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# ADDENDUM TO OUR CHARACTERIZATION OF THE UNIT POLYDISC

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**Abstract**

In 2008, we obtained an intrinsic characterization of the unit polydisc $\Delta^n$ in $\mathbf{C}^n$ from the viewpoint of the holomorphic automorphism group. In connection with this, A. V. Isaev investigated the structure of a complex manifold $M$ with the property that every isotropy subgroup of the holomorphic automorhism group of $M$ is compact, and obtained the same characterization of $\Delta^n$ as ours among the class of all such manifolds. In this paper, we establish some extensions of these results. In particular, Isaev's characterization of the unit polydisc $\Delta^n$ is extended to that of any bounded symmetric domain in $\mathbf{C}^n$.

## 1. Introduction

This is a continuation of our previous paper [8], and we retain the terminology and notation there.

Let $M$ be a connected complex manifold and $\mathrm{Aut}(M)$ the group of all biholomorphic automorphisms of $M$. Then, equipped with the compact-open topology, $\mathrm{Aut}(M)$ is a topological group acting continuously on $M$. It should be remarked here that $\mathrm{Aut}(M)$ does not have the structure of a Lie group, in general; this often causes difficulties in studying various problems related to $\mathrm{Aut}(M)$.

In 1907, it was shown by Poincaré [10] that the Riemann mapping theorem does not hold in the higher dimensional case. In fact, he proved that *there exists no biholomorphic mapping from the unit polydisc* $\Delta^2$ *onto the unit ball* $B^2$ *in* $\mathbf{C}^2$ by comparing carefully the topological structures of the isotropy subgroups of $\mathrm{Aut}(\Delta^2)$ and $\mathrm{Aut}(B^2)$ at the origin $o$ of $\mathbf{C}^2$. In view of this fact, for a given complex manifold $M$, it seems to be an interesting problem to bring out some complex analytic nature of $M$ under some topological conditions on $\mathrm{Aut}(M)$. Taking this into account, we asked the following question in [8]: *Let $M$ and $N$ be connected complex manifolds and assume that their holomorphic automorphism*

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*groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are isomorphic as topological groups. Then is M biholomorphically equivalent to N?* And, as our main result, we obtained the following intrinsic characterization of the unit polydisc $\Delta^n$ from the viewpoint of the holomorphic automorphism group:

Theorem A ([8, Theorem]). *Let $M$ be a connected complex manifold of dimension $n$ that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then $M$ is biholomorphically equivalent to $\Delta^n$.*

Later, related to this theorem, Isaev [6] investigated the structure of a complex manifold $M$ with the property that every isotropy subgroup of the $\mathrm{Aut}(M)$-action is compact, and showed the following:

Theorem B ([6, Theorem 1.2]). *Let $M$ be a connected complex manifold of dimension $n$ satisfying the following two conditions*:

(1) *The isotropy subgroup of* $\mathrm{Aut}(M)$ *at every point of $M$ is compact.*

(2) $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\Delta^n)$ *as topological groups.*

*Then $M$ is biholomorphically equivalent to $\Delta^n$.*

The main purpose of this paper is to establish the following extensions of Theorems A and B, which were announced at the 17th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications in Ho Chi Minh City, Vietnam, August 2009:

Theorem 1. *Let $M$ be a connected complex manifold of dimension $n$ that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that there exists a topological subgroup $G$ of* $\mathrm{Aut}(M)$ *that is isomorphic to the identity component of* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then $M$ is biholomorphically equivalent to $\Delta^n$.*

This theorem will be proved in Section 2 by modifying the proof of Theorem A.

