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KODAI MATH. J.
32 (2009), 256–323

COMPACTNESS CHARACTERIZATION OF COMMUTATORS FOR LITTLEWOOD-PALEY OPERATORS*

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Abstract

In this paper, the authors prove that the commutator $[b, L]$ is a compact operator in $L^p(\mathbf{R}^n)$ if and only if $b \in VMO(\mathbf{R}^n)$, where L denotes the Littlewood-Paley operators, such as the Littlewood-Paley g-function, Lusin area integral and Littlewood-Paley g_λ^* function.

1. Introduction and main results

For $\varphi \in \mathscr{S}(\mathbf{R}^n)$ with $\int \varphi(x)\,dx = 0$, then the following operators are well defined:

(i) Littlewood-Paley g-function:

$$g_\varphi(f)(x) = \left(\int_0^\infty |\varphi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2},$$

(ii) Lusin area integral:

$$S(f)(x) = \left(\iint_{\Gamma(x)} |\varphi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

(i) Littlewood-Paley g_λ^* function:

$$g_\lambda^* f(x) = \left(\iint_{\mathbf{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |\varphi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\varphi_t = t^{-n}\varphi(x/t)$ and $\Gamma(x) = \{(y,t) \in \mathbf{R}_+^{n+1} : |x-y| < t\}$ and $\lambda > 1$.

The operators g_φ, S and g_λ^* are called as Littlewood-Paley operators. It is well known that the Littlewood-Paley operators are very important tools in the singular integral operators theory, function spaces theory and PDE (see [28], [29],

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MR (2000) *Subject Classification*: 42B20, 42B25

Keywords: Littlewood-Paley operators, Commutators, Compactness, BMO, VMO.

* The research was supported by NSF of China (Grant: 10571015, 10826046) and SRFDP of China (Grant: 20050027025).

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Received July 3, 2008.

[31], [21], [6], [17], [22], for example). The L^p boundedness and the weighted L^p boundedness of the Littlewood-Paley operators have been studied by many authors (see [27], [3], [23], [19], [10], [11], [1], [35], for example).

On the other hand, the commutators of the Calderòn-Zygmund singular integral operator plays a very important role in characterizing function space ([5], [20], [25]) and studying the regularity of the solution of the second order elliptic equations ([8], [18]).

Let $S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$ be the unit sphere in $\mathbf{R}^n$ equipped with the Lebesgue measure $d\sigma$. For $b \in L_{loc}(\mathbf{R}^n)$, the commutator $[b, T_\Omega]$ formed by b and the singular integral operator T_Ω is defined by

$$[b, T_\Omega]f(x) = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^n}(b(x) - b(y))f(y)\,dy,$$

where Ω satisfies the following conditions:

(a) Ω is homogeneous function of degree zero on $\mathbf{R}^n \backslash \{0\}$, i.e.

$$\Omega(\lambda x) = \Omega(x) \quad \text{for any } \lambda > 0 \text{ and } x \in \mathbf{R}^n \backslash \{0\}. \tag{1.1}$$

(b) Ω has mean zero in S^{n-1}, i.e.

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0. \tag{1.2}$$

(c) $\Omega \in \mathrm{Lip}(S^{n-1})$, i.e. there exists constant $M > 0$ such that

$$|\Omega(x') - \Omega(y')| \le M|x' - y'| \quad \text{for any } x', y' \in S^{n-1}. \tag{1.3}$$

Before showing the known results, let us recall the definitions of $BMO(\mathbf{R}^n)$ and $VMO(\mathbf{R}^n)$. Denote

$$M(b, Q) = \frac{1}{|Q|}\int_Q |b(x) - b_Q|\,dx,$$

where Q is a cube in $\mathbf{R}^n$, $b_Q = \frac{1}{|Q|}\int_Q b(x)\,dx$. Then

$$BMO(\mathbf{R}^n) = \left\{ b \in L_{loc}(\mathbf{R}^n) : \|b\|_{BMO} = \sup_{Q \subset \mathbf{R}^n} M(b, Q) < \infty \right\}.$$

Moreover, denote by $VMO(\mathbf{R}^n)$ the BMO-closure of $C_c^\infty(\mathbf{R}^n)$, the set of C^∞-functions with compact support in $\mathbf{R}^n$.

In 1976, Coifman, Rochberg and Weiss [5] proved the following result:

THEOREM A ([5]). (i) *Suppose that* Ω *satisfies* (1.1), (1.2) *and* (1.3). *If* $b \in BMO(\mathbf{R}^n)$, *then* $[b, T_\Omega]$ *is bounded in* $L^p(\mathbf{R}^n)$ *for* $1 < p < \infty$.

(ii) *If* $[b, R_j]$ $(j = 1, \dots, n)$ *are bounded in* $L^p(\mathbf{R}^n)$ *for some* p, $1 < p < \infty$, *then* $b \in BMO(\mathbf{R}^n)$, *where* R_j *is the j-th Reisz transform.*

Using this result, the authors in [5] gave a decomposition of Hardy space.

In 1978, Uchiyama [34] showed that the Riesz transforms R_j in the conclusion (ii) of Theorem A can be replaced by the Calderón-Zygmund singular integral operator T_Ω.

Theorem B ([34]). *Suppose that Ω satisfies* (1.1), (1.2) *and* (1.3). *If $[b, T_\Omega]$ is bounded in $L^p(\mathbf{R}^n)$ for some p, $1 < p < \infty$, then $b \in BMO(\mathbf{R}^n)$.*

Combining Theorem B with the conclusion (i) of Theorem A, Uchiyama gave really a characterization of the L^p-boundedness of the commutator $[b, T_\Omega]$.

In 1990, Torchinsky and Wang [33] extended the conclusion (i) in Theorem A to the commutators $[b, g_\Omega]$ of the Littlewood-Paley g function g_Ω, where $g_\Omega := g_\varphi$ for $\varphi(x) = \Omega(x)|x|^{-n+1}\chi_{\{|x|\le 1\}}(x)$ with $\Omega \in L^1(S^{n-1})$ satisfying the condition (1.1) and (1.2). g_Ω is also called as Marcinkiewicz integral, which was first introduced by Stein [27] in 1958.

For $b \in L_{\mathrm{loc}}(\mathbf{R}^n)$, the commutator $[b, g_\Omega]$ formed by b and g_Ω is defined by

$$[b, g_\Omega]f(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Torchinsky and Wang proved the following conclusion:

Theorem C ([33]). *Suppose that $b \in BMO(\mathbf{R}^n)$ and Ω satisfies* (1.1), (1.2) *and* (1.3). *Then $[b, g_\Omega]$ is bounded in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Theorem C was improved by Ding, Lu and Yabuta [14] in 2002.

