u^2 > k^2(u-1)^2 + k^2 v^2; 0 \le k < \infty \}.$$

For details of the geometric description of Ω_k see [8, 9]. Taking cue from (1.6), we now define a new subclass of analytic functions by using the linear operator $\mathscr{I}_n$.

Definition 1. The function $f \in \mathscr{A}_0$ is said to be in the class $k-\mathscr{UCV}_n$ $(0 \le k < \infty; n \in \mathbf{N}_0)$ if $\mathscr{I}_n f \in k-\mathscr{SP}$. Or equivalently

$$\Re\left(\frac{z(\mathscr{I}_n f)'(z)}{(\mathscr{I}_n f)(z)}\right) > k\left|\frac{z(\mathscr{I}_n f)'(z)}{(\mathscr{I}_n f)(z)} - 1\right| \quad (z \in \mathscr{U}).$$

Note that the class $k-\mathscr{UCV}_n$ unifies many subclasses of $\mathscr{A}_0$ related to $\mathscr{S}$. In particular, for $k=0$, $n=0: 0-\mathscr{UCV}_0 := \mathscr{CV}$, the class of univalent convex functions (see [4]); for $k=1$, $n=0: 1-\mathscr{UCV}_0 := \mathscr{UCV}$, the class of uniformly convex functions (see [6]); for $k=0$, $n=1: 0-\mathscr{UCV}_1 := \mathscr{S}^*$, the class of univalent starlike functions (see [4]); for $k=1$, $n=1: 1-\mathscr{UCV}_1 := \mathscr{SP}$, the class of parabolic starlike functions (see [16]); for $k \ne 0$, $n=0: k-\mathscr{UCV}_0 := k-\mathscr{UCV}$ (see [8]); for $k \ne 0$, $n=1: k-\mathscr{UCV}_1 := k-\mathscr{SP}$ (see [8]).

It is well known (cf. [4]) that for $f \in \mathscr{S}$ and given by (1.1), the sharp inequality $|a_3 - a_2^2| \le 1$ holds. Fekete-Szegö, in their seminal work [5], found sharp bounds for $|\mu a_2^2 - a_3|$, $(\mu \in \mathbf{R}, 0 \le \mu < 1)$, for $f \in \mathscr{S}$. Thus, for any family $\mathscr{F}$ of functions in $\mathscr{A}_0$, the problem of finding sharp estimates on the nonlinear functional $|\mu a_2^2 - a_3|$, $(\mu \in \mathbf{R}$ or $\mu \in \mathbf{C})$ is popularly known as the Fekete-Szegö problem for $\mathscr{F}$. Recently the present authors [12] settled this problem for the classes $k-\mathscr{UCV}$, $k-SP$ and some related classes defined using fractional calculus. One may also see [7, 10, 11, 17, 18] for some interesting results on this topic. In the present paper the Fekete-Szegö problem for the class $k-\mathscr{UCV}_n$ $(0 \le k < \infty; n \in \mathbf{N}_0)$ is settled completely. For $k=1$, $n=0$ and $n=0$; $n=1$, the result on Fekete-Szegö inequalities obtained here include, respectively, earlier results of Ma and Minda [11] on $\mathscr{UCV}$ and recent results of the authors on $k-\mathscr{UCV}$ and $k-\mathscr{SP}$ in [12]. For values of $n > 1$ our results provide new information.

In the present investigation we also need the following definitions and notations, for the presentation of our results.

The *Jacobi elliptic integral* (or normal elliptic integral) of first kind (*cf.* [1], [2], also see [19, p. 50]) is defined by

$$\mathscr{F}(\omega, t) = \int_0^{\omega} \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}} \quad (0 < t < 1).$$

The function $\mathscr{F}(1,t) := \mathscr{K}(t)$ is called the complete elliptic integral of the first kind. Changing to the variable $t' = \sqrt{1-t^2}$, $t \in (0,1)$, we write $\mathscr{K}'(t) := \mathscr{K}(t')$. It should be emphasized here that the symbol $\prime$ (prime) does not stand for derivative. The following properties of $\mathscr{K}(t)$ and $\mathscr{K}'(t)$ are well known (*cf.* [7]).

$$\lim_{t \to 0^+} \mathscr{K}(t) = \frac{\pi}{2} \quad \lim_{t \to 1^-} \mathscr{K}(t) = \infty.$$

Moreover the function

$$v(t) = \frac{\pi}{2}\frac{\mathcal{K}'(t)}{\mathcal{K}(t)}, \quad (t \in (0,1))$$

strictly decreases from ∞ to 0 as t moves from 0 to 1. Therefore every positive number k can be expressed as

$$k = \cosh(v(t)) \tag{1.7}$$

for some unique $t \in (0,1)$.

Define the function $\mathcal{G}$ on $\mathcal{U}$ by

$$\mathcal{G}(z) = [z_2F_1(n+1,1;2;z)] * \left\{ z \exp\left(\int_0^z \frac{q_k(\zeta)-1}{\zeta}\, d\zeta \right) \right\}, \quad (z \in \mathcal{U}), \tag{1.8}$$

where q_k is the function defined in Lemma 1 (below) and write the function $\psi(z,\theta,\eta)$ in $k-\mathcal{UCV}_n$ by

$$\psi(z,\theta,\eta) = [z_2F_1(n+1,1;2;z)] * z \exp\left(\int_0^z \left[q_k\left(\frac{e^{i\theta}\zeta(\zeta+\eta)}{1+\eta\zeta} \right) - 1 \right] \frac{d\zeta}{\zeta} \right) \tag{1.9}$$

$$(0 \le \theta \le 2\pi; 0 \le \eta \le 1).$$

Note that $\psi(z,0,1) = \mathcal{G}(z)$ defined by (1.8) and $\psi(z,\theta,0)$ is an odd function.