Let $W$ be an arbitrary domain in $\mathbf{C}^n$. Then it is well-known that $W$ admits a smooth envelope of holomorphy (cf. [9]). Hence, as an immediate consequence of this theorem, we obtain the following:

Corollary 1. *Let $M$ be a connected Stein manifold of dimension $n$ or a domain in $\mathbf{C}^n$. Assume that there exists a topological subgroup $G$ of* $\mathrm{Aut}(M)$ *that is isomorphic to the identity component of* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then $M$ is biholomorphically equivalent to $\Delta^n$.*

A bounded domain $D$ in $\mathbf{C}^n$ is called *symmetric* if, for each point $p \in D$, there exists an element $s_p \in \mathrm{Aut}(D)$ such that $s_p \circ s_p = \mathrm{id}_D$, $s_p \neq \mathrm{id}_D$ and $p$ is an isolated fixed point of $s_p$. Clearly, the unit polydisc $\Delta^n$ as well as the unit ball

$B^n$ in $\mathbf{C}^n$ is a typical example of bounded symmetric domains. As a natural generalization of Theorem B, we can prove the following theorem in Section 3:

THEOREM 2. *Let $M$ be a connected complex manifold of dimension $n$ and let $D$ be a bounded symmetric domain in $\mathbf{C}^n$. Assume that there exists a topological subgroup $G$ of* $\mathrm{Aut}(M)$ *satisfying the following two conditions*:

(1) *The isotropy subgroup of $G$ at every point of $M$ is compact.*

(2) *$G$ is isomorphic to the identity component of* $\mathrm{Aut}(D)$ *as topological groups.*

*Then $M$ is biholomorphically equivalent to $D$.*

Recall that the isotropy subgroup of $\mathrm{Aut}(M)$ at every point of $M$ is compact, provided that $M$ is hyperbolic in the sense of Kobayashi [7]. Hence we have the following:

COROLLARY 2. *Let $M$ be a connected hyperbolic manifold of dimension $n$ and let $D$ be a bounded symmetric domain in $\mathbf{C}^n$. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D)$ *as topological groups. Then $M$ is biholomorphically equivalent to $D$.*

Finally, it should be remarked that, for a given connected complex manifold $M$, the following conditions (A) and (B) are mutually independent (for the detail, see Section 4):

(A) $M$ is holomorphically separable and admits a smooth envelope of holomorphy.

(B) The isotropy subgroup of $\mathrm{Aut}(M)$ at every point of $M$ is compact.

In this sense, our Theorems 1 and 2 may be considered as characterizations of model domains from different viewpoints.

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## 2. Proof of Theorem 1

Our proof is based on the argument developed in our previous paper [8]. Although there are some overlaps with that paper, we carry out the proof for the sake of completeness and self-containedness.

Let us start with fixing a coordinate system $z = (z_1, \dots, z_n)$ in $\mathbf{C}^n$ and setting

$$\Delta_j = \{z_j \in \mathbf{C} \mid |z_j| < 1\} \ (1 \le j \le n) \quad \text{and} \quad \Delta^n = \Delta_1 \times \cdots \times \Delta_n.$$

Recall that $\mathrm{Aut}(\Delta_j)$ is a connected, real simple Lie group of dimension 3 with trivial center. Let $\mathrm{Aut}^o(\Delta^n)$ be the identity component of $\mathrm{Aut}(\Delta^n)$. Then we know that $\mathrm{Aut}^o(\Delta^n)$ can be identified with the direct product of $\mathrm{Aut}(\Delta_j)$: $\mathrm{Aut}^o(\Delta^n) = \mathrm{Aut}(\Delta_1) \times \cdots \times \mathrm{Aut}(\Delta_n)$. Let $\mathfrak{g}(\Delta_j)$ and $\mathfrak{g}(\Delta^n)$, respectively, denote the real Lie algebras consisting of all complete holomorphic vector fields on $\Delta_j$ and on $\Delta^n$. Then it is well-known that these Lie algebras are canonically identified with the Lie algebras of $\mathrm{Aut}(\Delta_j)$ and $\mathrm{Aut}(\Delta^n)$, respectively. Therefore we have

$$(2.1)\quad \mathfrak{g}(\Delta^n) = \mathfrak{g}(\Delta_1) \oplus \cdots \oplus \mathfrak{g}(\Delta_n), \quad [\mathfrak{g}(\Delta_i), \mathfrak{g}(\Delta_j)] = \{0\} \quad \text{for } 1 \le i, j \le n,\ i \ne j.$$