Theorem D ([14]). *Suppose that $\Omega \in L^q(S^{n-1})$ $(q > 1)$ and Ω satisfies* (1.1) *and* (1.2). *If $b \in BMO(\mathbf{R}^n)$, then $[b, g_\Omega]$ is bounded in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Recently, we considered the converse problem for the commutator $[b, g_\Omega]$ in [7]. That is, we showed that the conclusion of Theorem B is also true for the commutator $[b, g_\Omega]$.

Theorem E ([7]). *Suppose that Ω satisfies* (1.1), (1.2) *and the following condition*:

$$(1.4)\quad |\Omega(x') - \Omega(y')| \le \frac{C_1}{\left(\log \dfrac{2}{|x'-y'|}\right)^\gamma} \qquad C_1 > 0,\ \gamma > 1 \ \text{and}\ x', y' \in S^{n-1}.$$

If the commutator $[b, g_\Omega]$ is bounded in $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$, then $b \in BMO(\mathbf{R}^n)$.

Remark 1.1. It is easy to see that the condition (1.4) is weaker than $\mathrm{Lip}_\alpha(S^{n-1})$ for any $0 < \alpha \le 1$. Therefore Theorem E can be seen as an improvement of Theorem B in some sense.

Now let us turn to the compactness problem about the commutators. Uchiyama gave also a characterization of the commutator $[b, T_\Omega]$ which is compact operator in $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ in [34].

Theorem F ([34]). *Suppose that* Ω *satisfies* (1.1), (1.2) *and* (1.3).

(i) *If* $b \in L_{loc}(\mathbf{R}^n)$ *and* $[b, T_\Omega]$ *is a compact operator in* $L^p(\mathbf{R}^n)$ *for some* $1 < p < \infty$, *then* $b \in VMO(\mathbf{R}^n)$.

(ii) *If* $b \in VMO(\mathbf{R}^n)$, *then* $[b, T_\Omega]$ *is a compact operator in* $L^p(\mathbf{R}^n)$ *for* $1 < p < \infty$.

In 1993, Beatrous and Li [2] discussed also the boundedness and compactness for the commutators of mulitiplication operator related to Hankel type operator.

Therefore, an interesting question arises naturally. That is, is true the conclusion in Theorem F if replacing the commutator $[b, T_\Omega]$ by $[b, g_\Omega]$?

The purpose of this paper is to give a positive answer to above question. More precisely, under more weaker kernel conditions than one in Theorem F, we give a characterization for the commutators $[b, L]$ of the Littlewood-Paley operators which is a compact operator in $L^p(\mathbf{R}^n)$, where L expresses not only the Littlewood-Paley g function g_Ω, but also the Lusin area integral and Littlewood-Paley g_λ^* function with homogenous kernels.

To show our results, let us give some notations and definitions.

Definition 1 ([4]). Let X and Y be Banach spaces and U be a subset of X. Then operator $T : U \mapsto Y$ is said to be a compact operator if T is continuous and maps bounded subsets of U into strongly pre-compact subsets of Y.

Definition 2. Suppose that $\Omega(x') \in L^q(S^{n-1})$ for $q \ge 1$. Then the *integral modulus* $\omega_q(\delta)$ *of continuity of order* q of Ω is defined by

$$\omega_q(\delta) = \sup_{\|\tau\| \le \delta} \left(\int_{S^{n-1}} |\Omega(\tau x') - \Omega(x')|^q \, d\sigma(x') \right)^{1/q},$$

where τ denotes the rotation on $\mathbf{R}^n$ and $\|\tau\| = \sup_{x' \in S^{n-1}} |\tau x' - x'|$. The function Ω is said to satisfy the L^q-Dini condition, if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} \, d\delta < \infty.$$

Now, we give our first result as follows.

Theorem 1. *Suppose that* Ω *satisfies* (1.1), (1.2) *and* (1.4). *If* $b \in L_{loc}(\mathbf{R}^n)$ *and* $[b, g_\Omega]$ *is a compact operator in* $L^p(\mathbf{R}^n)$ *for some* $1 < p < \infty$, *then* $b \in VMO(\mathbf{R}^n)$.

However, the converse part of Theorem 1 holds also even under a condition which is much weaker than (1.4).

THEOREM 2. *Suppose that* $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1), (1.2) *and*

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}(1 + |\log \delta|)\, d\delta < \infty. \tag{1.5}$$

If $b \in VMO(\mathbf{R}^n)$, *then* $[b, g_\Omega]$ *is a compact operator on* $L^p(\mathbf{R}^n)$ *for* $1 < p < \infty$.

Remark 1.2. It is easy to see if Ω satisfies (1.4) for some $\gamma > 1$, then Ω satisfies also (1.5). In addition, it is also obvious to see that the condition (1.4) or (1.5) is weaker than $\mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$. Therefore, Theorems 1 and 2 may be seen as an improvement of Theorem F in this sense.

Now let us turn to the area integral and Littlewood-Paley g_λ^* function. Let $0 < \rho < n$ and denote $\varphi^\rho(x) = \Omega(x)|x|^{-n+\rho}\chi_{\{|x| \le 1\}}(x)$ with Ω satisfies the conditions (1.1)–(1.3). Then the parameterized area integral and parameterized Littlewood-Paley g_λ^* function are defined respectively by

$$S^\rho f(x) = \left(\iint_{\Gamma(x)} |\varphi_t^\rho * f(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

and

$$g_\lambda^{*,\rho} f(x) = \left(\iint_{\mathbf{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} |\varphi_t^\rho * f(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma(x) = \{(y, t) \in \mathbf{R}_+^{n+1} : |x - y| < t\}$ and $\lambda > 1$.

The parameterized Littlewood-Paley g-function first was discussed by Hörmander [23] in 1960. In 1990, Torchinsky and Wang [33] gave the weighted $L^2(\mathbf{R}^n)$ boundedness of S^ρ and $g_\lambda^{*,\rho}$ for $\rho = 1$ and $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$. For general ρ, in 1999, Sakamoto and Yabuta [26] gave the L^p boundedness of S^ρ and $g_\lambda^{*,\rho}$.

Now we give the definitions of the commutator $[b, S^\rho]$ and $[b, g_\lambda^{*,\rho}]$. Let $b \in L_{\mathrm{loc}}(\mathbf{R}^n)$, $0 < \rho < n$ and $\lambda > 1$. Then the commutator $[b, S^\rho]$ and $[b, g_\lambda^{*,\rho}]$ are defined respectively by

$$[b, S^\rho] f(x) = \left(\iint_{\Gamma(x)} \left| \frac{1}{t^\rho} \int_{|y-z| \le t} \frac{\Omega(y - z)}{|y - z|^{n-\rho}} (b(x) - b(z)) f(z)\, dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

and

$$[b, g_\lambda^{*,\rho}] f(x)$$

$$= \left(\iint_{\mathbf{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \left| \frac{1}{t^\rho} \int_{|y-z| \le t} \frac{\Omega(y - z)}{|y - z|^{n-\rho}} (b(x) - b(z)) f(z)\, dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$

In [15], Ding and Xue gave the L^p-boundedness of the commutators $[b, S^\rho]$ and $[b, g_\lambda^{*,\rho}]$.