2. Preliminary lemmas

We need the following results in our investigation:

Lemma 1 [7]. *Let $k \in [0,\infty)$ be fixed and q_k be the Riemann map of $\mathcal{U}$ on to Ω_k, satisfying $q_k(0) = 1$ and $q_k'(0) > 0$. If*

$$q_k(z) = 1 + Q_1 z + Q_2 z^2 + \cdots, \quad (z \in \mathcal{U}) \tag{2.1}$$

then

$$Q_1 = \begin{cases} \dfrac{2A^2}{1-k^2}; & 0 < k < 1, \\ \dfrac{8}{\pi^2}; & k = 1, \\ \dfrac{\pi^2}{4(k^2-1)\mathcal{K}^2(t)\sqrt{t}(1+t)}; & k > 1, \end{cases}$$

$$Q_2 = \begin{cases} \dfrac{(A^2+2)}{3} Q_1; & 0 < k < 1, \\ \dfrac{2}{3} Q_1; & k = 1, \\ \dfrac{(4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2)}{24\mathcal{K}^2(t)\sqrt{t}(1+t)} Q_1; & k > 1, \end{cases}$$

where

$$A = \frac{2}{\pi} \arccos k, \tag{2.2}$$

and $\mathcal{K}(t)$ *is the complete elliptic integral of first kind.*

Lemma 2 [10]. *Let h be analytic, with* $h(0) = 1$, $\Re(h(z)) > 0$ *and given by the series*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathcal{U}). \tag{2.3}$$

Then

$$|c_n| \le 2 \quad (n \in \mathbf{N}), \tag{2.4}$$

$$|c_2 - c_1^2| \le 2 \quad \text{and} \quad \left| c_2 - \frac{1}{2} c_1^2 \right| \le 2 - \frac{|c_1|^2}{2}. \tag{2.5}$$

3. Fekete-Szegö inequalities

The following calculations shall be used in each of the proofs of Theorems 1, 2 and 3.

By Definition 1 there exists a function $w \in \mathcal{A}$ satisfying the conditions of the Schwarz lemma such that

$$\frac{z(\mathcal{I}_n f(z))'}{(\mathcal{I}_n f(z))} = q_k(w(z)) \quad (z \in \mathcal{U}), \tag{3.1}$$

where q_k is the function defined as in Lemma 1.

Let the function p_1 be defined by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in \mathcal{U}),$$

so that $\Re(p_1(z)) > 0$. This gives

$$w(z) = \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots.$$

Substituting this in the series (2.1) we get

$$\begin{aligned} q_k(w(z)) &= 1 + Q_1 \left\{ \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right\} \\ &\quad + Q_2 \left\{ \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right\}^2 + \cdots \\ &= 1 + \frac{Q_1 c_1}{2} z + \left\{ \frac{1}{2}\left(c_2 - \frac{c_1^2}{2} \right) Q_1 + \frac{1}{4} c_1^2 Q_2 \right\} z^2 + \cdots, \end{aligned} \tag{3.2}$$

Using the expansion (1.5) in (3.1) and equating coefficients we find that

$$a_2 = \frac{(n+1)Q_1 c_1}{4} \tag{3.3}$$

and

$$a_3 = \frac{(n+1)(n+2)}{12}\left[\frac{Q_1^2 c_1^2}{4} + \frac{Q_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{Q_2 c_1^2}{4}\right]. \tag{3.4}$$

We have the following:

THEOREM 1. *Let the function f given by* (1.1) *be in the class $k - \mathcal{UCV}_n$* $(0 \le k < 1)$. *Then*

$$|\mu a_2^2 - a_3| \le \begin{cases} \dfrac{(n+1)(n+2)A^2}{6(1-k^2)}\left(\dfrac{6(n+1)A^2\mu}{(n+2)(1-k^2)} - \dfrac{(2+A^2)}{3} - \dfrac{2A^2}{(1-k^2)}\right); & \mu \ge \alpha_1, \\ \dfrac{(n+1)(n+2)A^2}{6(1-k^2)}; & \alpha_2 \le \mu \le \alpha_1, \\ \dfrac{(n+1)(n+2)A^2}{6(1-k^2)}\left(\dfrac{2A^2}{(1-k^2)} + \dfrac{(2+A^2)}{3} - \dfrac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right); & \mu \le \alpha_2, \end{cases} \tag{3.5}$$

where the constant A is given by (2.2),

$$\alpha_1 := \alpha_1(k) = \frac{(n+2)}{(n+1)}\left(\frac{(5+A^2)(1-k^2)}{18A^2} + \frac{1}{3}\right) \tag{3.6}$$

$$\text{and} \quad \alpha_2 := \alpha_2(k) = \frac{(n+2)}{(n+1)}\left(\frac{1}{3} - \frac{(1-A^2)(1-k^2)}{18A^2}\right).$$