Moreover, we see that $\mathfrak{g}(\Delta_j)$ contains the holomorphic vector fields

$$H_j := \sqrt{-1} z_j \partial/\partial z_j \quad \text{and} \quad V_j := (1 - z_j^2)\partial/\partial z_j$$

induced by the one-parameter subgroups

$$z_j \mapsto (\exp \sqrt{-1} t) z_j \quad \text{and} \quad z_j \mapsto \frac{(\cosh t) z_j + \sinh t}{(\sinh t) z_j + \cosh t}$$

($t \in \mathbf{R}$) of $\mathrm{Aut}(\Delta_j)$, respectively. Then, putting $W_j = [H_j, V_j]$, we have

$$(2.2)\quad \mathfrak{g}(\Delta_j) = \mathbf{R}\{H_j, V_j, W_j\} \quad \text{and} \quad [H_j, [H_j, V_j]] = -V_j, \quad [W_j, V_j] = 4H_j$$

for $1 \le j \le n$. These bracket relations will be very important in our proof.

As in Theorem 1 in the introduction, let $M$ be a connected complex manifold of dimension $n$ that is holomorphically separable and admits a smooth envelope of holomorphy and assume that there exists a topological group isomorphism $\Phi : \mathrm{Aut}^o(\Delta^n) \to G$, where $G$ is the given topological subgroup of $\mathrm{Aut}(M)$. Since $\Delta^n$ is a Reinhardt domain in $\mathbf{C}^n$, the $n$-dimensional torus $T^n$ acts naturally on $\Delta^n$ as a connected Lie transformation group, so that, via the isomorphism $\Phi$, $T^n$ now acts effectively and continuously on $M$ by biholomorphic transformations. Hence this action is necessarily real analytic by a classical result of Bochner and Montgomery [3]. Therefore, by a well-known fact due to Barrett, Bedford and Dadok [1], we may assume that $M$ is a Reinhardt domain $D$ in $\mathbf{C}^n$ and that there exists a topological group isomorphism $\Phi : \mathrm{Aut}^o(\Delta^n) \to G \subset \mathrm{Aut}(D)$ such that $\Phi(T(\Delta^n)) = T(D)$, where $T(\Delta^n)$ and $T(D)$, respectively, denote the subgroups of $\mathrm{Aut}(\Delta^n)$ and of $\mathrm{Aut}(D)$ induced by the restrictions of the standard $T^n$-action on $\mathbf{C}^n$ to $\Delta^n$ and to $D$.

Now, the group $G$ can be turned into a Lie group by transferring the Lie group structure from $\mathrm{Aut}^o(\Delta^n)$ by means of $\Phi$. Since the Lie group $G$ endowed with the compact-open topology acts continuously on $D$ by biholomorphic transformations, the action is real analytic with respect to the Lie group structure induced from $\mathrm{Aut}^o(\Delta^n)$ (cf. [3]). Thus $G$ is now a Lie transformation group of $D$ acting effectively on $D$ by biholomorphic transformations; accordingly, the Lie algebra of $G$ can be identified with the Lie algebra $\mathfrak{g}$ consisting of all holomorphic vector fields on $D$ induced by one-parameter subgroups of $G$ (so-called