Theorem G ([15]). *Suppose that $\Omega \in L^2(S^{n-1})$ satisfying* (1.1), (1.2) *and the following condition*:

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}(1 + |\log \delta|)^\sigma \, d\delta < \infty, \quad \sigma > 2, \tag{1.6}$$

where ω_2 denotes the integral modulus of continuity of order 2 of Ω. If $n/2 < \rho < n$ and $b \in BMO(\mathbf{R}^n)$, then for $1 < p < \infty$ there exists a constant $C > 0$ such that for any $f \in L^p(\mathbf{R}^n)$

(i) $\|[b, S^\rho](f)\|_p \le C\|b\|_{BMO}\|f\|_p$;

(ii) $\|[b, g_\lambda^{*,\rho}](f)\|_p \le C\|b\|_{BMO}\|f\|_p$ $(\lambda > 2)$.

Recently, we gave the converse of Theorem G in some sense in [7].

Theorem H ([7]). *Suppose that Ω satisfies* (1.1), (1.2) *and* (1.4). *For $0 < \rho < n$ and $\lambda > 1$, if $[b, S^\rho]$ or $[b, g_\lambda^{*,\rho}]$ is a bounded operator in $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$, then $b \in BMO(\mathbf{R}^n)$.*

Below we give the characterization of the compactness for the commutators $[b, S^\rho]$ and $[b, g_\lambda^{*,\rho}]$ in $L^p(\mathbf{R}^n)$. In other words to say, we show that the conclusions of Theorems 1 and 2 hold for the commutators $[b, S^\rho]$ and $[b, g_\lambda^{*,\rho}]$.

Theorem 3. *Suppose that Ω satisfies* (1.1), (1.2) *and* (1.4). *If $n/2 < \rho < n$ and $[b, S^\rho]$ is a compact operator in $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$, then $b \in VMO(\mathbf{R}^n)$.*

Theorem 4. *Suppose that $\Omega \in L^2(S^{n-1})$ satisfying* (1.1), (1.2) *and* (1.6). *If $n/2 < \rho < n$ and $b \in VMO(\mathbf{R}^n)$, then $[b, S^\rho]$ is a compact operator in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Similarly, for the commutator $[b, g_\lambda^{*,\rho}]$ of the parameterized Littlewood-Paley $g_\lambda^{*,\rho}$ function, we have

Theorem 5. *Suppose that Ω satisfies* (1.1), (1.2) *and* (1.4). *For $n/2 < \rho < n$ and $\lambda > 2$, if $[b, g_\lambda^{*,\rho}]$ is a compact operator in $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$, then $b \in VMO(\mathbf{R}^n)$.*

Theorem 6. *Suppose that $\Omega \in L^2(S^{n-1})$ satisfying* (1.1), (1.2) *and* (1.6). *If $n/2 < \rho < n$, $\lambda > 2$ and $b \in VMO(\mathbf{R}^n)$, then $[b, g_\lambda^{*,\rho}]$ is a compact operator in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

To end this introduction, we give two remarks as follows.

Remark 1.3. Do not like the singular integral, all Littlewood-Paley operators, such as g_Ω, S^ρ, $g_\lambda^{*,\rho}$ considered in this paper, are sublinear operators. It is well known that the compactness is very important property for many nonlinear operators arising in mathematical physics and differential geometry, hence the results presented in this paper have their important significance. Moreover, as far as know, this is the first paper to discuss the compactness for the commutators of the Littlewood-Paley operators. In this paper, we use some new idea to overcome the nonlinearity of the Littlewood-Paley operators.

Remark 1.4. It is easy to check the following pointwise relationship

$$[b, S^\rho]f(x) \le C[b, g_\lambda^{*,\rho}]f(x) \tag{1.7}$$

holds for any fixed functions f and b. However, (1.7) is unable to assure that the compactness of $[b, S^\rho]$ in L^p is implied by the compactness of $[b, g_\lambda^{*,\rho}]$ in L^p. Hence, Theorem 4 is not a consequence of Theorem 6, and similarly, Theorem 5 is also not a consequence of Theorem 3. So, we need to prove these theorems one by one.

This paper is organized as follows. In Section 2, we give some lemmas, which will be applied in proving theorems. In Section 3, we give the proofs of compactness characterization of the commutator $[b, g_\Omega]$. The compactness characterization of $[b, S^\rho]$ is showed in Section 4 and the conclusion about $[b, g_\lambda^{*,\rho}]$ is arranged in the final section. Throughout this paper the letter C will stand for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence. Moreover, $|E|$ denotes the Lebesgue measure of the measurable set E in $\mathbf{R}^n$. As usual, for $p \ge 1$, $p' = p/(p-1)$ denotes the dual exponent of p.

2. Some lemmas

In this section, we list some known results, which will be employed in the forthcoming considerations.

Lemma 2.1 ([28]). *If* $|x| > 2|y|$, *then* $|(x-y)' - x'| \le \dfrac{2|y|}{|x|}$, *where* $x' = \dfrac{x}{|x|}$ *for* $x \ne 0$.

Lemma 2.2 (see [9]). *If* Ω *satisfies conditions* (1.1), (1.2) *and* (1.4). *Let* $\beta > 0$. *Then for* $|x| > 2|y|$

$$\left|\frac{\Omega(x-y)}{|x-y|^\beta} - \frac{\Omega(x)}{|x|^\beta}\right| \le \frac{C}{|x|^\beta \left(\log \frac{|x|}{|y|}\right)^\gamma}.$$

LEMMA 2.3 ([32]). *If $f \in BMO(\mathbf{R}^n)$ and $1 \le p < \infty$, then*

$$\left(\frac{1}{|Q|}\int_Q |f(y) - f_Q|^p \, dy\right)^{1/p} \le C\|f\|_{BMO}$$

for any cube Q.