Each of the estimates in (3.5) *is sharp.*

Proof. Putting the value of Q_1 and Q_2 for $0 \le k < 1$ from Lemma 1 in (3.3) and (3.4) we get

$$a_2 = \frac{(n+1)A^2}{2(1-k^2)} c_1$$

and

$$a_3 = \frac{(n+1)(n+2)A^2}{12(1-k^2)}\left\{c_2 - \frac{1}{6}\left((1-A^2) - \frac{6A^2}{1-k^2}\right)c_1^2\right\}.$$

Therefore

$$|\mu a_2^2 - a_3| = \left| \frac{(n+1)^2 A^4 c_1^2}{4(1-k^2)^2}\mu - \frac{(n+1)(n+2)A^2}{12(1-k^2)}\left\{ c_2 \right.\right.$$

$$\left.\left. -\frac{1}{6}\left((1-A^2) - \frac{6A^2}{1-k^2}\right)c_1^2 \right\}\right|$$

$$= \left| \left\{ \frac{(n+1)^2 A^4}{4(1-k^2)^2}\mu + \frac{(n+1)(n+2)A^2}{72(1-k^2)}\left((1-A^2)\right.\right.\right.$$

$$\left.\left.\left. - \frac{6A^2}{(1-k^2)}\right)\right\} c_1^2 - \frac{(n+1)(n+2)A^2}{12(1-k^2)}c_2 \right|$$

$$= \frac{(n+1)(n+2)A^2}{24(1-k^2)} \left| \left\{ \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} \right.\right.$$

$$\left.\left. + \frac{1}{3}\left((1-A^2) - \frac{6A^2}{(1-k^2)}\right)\right\} c_1^2 - 2c_2 \right|$$

$$= \frac{(n+1)(n+2)A^2}{24(1-k^2)} \left| \left\{ \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} \right.\right. \tag{3.7}$$

$$\left.\left. + \frac{1-A^2}{3} - \frac{2A^2}{(1-k^2)} - 2 \right\} c_1^2 + 2c_1^2 - 2c_2 \right|$$

$$\leq \frac{(n+1)(n+2)A^2}{24(1-k^2)} \left\{ \left| \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} \right.\right. \tag{3.8}$$

$$\left.\left. - \frac{5+A^2}{3} - \frac{2A^2}{(1-k^2)} \right| |c_1^2| + 2|c_1^2 - c_2| \right\}.$$

Now we note that

$$\frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} - \frac{5+A^2}{3} - \frac{2A^2}{(1-k^2)} \geq 0 \tag{3.9}$$

$$\text{provided} \quad \mu \geq \frac{(n+2)}{(n+1)}\left(\frac{(5+A^2)(1-k^2)}{18A^2} + \frac{1}{3}\right) = \alpha_1.$$

Thus if $\mu \geq \alpha_1$, the expression inside the first modulus symbol on the right hand side of (3.8) is non negative. An application of Lemma 2 yields,

$$|\mu a_2^2 - a_3| \leq \frac{(n+1)(n+2)A^2}{24(1-k^2)} \left\{ \left(\frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} \right.\right. \tag{3.10}$$

$$\left.\left. - \frac{5+A^2}{3} - \frac{2A^2}{(1-k^2)} \right) 4 + 4 \right\}$$

$$\leq \frac{(n+1)(n+2)A^2}{6(1-k^2)}\left\{\frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} - \frac{2+A^2}{3} - \frac{2A^2}{(1-k^2)}\right\}.$$

which is the first part of assertion (3.5).

Rewriting (3.7)

$$(3.11)\qquad |\mu a_2^2 - a_3| = \frac{(n+1)(n+2)A^2}{24(1-k^2)}\left|-2c_2 - \left\{\frac{2A^2}{(1-k^2)} - \frac{1-A^2}{3} - \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right\}c_1^2\right|$$
$$= \frac{(n+1)(n+2)A^2}{24(1-k^2)}\left|\left\{\frac{2A^2}{(1-k^2)} - \frac{1-A^2}{3} - \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right\}c_1^2 + 2c_2\right|.$$

Now if $\mu \leq \alpha_2$, where α_2 is given by (3.6), then

$$|\mu a_2^2 - a_3| \leq \frac{(n+1)(n+2)A^2}{24(1-k^2)}\left\{\left(\frac{2A^2}{(1-k^2)} - \frac{1-A^2}{3} - \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right)|c_1^2| + 2|c_2|\right\}$$

Applying Lemma 2, we get

$$(3.12)\qquad |\mu a_2^2 - a_3| \leq \frac{(n+1)(n+2)A^2}{24(1-k^2)}\left\{\left(\frac{2A^2}{(1-k^2)} - \frac{1-A^2}{3} - \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right)4 + 4\right\}$$
$$= \frac{(n+1)(n+2)A^2}{6(1-k^2)}\left(\frac{2A^2}{(1-k^2)} + \frac{2+A^2}{3} - \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right)$$

which is the third part of the assertion (3.5).