*G-vector fields* on $D$). We thus obtain the Lie algebra isomorphism $d\Phi : \mathfrak{g}(\Delta^n) \to \mathfrak{g}$ induced by $\Phi$. From now on, for the sake of simplicity, let us put

$$G_j = \Phi(\mathrm{Aut}(\Delta_j)), \quad \mathfrak{g}_j = d\Phi(\mathfrak{g}(\Delta_j)) \quad \text{and}$$
$$I_j = d\Phi(H_j), \quad X_j = d\Phi(V_j), \quad Y_j = d\Phi(W_j)$$

for $1 \le j \le n$. Then $G = G_1 \times \cdots \times G_n$ and, by (2.1) and (2.2), we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\} \quad \text{for } 1 \le i, j \le n,\ i \ne j; \tag{2.3}$$

$$\mathfrak{g}_j = \mathbf{R}\{I_j, X_j, Y_j\} \quad \text{and} \quad [I_j, [I_j, X_j]] = -X_j, \quad [Y_j, X_j] = 4I_j \tag{2.4}$$

for every $1 \le j \le n$.

Put $D^* = D \cap (\mathbf{C}^*)^n$ and, for a point $z \in D$, let $(\mathfrak{g}_j)_z$ denote the subspace of the tangent space to $D$ at $z$ that consist of the values of the elements of $\mathfrak{g}_j$ at $z$. Then, using the bracket relations (2.3) and (2.4), one can verify the following assertion:

1) *For every point $z_o \in D^*$, there exist a local holomorphic coordinate system $(U, w_1, \ldots, w_n)$ on $D^*$, centered at $z_o$, and a nowhere dense real analytic subset $\mathscr{A}$ of $U$ such that $(\mathfrak{g}_j)_p = \mathbf{C}\{(\partial/\partial w_j)_p\}$ for $p \in U \backslash \mathscr{A}$ and $1 \le j \le n$.*

Therefore, if we choose a point $p \in U\backslash\mathscr{A}$ and consider the orbits

$$D_p := G \cdot p \quad \text{and} \quad S_j := G_j \cdot p \ \ (1 \le j \le n)$$

of $G$ and of $G_j$ passing through $p$, then the assertion 1) together with (2.3) guarantees us that every $S_j$ is a complex submanifold of $D$ and $D_p$ is an open subset of $D$. Hence $D_p$ is a Reinhardt domain in $\mathbf{C}^n$, because $G$ is connected and contains the torus $T(D) = T^n$. More precisely, in exactly the same way as in the proof of [8, Theorem], it can be shown that

2) *every $S_j$ is biholomorphically equivalent to the unit disc $\Delta_j$*;

3) *$D_p$ is biholomorphically equivalent to the unit polydisc $\Delta^n$*; and

4) *$D$ is a bounded domain in $\mathbf{C}^n$ and $D_p$ is an open dense subset of $D$.*

Thus the proof of Theorem 1 is now reduced to showing that $D_p$ is also closed in $D$. If $G$ is a closed subgroup of $\mathrm{Aut}(D)$, then $G$ acts properly on $D$, as seen in the proof of [8; Theorem]. Consequently, the orbit $D_p = G \cdot p$ has to be closed in $D$ in this case. Here, whether or not $G$ is closed in $\mathrm{Aut}(D)$, we want to verify the closedness of $D_p$ in $D$. To this end, assume the contrary that there exists a boundary point $q \in \partial D_p$ in $D$. Let $d_D$ denote the Kobayashi distance on $D$ and let $K(x; r) = \{y \in D \mid d_D(x, y) < r\}$ be the Kobayashi ball of radius $r > 0$ with center $x \in D$. Since $d_D$ induces the standard topology of $D$ (cf. [2], [12]) and $p$ is an interior point of $D_p$, one can pick a small $r > 0$ in such a way that $K(p; r) \subset D_p$. For such an $r > 0$, choose a point $x_o \in D_p \cap K(q; r)$ arbitrarily and let $g_o$ be an element of $G$ such that $x_o = g_o \cdot p$. Then, since $d_D$ is invariant under the action of $G \subset \mathrm{Aut}(D)$, we have

$$d_D(g_o^{-1} \cdot q, p) = d_D(q, g_o \cdot p) = d_D(q, x_o) < r,$$

which means that $g_o^{-1} \cdot q \in K(p;r) \subset D_p$ and hence $q \in g_o \cdot D_p = D_p$, a contradiction to $q \in \partial D_p$. Therefore $D_p$ is, in fact, closed in $D$ and accordingly $D = D_p$ is biholomorphically equivalent to $\Delta^n$; completing the proof of Theorem 1. $\square$

## 3. Proof of Theorem 2

We shall use several fundamental facts on symmetric spaces without proofs. For the details, the reader may consult, for instance, Helgason's book [4].