LEMMA 2.4 ([32]). *If $b(x) \in BMO(\mathbf{R}^n)$, $C_2 > C_1 > 2$ and Q is a cube centered at x_0 with diameter r, then exist positive constants C_3, C_4, C_5 (depend on C_1, C_2 and b), such that*

$$|\{C_1 r < |x - x_0| < C_2 r : |b(x) - b_Q| > v + C_3\}| \le C_4|Q|e^{-C_5 v}, \quad (0 < v < \infty).$$

LEMMA 2.5 ([30]). *Suppose that $f(x)$ is a measurable function, $\lambda(w) = |\{x \in \mathbf{R}^n : |f(x)| > w > 0\}|$ and E is a measurable set. Define $f^*(t) = \inf\{w : \lambda(w) \le t\}$, $t > 0$, then*

$$\int_E |f(x)|^p \, dx \le \int_0^{|E|} |f^*(t)|^p \, dt, \quad 1 \le p < \infty.$$

LEMMA 2.6 ([34]). *For $f \in BMO(\mathbf{R}^n)$, then $f \in VMO(\mathbf{R}^n)$ if and only if f satisfies the following three conditions:*

(i) $\lim_{a\to 0} \sup_{|Q|=a} M(f, Q) = 0$.

(ii) $\lim_{a\to\infty} \sup_{|Q|=a} M(f, Q) = 0$.

(iii) $\lim_{x\to\infty} M(f, Q + x) = 0$, *for each Q.*

LEMMA 2.7 (see [12]). *Suppose that $0 < \beta < n$, Ω satisfies (1.1) and the L^q-Dini condition (1.5) for $q \ge 1$. If there exists a constant $0 < \theta < 1/2$ such that $|x| < \theta R$, then for any $k \in \mathbf{N}$*

$$\left(\int_{2^k R<|y|<2^{k+1}R} \left|\frac{\Omega(y - x)}{|y - x|^{n-\beta}} - \frac{\Omega(y)}{|y|^{n-\beta}}\right|^q dy\right)^{1/q}$$
$$\le C(2^k R)^{n/q-(n-\beta)}\left\{\frac{|x|}{2^k R} + \int_{|x|/2^{k+1}R}^{|x|/2^k R} \frac{\omega_q(\delta)}{\delta} \, d\delta\right\},$$

where the constant $C > 0$ is independent of R, k and x.

LEMMA 2.8 ([17]). *Suppose that $\Omega \in L^1(S^{n-1})$ satisfying (1.1). Define the rough maximal operator by*

$$M_\Omega f(x) = \sup_{x\in Q\subset\mathbf{R}^n} \frac{1}{|Q|}\int_Q |f(x - y)|\,|\Omega(y)| \, dy.$$

Then for $1 < p \le \infty$, there exists a constant $C > 0$ such that for any $f \in L^p(\mathbf{R}^n)$

$$\|M_\Omega f\|_p \le C\|f\|_p.$$

3. The compactness of the commutator $[b, g_\Omega]$

3.1. The proof of Theorem 1: $[b, g_\Omega]$ is a compact operator in $L^p \Rightarrow b \in VMO$

Since $[b, g_\Omega]$ is a compact operator on $L^p(\mathbf{R}^n)$, then $[b, g_\Omega]$ is bounded in $L^p(\mathbf{R}^n)$ by Definition 1. Hence applying Theorem H, we know that $b \in BMO(\mathbf{R}^n)$. Without loss of generality, we may assume $\|b\|_{BMO} = 1$. By Lemma 2.6, to prove that $b \in VMO(\mathbf{R}^n)$, it suffices to show that b satisfies the conditions (i), (ii) and (iii) in Lemma 2.6.

First suppose that b does not satisfy (i) of Lemma 2.6. Then there exist a $\zeta > 0$ and a sequence of cubes $\{Q_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} r_j = 0$, where r_j is the diameter of $Q_j := Q_j(y_j)$, such that and for every j

$$M(b, Q_j) = |Q_j|^{-1} \int_{Q_j} |b(y) - b_{Q_j}|\, dy > \zeta. \tag{3.1}$$

Let

$$f_j(y) = |Q_j|^{-1/p}\{[\operatorname{sgn}(b(y) - b_{Q_j}) - c_0]\chi_{Q_j}(y)\}, \tag{3.2}$$

where $c_0 = |Q_j|^{-1} \int_{Q_j} \operatorname{sgn}(b(y) - b_{Q_j})\, dy$. Since $\int_{Q_j} [b(y) - b_{Q_j}]\, dy = 0$, it is easy to see that $|c_0| < 1$. Thus f_j has the following properties:

$$\operatorname{supp} f_j \subset Q_j; \tag{3.3}$$

$$f_j(y)(b(y) - b_{Q_j}) > 0; \tag{3.4}$$

$$\int_{\mathbf{R}^n} f_j(y)\, dy = 0; \tag{3.5}$$

$$|f_j(y)| \le 2|Q_j|^{-1/p} \quad \text{for } y \in Q_j. \tag{3.6}$$

Obviously, $\{\|f_j\|_p\}_{j=1}^{\infty}$ are bounded uniformly in j. Now we show that $\{[b, g_\Omega](f_j)\}_{j=1}^{\infty}$ is not a strongly pre-compact subsets of L^p with the above choice of $\{f_j\}_{j=1}^{\infty}$.

Below for $i = 1, \dots, 13$, B_i denotes a positive constant depending only on Ω, γ, ζ and B_j $(1 \le j < i)$. By (1.2), there exists $0 < B_1 < 1$ such that

$$\sigma\left(\left\{x \in S^{n-1} : \Omega(x') \ge \frac{2C_1}{\left(\log \frac{2}{B_1}\right)^{\gamma}}\right\}\right) > 0.$$

If denote

$$\Lambda = \left\{x \in S^{n-1} : \Omega(x') \ge \frac{2C_1}{\left(\log \frac{2}{B_1}\right)^{\gamma}}\right\}, \tag{3.7}$$

then Λ is a closed set by (1.4). We now claim that

(3.8)
$$\text{if } x' \in \Lambda \text{ and } y' \in S^{n-1} \text{ satisfying } |x' - y'| \le B_1, \text{ then } \Omega(y') \ge \frac{C_1}{\left(\log \frac{2}{B_1}\right)^{\gamma}}.$$

In fact, since $|\Omega(x') - \Omega(y')| \le \frac{C_1}{\left(\log \frac{2}{|x'-y'|}\right)^{\gamma}} \le \frac{C_1}{\left(\log \frac{2}{B_1}\right)^{\gamma}}$ and $\Omega(x') \ge 2\frac{C_1}{\left(\log \frac{2}{B_1}\right)^{\gamma}}$ by $x' \in \Lambda$, we hence get $\Omega(y') \ge \frac{C_1}{\left(\log \frac{2}{B_1}\right)^{\gamma}}$.