Lastly from (3.11), write

$$(3.13)\qquad |\mu a_2^2 - a_3| = \frac{(n+1)(n+2)A^2}{24(1-k^2)}\left|2c_2 + \left\{\frac{2A^2}{(1-k^2)} - \frac{1-A^2}{3} - \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right\}c_1^2\right|$$

$$\leq \frac{(n+1)(n+2)A^2}{24(1-k^2)}\left\{2\left|c_2-\frac{1}{2}c_1^2\right|\right.$$
$$\left.+\left|\frac{2A^2}{(1-k^2)}+\frac{2+A^2}{3}-\frac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right||c_1^2|\right\}.$$

Observe that $\alpha_2 < \mu < \alpha_1$ gives

$$\left|\frac{2A^2}{(1-k^2)}+\frac{2+A^2}{3}-\frac{6(n+1)A^2\mu}{(n+2)(1-k^2)}\right| \leq 1. \tag{3.14}$$

Therefore an application of Lemma 2 in (3.13) gives

$$|\mu a_2^2 - a_3| \leq \frac{(n+1)(n+2)A^2}{6(1-k^2)} \tag{3.15}$$

which is the second part of assertion (3.5).

We next discuss the sharpness of (3.5).

If $\mu > \alpha_1$, equality holds in (3.5) if and only if equality holds in (3.10). This happens if and only if $|c_1| = 2$ and $|c_1^2 - c_2| = 2$. Thus $w(z) = z$. Equivalently the extremal function is $\mathscr{G}(z)$ defined by (1.8) or one of its rotations.

If $\mu < \alpha_2$ then equality holds in (3.12) if and only if $c_1^2 = -4$ and $c_2 = -2$ in (3.7) if and only if $c_1 = 2e^{i\pi/2}$ or $c_1 = 2e^{i3\pi/2}$ which also gives $c_2 = -2$. Thus $w(z) = e^{i\theta}z$ where $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ and the extremal function is $\psi(z, \theta, 1)$ or one of its rotations.

If $\mu = \alpha_2$, the equality holds if and only if $|c_2| = 2$. Equivalently, we have

$$p_1(z) = \frac{1+\eta}{2}\left(\frac{1+z}{1-z}\right)+\frac{1-\eta}{2}\left(\frac{1-z}{1+z}\right) \quad (0 < \eta < 1; z \in \mathscr{U}).$$

Thus the extremal function f is $\psi(z, 0, \eta)$ or one of its rotations.

Similarly if $\mu = \alpha_1$, is equivalent to

$$\frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} - \frac{5+A^2}{3} - \frac{2A^2}{(1-k^2)} = 0.$$

Therefore equality holds true in (3.10) if and only if $|c_1^2 - c_2| = 2$ in (3.8). This happens if and only if

$$\frac{1}{p_1(z)} = \frac{1+\eta}{2}\left(\frac{1+z}{1-z}\right)+\frac{1-\eta}{2}\left(\frac{1-z}{1+z}\right) \quad (0 < \eta < 1; z \in \mathscr{U}).$$

Thus the extremal function is $\psi(z, \pi, \eta)$ or one of its rotations.

Lastly if $\alpha_2 < \mu < \alpha_1$, then equality holds true if $|c_1| = 0$ and $|c_2| = 2$. Equivalently we have

$$p_1(z) = \frac{1+\eta z^2}{1-\eta z^2} \quad (0 \leq \eta \leq 1; z \in \mathscr{U}).$$

Thus the function f is $\psi(z, 0, 0)$ or one of its rotation. The proof of Theorem 1 is complete. ∎

Taking $n = 0$ and $n = 1$ in Theorem 1, we get the Fekete-Szegö inequalities for the classes $k - \mathcal{UCV}$ and $k - \mathcal{SP}$ $(0 \leq k < 1)$ respectively, obtained earlier by the authors [12].

Corollary 1. *Let the function f given by* (1.1) *be in the class $k - \mathcal{UCV}$ $(0 \leq k < 1)$. Then*

$$|\mu a_2^2 - a_3| \leq \begin{cases} \dfrac{2A^2}{3(1-k^2)}\left(\dfrac{3A^2\mu}{2(1-k^2)} - \dfrac{(7-k^2)A^2}{6(1-k^2)} - \dfrac{1}{3}\right); & \mu \geq \alpha_1^*, \\ \dfrac{A^2}{3(1-k^2)}; & \alpha_2^* \leq \mu \leq \alpha_1^*, \\ \dfrac{2A^2}{3(1-k^2)}\left(\dfrac{1}{3} + \dfrac{(7-k^2)A^2}{6(1-k^2)} - \dfrac{3A^2\mu}{2(1-k^2)}\right); & \mu \leq \alpha_2^*, \end{cases}$$

where the constant A is given by (2.2),

$$\alpha_1^* = \frac{5(1-k^2)}{9A^2} + \frac{7-k^2}{9} \quad \textit{and} \quad \alpha_2^* = \frac{7-k^2}{9} - \frac{1-k^2}{9A^2}.$$

The result is best possible.