Let $M$ be a connected complex manifold of dimension $n$ and let $D$ be a bounded symmetric domain in $\mathbf{C}^n$. Let $\mathbf{G}$ be the identity component of $\mathrm{Aut}(D)$ and let $\mathfrak{G}$ be its Lie algebra. Fix a point $o \in D$ once and for all and let $\mathbf{K}$ be the isotropy subgroup of $\mathbf{G}$ at $o$. Then $\mathbf{G}$ is a semi-simple Lie group with trivial center that acts transitively on $D$ and $\mathbf{K}$ is a maximal compact subgroup of $\mathbf{G}$. Note that, since a maximal compact subgroup of a connected Lie group is always connected, $\mathbf{K}$ is a connected Lie subgroup of $\mathbf{G}$. Moreover, $D$ can now be represented as the coset space $D = \mathbf{G}/\mathbf{K}$. Consider here the involutive automorphism $\sigma : g \mapsto s_o g s_o$ of $\mathbf{G}$, where $s_o$ denotes the symmetry of $D$ with respect to $o$, and put $s = d\sigma$, the involutive automorphism of $\mathfrak{G}$ induced by $\sigma$. Let $\mathfrak{K}$ and $\mathfrak{P}$ be the eigenspaces of $s$ for the eigenvalues $+1$ and $-1$, respectively. Then $\mathfrak{K}$ coincides with the Lie algebra of $\mathbf{K}$ and we have

$$\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}, \quad [\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{K}, \quad [\mathfrak{K}, \mathfrak{P}] \subset \mathfrak{P} \quad \text{and} \quad [\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{K}. \tag{3.1}$$

As usual, we identify $\mathfrak{P}$ with the tangent space $T_o(D)$ to $D$ at $o$; accordingly, $\mathfrak{P} = T_o(D)$ has the complex structure $J_o^D$ induced by the standard complex structure tensor $J^D$ on $D$. Thus $\mathfrak{P}$ can be regarded as a complex vector space. Moreover, under the identification $T_o(D) = \mathfrak{P}$, the linear isotropy group $\mathbf{K}^*$ of $\mathbf{G}$ at $o$ is just the group $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K})$, where $\mathrm{Ad}_{\mathbf{G}}$ is the adjoint representation of $\mathbf{G}$. We will often use this fact in the proof.

Assume now that there exists a topological group isomorphism $\Phi : \mathbf{G} \to G$, where $G$ is the given topological subgroup of $\mathrm{Aut}(M)$ in Theorem 2. Since $\mathbf{G}$ is a Lie group, $G$ has a unique Lie group structure with respect to which $\Phi : \mathbf{G} \to G$ is a Lie group isomorphism. Thus, by the same reasoning as in the proof of Theorem 1, $G$ becomes a Lie transformation group of $M$ acting effectively on $M$ by biholomorphic transformations. We denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $d\Phi : \mathfrak{G} \to \mathfrak{g}$ the Lie algebra isomorphism induced by $\Phi$.