Now let $B_2 = \frac{3}{B_1} + 1$. Then $|x - y_j| \simeq |x - y|$ for $x \in (B_2 Q_j)^c$ and $y \in Q_j$. Note that $\Omega \in L^\infty(S^{n-1})$, by (3.3), (3.6) and Lemma 2.3 we obtain

$$\begin{aligned}
(3.9)\quad & |g_\Omega((b - b_{Q_j})f_j)(x)| \\
&= \left\{ \int_0^\infty \left| \int_{|x-y|\le t} (b(y) - b_{Q_j}) f_j(y) \frac{\Omega(x-y)}{|x-y|^{n-1}}\, dy \right|^2 \frac{dt}{t^3} \right\}^{1/2} \\
&\le \int_{Q_j} |b(y) - b_{Q_j}|\, |f_j(y)| \frac{|\Omega(x-y)|}{|x-y|^{n-1}} \left(\int_{|x-y|\le t} \frac{dt}{t^3} \right)^{1/2} dy \\
&\le C \int_{Q_j} \frac{|b(y) - b_{Q_j}|\, |f_j(y)|}{|x-y|^n}\, dy \\
&\le C|Q_j|^{1/p'} \left(\frac{1}{|Q_j|} \int_{Q_j} |b(y) - b_{Q_j}|^{p'}\, dy \right)^{1/p'} \left(\int_{Q_j} \frac{|f_j(y)|^p}{|x-y|^{np}}\, dy \right)^{1/p} \\
&\le CQ_j|^{1/p'} |x - y_j|^{-n}.
\end{aligned}$$

Since $|x - y_j| > B_2|y - y_j|$ for $x \in (B_2 Q_j)^c \cap \{x : (x - y_j)' \in \Lambda\}$ and $y \in Q_j$, hence by Lemma 2.1

$$|(x-y)' - (x-y_j)'| \le 2\frac{|y - y_j|}{|x - y_j|} < B_1.$$

Applying (3.8), we get $\Omega((x-y)') \ge \frac{C_1}{\left(\log \frac{2}{B_1}\right)^{\gamma}}$. Thus, when $x \in (B_2 Q_j)^c \cap \{x : (x - y_j)' \in \Lambda\}$ and $y \in Q_j$, by (3.3), (3.4) and Hölder's inequality, we have

$$
\begin{aligned}
(3.10)\qquad & |g_\Omega((b-b_{Q_j})f_j)(x)| \\
&\geq \frac{C_1}{\left(\log \frac{2}{B_1}\right)^\gamma}\left\{\int_0^\infty\left(\int_{Q_j}\frac{(b(y)-b_{Q_j})f_j(y)\chi_{\{|x-y|\leq t\}}}{|x-y|^{n-1}}\,dy\right)^2\frac{dt}{t^3}\right\}^{1/2} \\
&\geq C\int_{|x-y_j|}^\infty\int_{Q_j}\frac{(b(y)-b_{Q_j})f_j(y)\chi_{\{|x-y|\leq t\}}}{|x-y|^{n-1}}\,dy\frac{dt}{t^3}\left(\int_{|x-y_j|}^\infty\frac{dt}{t^3}\right)^{-1/2} \\
&= C|x-y_j|\int_{Q_j}|x-y|^{1-n}(b(y)-b_{Q_j})f_j(y)\int_{\substack{|x-y_j|\leq t\\|x-y|\leq t}}\frac{dt}{t^3}\,dy \\
&\geq C|x-y_j|^{-n}\int_{Q_j}(b(y)-b_{Q_j})f_j(y)\,dy \\
&= C|x-y_j|^{-n}|Q_j|^{-1/p}\int_{Q_j}(b(y)-b_{Q_j})[sgn(b(y)-b_{Q_j})-c_0]\,dy \\
&= C|x-y_j|^{-n}|Q_j|^{-1/p}\int_{Q_j}|b(y)-b_{Q_j}|\,dy \\
&\geq C\zeta|Q_j|^{1/p'}|x-y_j|^{-n}.
\end{aligned}
$$

On the other hand, for $x\in(B_2Q_j)^c$, by (3.3), (3.5), (3.6) and $|x-y_j|\simeq|x-y|$ when $y\in Q_j$, we have

$$
\begin{aligned}
(3.11)\qquad & |(b(x)-b_{Q_j})g_\Omega(f_j)(x)| \\
&= |b(x)-b_{Q_j}|\left(\int_0^\infty\left|\int_{\mathbf{R}^n}\left(\frac{\Omega(x-y)}{|x-y|^{n-1}}\chi_{\{|x-y|\leq t\}}\right.\right.\right. \\
&\quad \left.\left.\left.-\frac{\Omega(x-y_j)}{|x-y_j|^{n-1}}\chi_{\{|x-y_j|\leq t\}}\right)f_j(y)\,dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\
&\leq |b(x)-b_{Q_j}|\left(\int_0^\infty\left|\int_{\substack{|x-y|\leq t\\|x-y_j|\leq t}}\left(\frac{\Omega(x-y)}{|x-y|^{n-1}}\right.\right.\right. \\
&\quad \left.\left.\left.-\frac{\Omega(x-y_j)}{|x-y_j|^{n-1}}\right)f_j(y)\,dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\
&\quad +|b(x)-b_{Q_j}|\left(\int_0^\infty\left|\int_{\substack{|x-y|\leq t\\|x-y_j|>t}}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)\,dy\right|^2\frac{dt}{t^3}\right)^{1/2}
\end{aligned}
$$

$$+ |b(x) - b_{Q_j}| \left(\int_0^\infty \left| \int_{\substack{|x-y|>t \\ |x-y_j|\le t}} \frac{\Omega(x-y_j)}{|x-y_j|^{n-1}} f_j(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{1/2}$$

$$\le |b(x) - b_{Q_j}| \int_{Q_j} |f_j(y)| \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-y_j)}{|x-y_j|^{n-1}} \right| \left(\int_{\substack{|x-y|\le t \\ |x-y_j|\le t}} \frac{dt}{t^3} \right)^{1/2} dy$$

$$+ |b(x) - b_{Q_j}| \int_{Q_j} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_j(y)| \left(\int_{\substack{|x-y|\le t \\ |x-y_j|>t}} \frac{dt}{t^3} \right)^{1/2} dy$$

$$+ |b(x) - b_{Q_j}| \int_{Q_j} \frac{|\Omega(x-y_j)|}{|x-y_j|^{n-1}} |f_j(y)| \left(\int_{\substack{|x-y|>t \\ |x-y_j|\le t}} \frac{dt}{t^3} \right)^{1/2} dy$$

$$\le C|b(x) - b_{Q_j}| \left(\int_{Q_j} \frac{|f_j(y)|}{|x-y_j|^n \left(\log \frac{|x-y_j|}{r_j}\right)^\gamma} dy \right.$$

$$\left. + {r_j}^{1/2} \int_{Q_j} \frac{|f_j(y)|}{|x-y_j|^{n+1/2}} dy \right)$$

$$\le C|Q_j|^{1/p'} \frac{|b(x) - b_{Q_j}|}{|x-y_j|^n \left(\log \frac{|x-y_j|}{r_j}\right)^\gamma}.$$