Corollary 2. *Let the function f given by* (1.1) *be in the class $k - \mathcal{SP}$ $(0 \leq k < 1)$. Then*

$$|\mu a_2^2 - a_3| \leq \begin{cases} \dfrac{2A^2}{(1-k^2)}\left(\dfrac{2A^2\mu}{(1-k^2)} - \dfrac{(7-k^2)A^2}{6(1-k^2)} - \dfrac{1}{3}\right); & \mu \geq \alpha_1^{**}, \\ \dfrac{A^2}{(1-k^2)}; & \alpha_2^{**} \leq \mu \leq \alpha_1^{**}, \\ \dfrac{2A^2}{(1-k^2)}\left(\dfrac{1}{3} + \dfrac{(7-k^2)A^2}{6(1-k^2)} - \dfrac{2A^2\mu}{(1-k^2)}\right); & \mu \leq \alpha_2^{**}, \end{cases}$$

where the constant A is given by (2.2),

$$\alpha_1^{**} = \frac{5(1-k^2)}{12A^2} + \frac{7-k^2}{12} \quad \textit{and} \quad \alpha_2^{**} = \frac{(7-k^2)}{12} - \frac{1-k^2}{12A^2}.$$

The result is best possible.

Theorem 2. *Let the function f given by* (1.1) *be in the class $k - \mathcal{UCV}_n$ $(k = 1, n \in \mathbf{N})$. Then*

$$|\mu a_2^2 - a_3| \leq \begin{cases} \dfrac{4(n+1)(n+2)}{3\pi^2}\left(\dfrac{12(n+1)\mu}{(n+2)\pi^2} - \dfrac{1}{3} - \dfrac{4}{\pi^2}\right); & \mu \geq \beta_1, \\ \dfrac{2(n+1)(n+2)}{3\pi^2}; & \beta_2 \leq \mu \leq \beta_1, \\ \dfrac{4(n+1)(n+2)}{3\pi^2}\left(\dfrac{4}{\pi^2} + \dfrac{1}{3} - \dfrac{12(n+1)\mu}{(n+2)\pi^2}\right); & \mu \leq \beta_2, \end{cases} \tag{3.16}$$

where,

$$\beta_1 := \beta_1(k) = \frac{(n+2)}{(n+1)}\left(\frac{5\pi^2}{72}+\frac{1}{3}\right) \quad \textit{and} \quad \beta_2 := \beta_2(k) = \frac{(n+2)}{(n+1)}\left(\frac{1}{3}-\frac{\pi^2}{72}\right).$$

Each of the estimates in (3.16) *is sharp.*

Proof. We put the values of Q_1 and Q_2 for $k = 1$ given in (3.5) and (3.6). By following the lines of proof of Theorem 1 the result follows. ■

Taking $n = 0$ in Theorem 2, we get the Fekete-Szegö inequalities for the class $\mathcal{UCV}$ obtained earlier by Ma and Minda [11]. Similarly the choice $n = 1$ yields a result for the class $\mathcal{SP}$ obtained recently by the present authors [12], which is as follows:

Corollary 3. *Let the function f given by* (1.1) *be in the class $\mathcal{SP}$. Then*

$$|\mu a_2^2 - a_3| \le \begin{cases} \dfrac{8}{\pi^2}\left(\dfrac{8\mu}{\pi^2}-\dfrac{1}{3}-\dfrac{4}{\pi^2}\right); & \mu \ge \beta_1^*, \\ \dfrac{4}{\pi^2}; \quad \beta_2^* \le \mu \le \beta_1^*, \\ \dfrac{8}{\pi^2}\left(\dfrac{4}{\pi^2}+\dfrac{1}{3}-\dfrac{8\mu}{\pi^2}\right); & \mu \le \beta_2^*, \end{cases}$$

where,

$$\beta_1^* = \frac{3}{2}\left(\frac{5\pi^2}{72}+\frac{1}{3}\right) \quad \textit{and} \quad \beta_2^* = \frac{3}{2}\left(\frac{1}{3}-\frac{\pi^2}{72}\right).$$

The result is best possible.

Theorem 3. *Let the function f given by* (1.1) *be in the class $k - \mathcal{UCV}_n$ $(k > 1)$. Then*

$$(3.17) \qquad |\mu a_2^2 - a_3| \le \begin{cases} \dfrac{(n+1)(n+2)Q_1}{12}\left(\dfrac{3(n+1)\mu}{(n+2)}Q_1 - Q_1 \right. \\ \qquad \left. - \dfrac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)}\right); \quad \mu \ge \gamma_1, \\ \dfrac{(n+1)(n+2)Q_1}{12}; \quad \gamma_2 \le \mu \le \gamma_1, \\ \dfrac{(n+1)(n+2)Q_1}{12}\left(Q_1 + \dfrac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)} \right. \\ \qquad \left. - \dfrac{3(n+1)\mu}{(n+2)}Q_1\right); \quad \mu \le \gamma_2, \end{cases}$$

where $\mathscr{K}(t)$ is the complete elliptic integral of the first kind, Q_1 is given in (2.1),

$$\gamma_1 := \gamma_1(k) = \frac{(n+2)}{3Q_1(n+1)}\left(1 + Q_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right)$$

and

$$\gamma_2 := \gamma_2(k) = \frac{(n+2)}{3Q_1(n+1)}\left(Q_1 - 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right).$$

Each of the estimates in (3.17) *is sharp.*

Proof. We follow the lines of proof of Theorem 1 and give here only the essential steps. Putting the value of Q_2 for $k > 1$ from Lemma 1 in (3.4) we get

$$a_2 = \frac{Q_1(n+1)}{4}c_1,$$

$$a_3 = \frac{Q_1(n+1)(n+2)}{12}\left[\frac{Q_1c_1^2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\frac{c_1^2}{4}\right].$$