Fix a point $p \in M$ arbitrarily and denote by $K$ the isotropy subgroup of $G$ at $p$. Then, by our assumption, $K$ is a compact subgroup of $G$. Here, along the same line as in [6], we shall show that $G$ acts transitively on $M$; accordingly, $M$ can be written in the form $M = G/K$. To this end, choose a maximal compact subgroup $\hat{K}$ of $G$ containing $K$. Then, since any two maximal compact subgroups of $G$ are always conjugate under an inner automorphism of $G$, one can find an element $g_o \in G$ such that $\hat{K} = g_o \Phi(\mathbf{K}) g_o^{-1}$. Moreover, notice that the orbit $G \cdot p = G/K$ of $G$ passing through $p$ is a real analytic submanifold of $M$. Thus

$$2n \geq \dim G/K \geq \dim G/\hat{K} = \dim \mathbf{G}/\mathbf{K} = 2n,$$

from which we have $K = \hat{K}$, $\dim G/K = 2n$ and hence the orbit $G \cdot p = G/K$ is open in $M$. Since this is true for any point $q \in M$ with $q \neq p$ and since $M$ is connected, we conclude that $M = G/K$, as desired. Therefore, by replacing $\Phi$ by $g_o \Phi(\cdot) g_o^{-1}$ if necessary, one may assume that $\hat{K} = \Phi(\mathbf{K})$; consequently, $\Phi$ induces a real analytic diffeomorphism, say again,

$$\Phi : D = \mathbf{G}/\mathbf{K} \to G/K = M. \tag{3.2}$$

Put $\mathfrak{k} = d\Phi(\mathfrak{K})$ and $\mathfrak{p} = d\Phi(\mathfrak{P})$. Then $\mathfrak{k}$ is the Lie subalgebra of $\mathfrak{g}$ corresponding to $K$ and we have the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with the same properties as in (3.1). Let $J^M$ be the $G$-invariant complex structure tensor on $M$ and let $J_p^M$ be the complex structure on $T_p(M) = \mathfrak{p}$ induced by $J^M$. Then, since $J_p^M$ commutes with each element in the linear isotropy group $K^*$ of $G$ at $p$, so does with $\mathrm{Ad}_G(k)$ for all $k \in K$, where $\mathrm{Ad}_G$ is the adjoint representation of $G$.

In order to complete the proof of Theorem 2, we need to prove that, after a slight modification if necessary, the diffeomorphism $\Phi$ in (3.2) gives rise to a biholomorphic equivalence between $D$ and $M$. For this purpose, by using the fact that $d\Phi$ gives a linear isomorphism from $\mathfrak{P}$ onto $\mathfrak{p}$, let us define the endomorphism $J_o^*$ of $\mathfrak{P}$ by the formula

$$d\Phi(J_o^* X) = J_p^M(d\Phi(X)) \quad \text{for all } X \in \mathfrak{P}. \tag{3.3}$$

Then $J_o^* \circ J_o^* = -I$ and moreover, since

$$d\Phi(\mathrm{Ad}_{\mathbf{G}}(k)X) = \mathrm{Ad}_G(\Phi(k))\, d\Phi(X) \quad \text{for all } k \in \mathbf{K} \text{ and all } X \in \mathfrak{P},$$

it can be easily seen that $J_o^*$ commutes with $\mathrm{Ad}_{\mathbf{G}}(k)$ for all $k \in \mathbf{K}$. Therefore $D = \mathbf{G}/\mathbf{K}$ admits a unique almost complex structure tensor $J^*$ which coincides with $J_o^*$ at $o$ and is invariant under the action of $\mathbf{G}$. The proof is now divided into two cases as follows:

Case 1. *$D$ is irreducible.* In this case, $\mathbf{G}$ is a simple Lie group and $\mathbf{K}$ is a maximal compact subgroup of $\mathbf{G}$ with one-dimensional center isomorphic to the circle group $S^1$. By definition of the irreducibility, $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K})$ now acts irreducibly on $\mathfrak{P}$. Hence, Schur's lemma implies that $J_o^* = cI$ with some constant $c \in \mathbf{C}$; accordingly $J_o^* = \pm\sqrt{-1}I = \pm J_o^D$ and $J^* = \pm J^D$, because $(J_o^*)^2 = -I$. Moreover, we would like to assert here the following: one may assume, without loss of generality, that $D$ is invariant under the complex conjugation $\psi : z \to \bar{z}$ of $\mathbf{C}^n$ with respect to $\mathbf{R}^n$. Indeed, in the case where $D$ is one of the four classical domains, it is well-known that $D$ can be realized as a subdomain $\tilde{D}$ in some complex matrix space (cf. [5]). Then, a glance at $\tilde{D}$ tells us that it is invariant under the complex conjugation $\psi$. On the other hand, in the case where $D$ is an exceptional bounded symmetric domain, it is shown in Roos [11; Section 3] that its Harish-Chandra realization $\tilde{D}$ has an explicit algebraic and geometric description using exceptional Jordan triple systems; from which it follows at once that $\tilde{D}$ is invariant under the complex conjugation $\psi$, as asserted. Thus, taking the diffeomorphism $\Phi \circ \psi$ instead of $\Phi$ in (3.2) if necessary, we may assume that $J^* = J^D$. This combined with (3.3) yields that $\Phi : D \to M$ is holomorphic;

consequently, it gives a biholomorphic equivalence between $D$ and $M$, as required.

Case 2. *$D$ is reducible*. In this case, $D$ can be uniquely (up to an order) decomposed into the direct product

$$D = D_1 \times \cdots \times D_r, \tag{3.4}$$

where the factors $D_i$ are irreducible bounded symmetric domains in $\mathbf{C}^{n_i}$ with $n_1 + \cdots + n_r = n$. Here, as in Case 1, one may assume that each $D_i$ is invariant under the complex conjugation. Let $\mathbf{G}$ and $\mathbf{G}_i$ be the identity components of $\mathrm{Aut}(D)$ and of $\mathrm{Aut}(D_i)$. And, writing $o = (o_1, \dots, o_r)$ with $o_i \in D_i$ according to the decomposition (3.4), we denote by $\mathbf{K}$ and $\mathbf{K}_i$ the isotropy subgroups of $\mathbf{G}$ and of $\mathbf{G}_i$ at $o$ and at $o_i$, respectively. Then, as mentioned in Case 1, each $\mathbf{G}_i$ is a simple Lie group with $\mathbf{K}_i$ as a maximal compact subgroup of it and $D_i$ is a homogeneous space of $\mathbf{G}_i$. Moreover, we have $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ and $\mathbf{K} = \mathbf{K}_1 \times \cdots \times \mathbf{K}_r$, so that $D$ can be expressed as

$$D = \mathbf{G}/\mathbf{K} = \mathbf{G}_1/\mathbf{K}_1 \times \cdots \times \mathbf{G}_r/\mathbf{K}_r. \tag{3.5}$$

Let $\mathfrak{G}_i$ be the Lie algebra of $\mathbf{G}_i$. Let $\sigma_i$ be the involutive automorphism $g \mapsto s_{o_i} g s_{o_i}$ of $\mathbf{G}_i$ and put $s_i = d\sigma_i$. Then, denoting by $\mathfrak{K}_i$ and $\mathfrak{P}_i$, respectively, the eigenspaces of $s_i$ for the eigenvalues $+1$ and $-1$, we obtain the direct sum decomposition $\mathfrak{G}_i = \mathfrak{K}_i \oplus \mathfrak{P}_i$ as in (3.1). As before, we identify $\mathfrak{P}_i = T_{o_i}(D_i)$ and we denote also by $J^{D_i}$ the standard complex structure tensor on $D_i$. Let $J_o^*$ be the complex structure on $\mathfrak{P} = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_r$ defined by (3.3). Then, since $J_o^*$ commutes with $\mathrm{Ad}_{\mathbf{G}}(k)$ for all $k \in \mathbf{K}$ and since $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K}_i)$ acts irreducibly on $\mathfrak{P}_i$ and trivially on $\mathfrak{P}_j$ for $j \neq i$, it follows that each $\mathfrak{P}_i$ is invariant under $J_o^*$. Thus $J_o^*$ is decomposed $J_o^* = J_{o_1}^* \times \cdots \times J_{o_r}^*$, where each $J_{o_i}^*$ is the restriction of $J_o^*$ to $\mathfrak{P}_i$. Therefore, letting $J_i^*$ be the unique $\mathbf{G}_i$-invariant almost complex structure tensor on $D_i$ which coincides with $J_{o_i}^*$ at $o_i$, we have $J^* = J_1^* \times \cdots \times J_r^*$. Moreover, since $\mathrm{Ad}_{\mathbf{G}_i}(\mathbf{K}_i)$ acts now irreducibly on $\mathfrak{P}_i$, Schur's lemma again implies that $J_i^* = \pm J^{D_i}$ for each $1 \le i \le r$. Finally, consider a real analytic diffeomorphism $\hat{\Phi} : D = D_1 \times \cdots \times D_r \to M$ given by