In the above estimate we have used Lemma 2.2. Before proceeding further, let us point out the following inequality:

$$\left(\int_{2^s r_j < |x-y_j| < 2^{s+1} r_j} |b(x) - b_{Q_j}|^p \, dx \right)^{1/p} \le C 2^{sn/p} s |Q_j|^{1/p}.$$

For $v > B_2$, using (3.11) and the above inequality we obtain

$$\text{(3.12)} \quad \left(\int_{|x-y_j|>vr_j} |(b(x) - b_{Q_j}) g_\Omega(f_j)(x)|^p \, dx \right)^{1/p}$$

$$\le C|Q_j|^{1/p'} \left(\int_{|x-y_j|>vr_j} \frac{|b(x) - b_{Q_j}|^p}{|x-y_j|^{np} \left(\log \frac{|x-y_j|}{r_j}\right)^{\gamma p}} dx \right)^{1/p}$$

$$\leq C|Q_j|^{1/p'}\sum_{s=[\log_2 v]}^{\infty}\left(\int_{2^s r_j<|x-y_j|\leq 2^{s+1}r_j}\frac{|b(x)-b_{Q_j}|^p}{|x-y_j|^{np}\left(\log\frac{|x-y_j|}{r_j}\right)^{\gamma p}}\,dx\right)^{1/p}$$

$$\leq C\sum_{s=[\log_2 v]}^{\infty}s^{1-\gamma}2^{-sn+sn/p}$$

$$\leq C(\log v)^{1-\gamma}v^{-n+n/p}.$$

Then for $\eta > v > B_2$, using (3.10) and (3.12) we get

$$\left(\int_{\{vr_j<|x-y_j|<\eta r_j\}}|[b,g_\Omega]f_j(x)|^p\,dx\right)^{1/p} \tag{3.13}$$

$$\geq\left(\int_{\{vr_j<|x-y_j|<\eta r_j\}\cap\{x:(x-y_j)'\in\Lambda\}}|g_\Omega((b-b_{Q_j})f_j)(x)|^p\,dx\right)^{1/p}$$

$$-\left(\int_{|x-y_j|>vr_j}|(b(x)-b_{Q_j})g_\Omega f_j(x)|^p\,dx\right)^{1/p}$$

$$\geq C\zeta|Q_j|^{1/p'}\left(\int_{\{vr_j<|x-y_j|<\eta r_j\}\cap\{x:(x-y_j)'\in\Lambda\}}\frac{1}{|x-y_j|^{np}}\,dx\right)^{1/p}$$

$$-C(\log v)^{1-\gamma}v^{-n+n/p}$$

$$\geq B_3\zeta(v^{-pn+n}-\eta^{-pn+n})^{1/p}-B_4(\log v)^{1-\gamma}v^{-n+n/p}.$$

From (3.9) and the proof of (3.12), we have

$$\left(\int_{|x-y_j|>\eta r_j}|[b,g_\Omega]f_j(x)|^p\,dx\right)^{1/p} \tag{3.14}$$

$$\leq\left(\int_{|x-y_j|>\eta r_j}|g_\Omega((b-b_{Q_j})f_j)(x)|^p\,dx\right)^{1/p}$$

$$+\left(\int_{|x-y_j|>\eta r_j}|(b(x)-b_{Q_j})g_\Omega(f_j)(x)|^p\,dx\right)^{1/p}$$

$$\leq B_5\eta^{-n+n/p}+B_6(\log\eta)^{1-\gamma}\eta^{-n+n/p}.$$

By (3.13) and (3.14), there exist B_7, $k=k(\Omega,p,\gamma,\zeta,B_3,B_4,B_5,B_6)>1$ and B_9 satisfying $B_2<B_7$,

$$\left(\int_{B_7 r_j<|x-y_j|<kB_7 r_j} |[b, g_\Omega]f_j|^p \, dy\right)^{1/p} \geq B_9 \tag{3.15}$$

and

$$\left(\int_{|x-y_j|>kB_7 r_j} |[b, g_\Omega]f_j|^p \, dy\right)^{1/p} \leq B_9/4. \tag{3.16}$$

Denote $B_8 = kB_7$. Let $E \subset \{x : B_7 r_j < |x - y_j| < B_8 r_j\}$ be an arbitrary measurable set. Then by (3.9), (3.11) and the Minkowski inequality, we have

$$\begin{aligned}
&\left(\int_E |[b, g_\Omega]f_j(x)|^p \, dx\right)^{1/p} \\
&\leq \left(\int_E |g_\Omega((b - b_{Q_j})f_j)(x)|^p \, dx\right)^{1/p} + \left(\int_E |(b(x) - b_{Q_j})g_\Omega f_j(x)|^p \, dx\right)^{1/p} \\
&\leq C|Q_j|^{1/p'}\left(\int_E |x - y_j|^{-np} \, dx\right)^{1/p} \\
&\quad + C|Q_j|^{1/p'}\left(\int_E \frac{|b(x) - b_{Q_j}|^p}{|x - y_j|^{np}\left(\log \frac{|x - y_j|}{r_j}\right)^{\gamma p}} \, dx\right)^{1/p} \\
&\leq C\left\{\frac{|E|^{1/p}}{|Q_j|^{1/p}} + \left(\frac{1}{|Q_j|}\int_E |b(x) - b_{Q_j}|^p \, dx\right)^{1/p}\right\}.
\end{aligned} \tag{3.17}$$

Let $h_j(x) = b(x) - b_{Q_j}$. Denote

$$\lambda_{h_j}(\omega) = |\{B_7 r_j < |x - y_j| < B_8 r_j : |h_j(x)| > \omega\}|, \quad 0 < \omega < \infty.$$

By Lemma 2.4 there exist constants B_{10}, B_{11} and B_{12}, such that for $0 < \omega < \infty$

$$\lambda_{h_j}(\omega + B_{10}) = |\{B_7 r_j < |x - y_j| < B_8 r_j : |h_j(x)| > \omega + B_{10}\}| \leq B_{11}|Q_j|e^{-B_{12}\omega}.$$

That is, $\lambda_{h_j}(\omega) \leq B_{11}|Q_j|e^{-B_{12}(\omega - B_{10})}$. Let $h_j^*(t) = \inf\{\omega : \lambda_{h_j}(\omega) \leq t\}$ $(t > 0)$. Then we get

$$h_j^*(t) \leq \frac{1}{B_{12}} \log \frac{B_{11}|Q_j|}{t} + B_{10}, \quad (0 < t < B_{11}|Q_j|). \tag{3.18}$$