Therefore

$$|\mu a_2^2 - a_3| = \frac{(n+1)(n+2)Q_1}{48}\left|\left\{\frac{3Q_1(n+1)\mu}{(n+2)} - \left(Q_1 - 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right)\right\}c_1^2 - 2c_2\right| \tag{3.18}$$

$$\leq \frac{(n+1)(n+2)Q_1}{48}\left\{\left|\frac{3Q_1(n+1)\mu}{(n+2)} - 1 - Q_1 - \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right||c_1^2| + 2|c_1^2 - c_2|\right\}. \tag{3.19}$$

Let

$$\mu \geq \frac{(n+2)}{3Q_1(n+1)}\left(1 + Q_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right) = \gamma_1 \quad \text{(say)} \tag{3.20}$$

Observe that if $\mu \geq \gamma_1$, then the expression inside the first modulus of right-hand side (3.19) is non negative. Thus by applying Lemma 2 we get

$$|\mu a_2^2 - a_3| \leq \frac{(n+1)(n+2)Q_1}{12}\left(\frac{3Q_1(n+1)\mu}{(n+2)} - Q_1 - \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right) \tag{3.21}$$

which is the first part of the assertion (3.17).

Rewriting (3.18)

$$|\mu a_2^2 - a_3| = \frac{(n+1)(n+2)Q_1}{48}\Bigg| -2c_2 - \Bigg\{ Q_1 - 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} - \frac{3Q_1(n+1)\mu}{(n+2)} \Bigg\} c_1^2 \Bigg|$$

$$= \frac{(n+1)(n+2)Q_1}{48}\Bigg| 2c_2 + \Bigg\{ Q_1 - 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} - \frac{3Q_1(n+1)\mu}{(n+2)} \Bigg\} c_1^2 \Bigg| \tag{3.22}$$

$$\leq \frac{(n+1)(n+2)Q_1}{48}\Bigg\{ \Bigg| Q_1 - 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} - \frac{3Q_1(n+1)\mu}{(n+2)} \Bigg| |c_1^2| + 2|c_2| \Bigg\} \tag{3.23}$$

Let,

$$\mu \leq \frac{(n+2)}{3Q_1(n+1)}\left(Q_1 - 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} \right) = \gamma_2 \quad \text{say} \tag{3.24}$$

If $\mu \leq \gamma_2$, then the expression inside the first modulus of right hand side of (3.23) is non negative. Thus by applying Lemma 2 we get

$$|\mu a_2^2 - a_3| = \frac{(n+1)(n+2)Q_1}{12}\left(Q_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} - \frac{3Q_1(n+1)\mu}{(n+2)} \right). \tag{3.25}$$

Which is the third part of assertion (3.17). Lastly (3.22) gives

$$|\mu a_2^2 - a_3| = \frac{(n+1)(n+2)Q_1}{48}\Bigg| 2\left(c_2 - \frac{1}{2}c_1^2 \right) + \left(Q_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} - \frac{3Q_1(n+1)\mu}{(n+2)} \right) c_1^2 \Bigg| \tag{3.26}$$

$$\leq \frac{(n+1)(n+2)Q_1}{48}\Bigg\{ 2\left| c_2 - \frac{1}{2}c_1^2 \right| + \left| Q_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} - \frac{3Q_1(n+1)\mu}{(n+2)} \right| |c_1^2| \Bigg\}.$$

Notably $\gamma_2 < \mu < \gamma_1$ gives

$$\left| Q_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} - \frac{3Q_1(n+1)\mu}{(n+2)} \right| \le 1.$$

Therefore, if $\gamma_2 < \mu < \gamma_1$, an application of Lemma 2 in (3.26) yields

$$\begin{aligned} |\mu a_2^2 - a_3| &\le \frac{(n+1)(n+2)Q_1}{48}\left\{2\left|c_2 - \frac{1}{2}c_1^2\right| + |c_1^2|\right\} \\ &\le \frac{(n+1)(n+2)Q_1}{12}. \end{aligned}$$

This is the middle part of the assertions in (3.17). The sharpness of each estimate in (3.17) can be established as in Theorem 1. The proof of Theorem 3 is complete. ■

Taking $n = 0$ and $n = 1$ in Theorem 3, we get the Fekete-Szegö inequalities for the classes $k - \mathscr{UCV}$ and $k - \mathscr{SP}$ respectively obtained recently by the present authors [12]. We list them here for the sake of completeness.

COROLLARY 4. *Let the function f given by* (1.1) *be in the class $k - \mathscr{UCV}$* $(k > 1)$. *Then*

$$|\mu a_2^2 - a_3| \le \begin{cases} \dfrac{Q_1}{6}\left(\dfrac{3Q_1\mu}{2} - Q_1 - \dfrac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right); & \mu \ge \gamma_1^*, \\ \dfrac{Q_1}{6}; & \gamma_2^* \le \mu \le \gamma_1^*, \\ \dfrac{Q_1}{6}\left(Q_1 + \dfrac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)} - \dfrac{3Q_1\mu}{2}\right); & \mu \le \gamma_2^*, \end{cases}$$

where $\mathscr{K}(t)$ is the complete elliptic integral of the first kind, Q_1 is given in (2.1),

$$\gamma_1^* = \frac{2}{3Q_1}\left(1 + Q_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right)$$

and

$$\gamma_2^* = \frac{2}{3Q_1}\left(Q_1 - 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right).$$

Each of the estimates is sharp.