$$\hat{\Phi}(u) = \Phi(\gamma_1(u_1), \dots, \gamma_r(u_r)) \quad \text{for } u = (u_1, \dots, u_r) \in D_1 \times \cdots \times D_r = D,$$

where $\gamma_i(u_i) = u_i$ or $\gamma_i(u_i) = \bar{u}_i$, the complex conjugation in $\mathbf{C}^{n_i}$, for $1 \le i \le r$ and $\Phi$ is the diffeomorphism appearing in (3.2). Then, replacing $\Phi$ by a suitable $\hat{\Phi}$ if necessary, we have $J^* = J^D$. This means that $\Phi : D \to M$ is holomorphic. Therefore, we have shown that $\Phi$ gives a biholomorphic equivalence between $D$ and $M$; thereby completing the proof of Theorem 2. □

## 4. A concluding remark

In this section, we would like to illustrate that the conditions (A) and (B) stated in the introduction are mutually independent, in general, with concrete examples as follows:

*Example* 1. Consider the two-dimensional complex Euclidean space $\mathbf{C}^2$, for instance. Then, the condition (A) is trivially satisfied for $\mathbf{C}^2$. On the other hand, notice that the isotropy subgroup $\mathrm{Aut}_o(\mathbf{C}^2)$ of $\mathrm{Aut}(\mathbf{C}^2)$ at the origin $o$ of $\mathbf{C}^2$ contains the biholomorphic mappings $\varphi_\nu : \mathbf{C}^2 \to \mathbf{C}^2$ defined by

$$\varphi_\nu(z,w) = (z, w \exp(\nu z)), \quad (z,w) \in \mathbf{C}^2 \text{ for } \nu = 1,2,\ldots.$$

Clearly this says that $\mathrm{Aut}_o(\mathbf{C}^2)$ is not to be compact; hence, the condition (B) is not satisfied for $\mathbf{C}^2$.

*Example* 2. Take an arbitrary compact connected hyperbolic manifold $X$ of dimension $\geqq 2$ and consider the manifold $M$ obtained from $X$ by deletion of one point, say $M = X \setminus \{p\}$ $(p \in X)$. Then, being a complex submanifold of the hyperbolic manifold $X$, $M$ is also hyperbolic. Accordingly, the condition (B) is automatically satisfied for $M$. However, we assert that $M$ is not holomorphically separable and does not admit a smooth envelope of holomorphy. To verify this, note that any holomorphic function on $M$ can be holomorphically extended to $X$ and hence it must be constant, because $X$ is a compact connected complex manifold of dimension $\geqq 2$. Thus, $M$ is never holomorphically separable. Moreover, assume that there exists a smooth envelope of holomorphy of $M$. Then, since every Stein manifold can be realized as a closed complex submanifold of some $\mathbf{C}^N$, we have a holomorphic imbedding $F : M \to \mathbf{C}^N$. But, since any holomorphic function on $M$ is now constant as mentioned above, $F$ must be also constant. Clearly, this is a contradiction. Therefore the condition (A) is not satisfied for this manifold $M$.

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