Notice that $E \subset \{x : B_7 r_j < |x - y_j| < B_8 r_j\}$, applying Lemma 2.5 and (3.18), when $|E| < B_{11}|Q_j|$, we get

$$\frac{1}{|Q_j|}\int_E |b(x)-b_{Q_j}|^p\,dx \le \frac{1}{|Q_j|}\int_0^{|E|} |h_j^*(t)|^p\,dt \tag{3.19}$$
$$\le \frac{1}{|Q_j|}\int_0^{|E|}\left(-\frac{1}{B_{12}}\log\frac{t}{B_{11}|Q_j|}+B_{10}\right)^p dt$$
$$= B_{11}\int_0^{|E|/(B_{11}|Q_j|)}\left(B_{10}-\frac{1}{B_{12}}\log t\right)^p dt$$
$$\le C\frac{|E|}{|Q_j|}\left(1+\log\frac{B_{11}|Q_j|}{|E|}\right)^{[p]+1}.$$

Combing (3.17) and (3.19), there exists constant $B_{13}<\min\{B_{11}^{1/n},B_8\}$, such that

$$\left(\int_E |[b,g_\Omega]f_j(y)|^p\,dy\right)^{1/p} \le B_9/4 \tag{3.20}$$

for every measurable set E satisfying

$$E\subset\{x: B_7r_j<|x-y_j|<B_8r_j\} \quad\text{and}\quad |E|/|Q_j|<B_{13}^n.$$

Now we choose a subsequence $\{Q_{j(k)}\}$ with their diameters $\{r_{j(k)}\}$ to satisfy

$$r_{j(k+1)}/r_{j(k)}<B_{13}/B_8. \tag{3.21}$$

Then for $m>0$, we have

$$\|[b,g_\Omega]f_{j(k)}-[b,g_\Omega]f_{j(k+m)}\|_p \ge \left(\int_{G_1}|[b,g_\Omega]f_{j(k)}(x)-[b,g_\Omega]f_{j(k+m)}(x)|^p\,dy\right)^{1/p}$$
$$\ge\left(\int_{G_1}|[b,g_\Omega]f_{j(k)}(x)|^p\,dx\right)^{1/p}$$
$$-\left(\int_{G_2}|[b,g_\Omega]f_{j(k+m)}(x)|^p\,dx\right)^{1/p}$$

where

$$G_1=\{x: B_7r_{j(k)}<|x-y_{j(k)}|<B_8r_{j(k)}\}\backslash\{x:|x-y_{j(k+m)}|\le B_8r_{j(k+m)}\},$$

and

$$G_2=\{x:|x-y_{j(k+m)}|>B_8r_{j(k+m)}\}.$$

Denote $G=\{x: B_7r_{j(k)}<|x-y_{j(k)}|<B_8r_{j(k)}\}$, then $G_1=G-(G_2^c\cap G)$. Thus by (3.15), (3.16) we get

$$\begin{aligned}
&\|[b, g_\Omega]f_{j(k)} - [b, g_\Omega]f_{j(k+m)}\|_p \\
&\quad \geq \left(\int_G |[b, g_\Omega]f_{j(k)}(x)|^p \, dx - \int_{G_2^c \cap G} |[b, g_\Omega]f_{j(k)}(x)|^p \, dx \right)^{1/p} \\
&\qquad - \left(\int_{G_2} |[b, g_\Omega]f_{j(k+m)}(x)|^p \, dx \right)^{1/p} \\
&\quad \geq \left(B_9^p - \int_{G_2^c \cap G} |[b, g_\Omega]f_{j(k)}(x)|^p \, dx \right)^{1/p} - \frac{B_9}{4}.
\end{aligned}$$

By (3.21), we have

$$\frac{|G_2^c \cap G|}{|Q_{j(k)}|} \leq \frac{B_8^n r_{j(k+m)}^n}{r_{j(k)}^n} < B_8^n \left(\frac{B_{13}^n}{B_8^n} \right)^m < B_{13}^n.$$

Thus by (3.20), we get

$$\int_{G_2^c \cap G} |[b, g_\Omega]f_{j(k)}(x)|^p \, dx \leq \left(\frac{B_9}{4} \right)^p.$$

So we get

$$\|[b, g_\Omega]f_{j(k)} - [b, g_\Omega]f_{j(k+m)}\|_p \geq \left(B_9^p - \left(\frac{B_9}{4} \right)^p \right)^{1/p} - \frac{B_9}{4} \geq \frac{B_9}{4}. \tag{3.22}$$

(3.22) shows that $\{[b, g_\Omega]f_{j(k)}\}_{k=1}^\infty$ does not have any convergence subsequence in $L^p(\mathbf{R}^n)$. Therefore, $[b, g_\Omega]$ is not compact operator in $L^p(\mathbf{R}^n)$. This contradiction shows that b must satisfy the condition (i) of Lemma 2.6. Quite similarly, we can prove that if b does not satisfy the conditions (ii) or (iii) of Lemma 2.6, then $[b, g_\Omega]$ is not a compact operator in $L^p(\mathbf{R}^n)$.

In fact, if b does not satisfy (ii) of Lemma 2.6, then there exist a $\zeta > 0$ and a sequence of cubes $\{Q_j\}_{j=1}^\infty$ with $\lim_{j\to\infty} r_j = \infty$, where r_j is the diameter of $Q_j := Q_j(y_j)$, such that (3.1) holds. Thus, (3.15), (3.16) and (3.20) hold still for the function sequence $\{f_j\}$ defined in (3.2). As done above, we may choose a subsequence $\{Q_{j(k)}\}$ such that its diameter sequence $\{r_{j(k)}\}$ satisfies (3.21). Then for $m > 0$, we have

$$\begin{aligned}
\|[b, g_\Omega]f_{j(k)} - [b, g_\Omega]f_{j(k+m)}\|_p &\geq \left(\int_{G_1} |[b, g_\Omega]f_{j(k)}(x) - [b, g_\Omega]f_{j(k+m)}(x)|^p \, dy \right)^{1/p} \\
&\geq \left(\int_{G_1} |[b, g_\Omega]f_{j(k+m)}(x)|^p \, dx \right)^{1/p} \\
&\quad - \left(\int_{G_2} |[b, g_\Omega]f_{j(k)}(x)|^p \, dx \right)^{1/p}
\end{aligned}$$

where

$$G_1 = \{x : B_7 r_{j(k+m)} < |x - y_{j(k+m)}| < B_8 r_{j(k+m)}\} \backslash \{x : |x - y_{j(k)}| \le B_8 r_{j(k)}\},$$

and

$$G_2 = \{x : |x - y_{j(k)}| > B_8 r_{j(k)}\}.$$

Denote

$$G = \{x : B_7 r_{j(k+m)} < |x - y_{j(k+m)}| < B_8 r_{j(k+m)}\},$$

then $G_1 = G - (G_2^c \cap G)$. Thus by (3.15) and (3.16), we get

$$\begin{aligned}
(3.23)\qquad & \|[b, g_\Omega] f_{j(k)} - [b, g_\Omega] f_{j(k+m)}\|_p \\
&\ge \left(\int_G |[b, g_\Omega] f_{j(k+m)}(x)|^p \, dx - \int_{G_2^c \cap G} |[b, g_\Omega] f_{j(k+m)}(x)|^p \, dx \right)^{1/p} \\
&\quad - \left(\int_{G_2} |[b, g_\Omega] f_{j(k)}(x)|^p \, dx \right)^{1/p} \\
&\ge \left(B_9^p - \int_{G_2^c \cap G} |[b, g_\Omega] f_{j(k+m)}(x)|^p \, dx \right)^{1/p} - \frac{B_9}{4}.
\end{aligned}$$