Corollary 5. *Let the function f given by* (1.1) *be in the class $k-\mathcal{SP}$ $(k>1)$. Then*

$$|\mu a_2^2 - a_3| \le \begin{cases} \dfrac{Q_1}{2}\left(2Q_1\mu - Q_1 - \dfrac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)}\right); & \mu \ge \gamma_1^{**}, \\ \dfrac{Q_1}{2}; & \gamma_2^{**} \le \mu \le \gamma_1^{**}, \\ \dfrac{Q_1}{2}\left(Q_1 + \dfrac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)} - 2Q_1\mu\right); & \mu \le \gamma_2^{**}, \end{cases}$$

where $\mathcal{K}(t)$ is the complete elliptic integral of the first kind, Q_1 is given in (2.1),

$$\gamma_1^{**} = \frac{1}{2Q_1}\left(1 + Q_1 + \frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)}\right)$$

and

$$\gamma_2^{**} = \frac{1}{2Q_1}\left(Q_1 - 1 + \frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)}\right).$$

Each of the estimates is sharp.

4. Remarks on main results

In this section we discuss some improvements of the middle inequalities in (3.5), (3.16) and (3.17) respectively.

Remark 1. The second part of assertion in (3.5) can be improved as follows:

$$(4.1)\qquad |\mu a_2^2 - a_3| + \left\{\mu - \frac{(n+2)}{(n+1)}\left(\frac{1}{3} - \frac{(1-A^2)(1-k^2)}{18A^2}\right)\right\}|a_2^2| \le \frac{(n+1)(n+2)A^2}{6(1-k^2)}; \quad \alpha_2 \le \mu \le \alpha_3,$$

and

$$(4.2)\qquad |\mu a_2^2 - a_3| + \left\{\frac{(n+2)}{(n+1)}\left(\frac{1}{3} + \frac{(5+A^2)(1-k^2)}{18A^2}\right) - \mu\right\}|a_2^2| \le \frac{(n+1)(n+2)A^2}{6(1-k^2)}; \quad \alpha_3 \le \mu \le \alpha_1,$$

where α_3 is given by

$$(4.3)\qquad \alpha_3 := \frac{n+2}{n+1}\left(\frac{1}{3} + \frac{(2+A^2)(1-k^2)}{18A^2}\right).$$

Proof. Suppose $\alpha_2 \le \mu \le \alpha_3^*$. We continue with the estimate in (3.13) and write

$$
\begin{aligned}
&|\mu a_2^2 - a_3| + (\mu - \alpha_2)|a_2|^2 \\
&\quad = |\mu a_2^2 - a_3| + \left\{ \mu - \frac{(n+2)}{(n+1)} \left(\frac{1}{3} - \frac{(1-A^2)(1-k^2)}{18A^2} \right) \right\} |a_2^2| \\
&\quad \le \frac{(n+1)(n+2)A^2}{24(1-k^2)} \left\{ 2\left| c_2 - \frac{1}{2}c_1^2 \right| + \left| \frac{2A^2}{(1-k^2)} + \frac{2+A^2}{3} - \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} \right| |c_1^2| \right\} \\
&\qquad + \left\{ \mu - \frac{(n+2)}{(n+1)} \left(\frac{1}{3} - \frac{(1-A^2)(1-k^2)}{18A^2} \right) \right\} \left| \frac{(n+1)^2 A^4 c_1^2}{4(1-k^2)^2} \right| \\
&\quad \le \frac{(n+1)(n+2)A^2}{24(1-k^2)} \left\{ 2\left| c_2 - \frac{1}{2}c_1^2 \right| + \left(\frac{2A^2}{(1-k^2)} + \frac{2+A^2}{3} - \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} \right) |c_1^2| \right. \\
&\qquad \left. + \left\{ \frac{6(n+1)A^2\mu}{(n+2)(1-k^2)} - \frac{6A^2}{(1-k^2)} \left(\frac{1}{3} - \frac{(1-A^2)(1-k^2)}{18A^2} \right) \right\} |c_1^2| \right\} \\
&\quad \le \frac{(n+1)(n+2)A^2}{24(1-k^2)} \left\{ 2\left| c_2 - \frac{1}{2}c_1^2 \right| + |c_1^2| \right\} \\
&\quad = \frac{(n+1)(n+2)A^2}{6(1-k^2)}.
\end{aligned}
$$

We get (4.1). On the other hand suppose $\alpha_3 \le \mu \le \alpha_1$. In this case we estimate $|\mu a_2^2 - a_3| + (\alpha_1 - \mu)|a_2|^2$. Now following the lines of proof for (4.1) with obvious changes we get (4.2).

The proof of next two remarks are same as in Remark 1. Therefore we omit details.

Remark 2. The second part of assertion in (3.16) can be improved as follows:

$$
|\mu a_2^2 - a_3| + \left\{ \mu - \frac{n+2}{n+1} \left(\frac{1}{3} - \frac{\pi^2}{72} \right) \right\} |a_2^2| \le \frac{2(n+1)(n+2)}{3\pi^2}; \quad \beta_2 \le \mu \le \beta_3, \tag{4.4}
$$

and

$$
|\mu a_2^2 - a_3| + \left\{ \frac{n+2}{n+1} \left(\frac{5\pi^2}{72} + \frac{1}{3} \right) - \mu \right\} |a_2^2| \le \frac{2(n+1)(n+2)}{3\pi^2}; \quad \beta_3 \le \mu \le \beta_1, \tag{4.5}
$$

where β_3 is given by

$$
\beta_3 := \frac{n+2}{n+1} \left(\frac{1}{3} + \frac{\pi^2}{36} \right). \tag{4.6}
$$

Remark 3. The second part of assertion in (3.17) can be improved as follows:

$$(4.7)\quad |\mu a_2^2 - a_3| + \left\{ \mu - \frac{(n+2)}{3Q_1(n+1)}\left(Q_1 - 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right)\right\} \times |a_2^2|$$
$$\leq \frac{(n+1)(n+2)Q_1}{12}; \quad \gamma_2 \leq \mu \leq \gamma_3,$$

and

$$(4.8)\quad |\mu a_2^2 - a_3| + \left\{ \frac{(n+2)}{3Q_1(n+1)}\left(Q_1 + 1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathscr{K}^2(t)\sqrt{t}(1+t)}\right) - \mu \right\} \times |a_2^2|$$
$$\leq \frac{(n+1)(n+2)Q_1}{12}; \quad \gamma_3 \leq \mu \leq \gamma_1,$$

where γ_3 is given by

$$(4.9)\quad \gamma_3 := \frac{n+2}{n+1}\left(\frac{1}{3} + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{72Q_1\mathscr{K}^2(t)\sqrt{t}(1+t)}\right).$$

5. An alternate method

Let $\mathscr{P}(q_k)$ denote the subclass of $\mathscr{A}$, consisting of functions $h(z) \prec q_k(z)$ in $\mathscr{U}$, where $q_k(z)$ is as in Lemma 1 and $\prec$ denotes subordination. We need the following result:

LEMMA 3 [7]. *Let $k \in [0, \infty)$ be fixed and the function $h \in \mathscr{P}(q_k)$ be given by the series expansion*

$$(5.1)\quad h(z) = 1 + b_1 z + b_2 z^2 + \cdots, \quad (z \in \mathscr{U})$$

then

$$(5.2)\quad |b_2 - u b_1^2| \leq \begin{cases} Q_1(k) - uQ_1^2(k); & u \leq 0, \\ Q_1(k); & u \in (0,1], \\ Q_1(k) + (u-1)Q_1^2(k); & u \geq 1, \end{cases}$$

where $Q_1(k)$ is as in Lemma 1. If $0 < u < 1$ then equality holds in (5.2) if $h(z) = q_k(z^2)$.

Now replacing $q_k(w(z))$ by $h(z)$ in the equation (3.1) we get

$$a_2 = \left(\frac{n+1}{2}\right) b_1$$

$$a_3 = \frac{(n+1)(n+2)}{12}(b_1^2 + b_2)$$

and

$$\mu a_2^2 - a_3 = \frac{(n+1)(n+2)}{12}\left[\left\{\frac{3(n+1)\mu}{(n+2)} - 1\right\}b_1^2 - b_2\right]$$

An application of Lemma 3 with,

$$u = 3\left(\frac{n+1}{n+2}\right)\mu - 1$$

yields

$$|\mu a_2^2 - a_3| = \frac{(n+1)(n+2)}{12}Q_1(k) \quad \text{if} \quad \frac{(n+2)}{3(n+1)} < \mu \leq \frac{2}{3}\left(\frac{n+2}{n+1}\right) \tag{5.3}$$

Putting the values of $Q_1(k)$ from Lemma 1 for $0 \leq k < 1$, $k = 1$ and $k > 1$ the estimate (5.3) simplifies to the middle inequalities in (3.5), (3.16) and (3.17) respectively. However, following the lines of proof Theorem 3.1 in [7] it can be shown that

$$\alpha_1(k), \beta_1(k), \gamma_1(k) > \frac{2}{3}\left(\frac{n+2}{n+1}\right)$$

and

$$\alpha_2(k), \beta_2(k), \gamma_2(k) < \frac{1}{3}\left(\frac{n+2}{n+1}\right)$$

for every $0 \leq k < \infty$ where $\alpha_1(k), \alpha_2(k)$; $\beta_1(k), \beta_2(k)$ and $\gamma_1(k), \gamma_2(k)$ are defined as in Theorems 3.1, 3.2 and 3.3 respectively. Thus the methods of proof adopted for Theorems 3.1, 3.2 and 3.3 yield larger interval for μ than the interval obtained for the estimates of $|\mu a_2^2 - a_3|$ in (5.3).

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Akshaya Kumar Mishra
Department of Mathematics
Berhampur University
Bhanja Bihar
Berhampur-760007, Orissa
India
E-mail: akshayam2001@yahoo.co.in

Priyabrat Gochhayat
Department of Mathematics
Sambalpur University
Jyoti Vihar
Sambalpur-768019, Orissa
India
E-mail: pb_gochhayat@yahoo.com