By (3.21), we have

$$\frac{|G_2^c \cap G|}{|Q_{j(k+m)}|} \le \frac{B_8^n r_{j(k)}^n}{r_{j(k+m)}^n} < B_8^n \left(\frac{B_{13}^n}{B_8^n} \right)^m < B_{13}^n,$$

then by (3.20), we get

$$\int_{G_2^c \cap G} |[b, g_\Omega] f_{j(k+m)}(x)|^p \, dx \le \left(\frac{B_9}{4} \right)^p.$$

So, by (3.23) we get

$$(3.24)\qquad \|[b, g_\Omega] f_{j(k)} - [b, g_\Omega] f_{j(k+m)}\|_p \ge \left(B_9^p - \left(\frac{B_9}{4} \right)^p \right)^{1/p} - \frac{B_9}{4} \ge \frac{B_9}{4}.$$

Thus $\{[b, g_\Omega] f_{j(k)}\}_{k=1}^\infty$ does not have any convergence subsequence in $L^p(\mathbf{R}^n)$. So, b should satisfy the condition (ii) of Lemma 2.6.

Finally, if b does not satisfy the condition (iii) of Lemma 2.6, then there exist a cube Q and a sequence $\{y_j\}$, with $\lim_{j\to\infty} y_j = \infty$, such that (3.1) holds for $\{Q_j = Q + y_j\}$. Similarly, (3.15) and (3.16) hold for the function sequence $\{f_j\}$ defined in (3.2). Denote $E_j = \{x \in \mathbf{R}^n : |x - y_j| < B_8 r\}$, where r is the diameter of Q, and choose a subsequence $\{E_{j(k)}\}$ such that

$$E_{j(k)} \cap E_{j(l)} = \emptyset, \quad l \ne k.$$

Then for $m > 0$, we have

$$\begin{aligned}\|[b,g_\Omega]f_{j(k)}-[b,g_\Omega]f_{j(k+m)}\|_p &\ge \left(\int_{G_1}|[b,g_\Omega]f_{j(k)}(x)-[b,g_\Omega]f_{j(k+m)}(x)|^p\,dy\right)^{1/p}\\ &\ge \left(\int_{G_1}|[b,g_\Omega]f_{j(k)}(x)|^p\,dx\right)^{1/p}\\ &\quad-\left(\int_{G_2}|[b,g_\Omega]f_{j(k+m)}(x)|^p\,dx\right)^{1/p}\end{aligned}$$

where

$$G_1=\{x:B_7r<|x-y_{j(k)}|<B_8r\}\backslash\{x:|x-y_{j(k+m)}|\le B_8r\},$$

and

$$G_2=\{x:|x-y_{j(k+m)}|>B_8r\}.$$

If denote $G=\{x:B_7r<|x-y_{j(k)}|<B_8r\}$, then $G_1=G$. Thus by (3.15) and (3.16) we get

$$\begin{aligned}&\|[b,g_\Omega]f_{j(k)}-[b,g_\Omega]f_{j(k+m)}\|_p\\ &\quad\ge \left(\int_G|[b,g_\Omega]f_{j(k)}(x)|^p\,dx\right)^{1/p}-\left(\int_{G_2}|[b,g_\Omega]f_{j(k+m)}(x)|^p\,dx\right)^{1/p}\\ &\quad\ge B_9-\frac{B_9}{4}\ge\frac{B_9}{4}.\end{aligned}\tag{3.25}$$

Thus $\{[b,g_\Omega]f_{j(k)}\}_{k=1}^{\infty}$ does not have any convergence subsequence in $L^p(\mathbf{R}^n)$ by (3.25). So this contradiction show that b satisfies also the condition (iii) of Lemma 2.6.

3.2. The proof of Theorem 2: $b\in VMO\Rightarrow[b,g_\Omega]$ is a compact operator in L^p

In the proof of this part, we need to use the following results.

Theorem J (Frechet-Kolmogorov) ([36]). *A subset G of $L^p(\mathbf{R}^n)$ $(1\le p<\infty)$, is strongly pre-compact if and only if G satisfies the conditions*:

$$\sup_{f\in G}\|f\|_p<\infty;\tag{3.26}$$

$$\lim_{|y|\to0}\|f(\cdot+y)-f(\cdot)\|_p=0\quad\textit{uniformly in } f\in G;\tag{3.27}$$

$$\lim_{\beta\to\infty}\|f\chi_{E_\beta}\|_p=0,\quad\textit{uniformly in } f\in G,\ \textit{where } E_\beta=\{x\in\mathbf{R}^n:|x|>\beta\}.\tag{3.28}$$

Theorem K ([11]). *If $\Omega\in H^1(S^{n-1})$ satisfies (1.1) and (1.2), then μ_Ω is of type (p,p) for $1<p<\infty$, where $H^1(S^{n-1})$ denotes the Hardy space on S^{n-1}.*

Remark 3.1. We point out that on S^{n-1}, for any $q > 1$

$$L^q(S^{n-1}) \subset L\log^+ L(S^{n-1}) \subset H^1(S^{n-1}),$$

and all inclusions are proper.

Let us return to the proof of Theorem 2. Suppose that $b \in VMO(\mathbf{R}^n) \subset BMO(\mathbf{R}^n)$, by Theorem D, the commutator $[b, g_\Omega]$ is continuous in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ since (1.5) implies $\Omega \in L^q(S^{n-1})$. Thus, it suffices to prove that for any bounded set $\mathscr{F}$ in $L^p(\mathbf{R}^n)$, the set $\{[b, g_\Omega]f : f \in \mathscr{F}\}$ is strongly pre-compact in $L^p(\mathbf{R}^n)$.

We first show that if the set $\{[b, g_\Omega]f : f \in \mathscr{F}\}$ is strongly pre-compact in $L^p(\mathbf{R}^n)$ for $b \in C_c^\infty(\mathbf{R}^n)$, then the set $\{[b, g_\Omega]f : f \in \mathscr{F}\}$ is also strongly pre-compact in $L^p(\mathbf{R}^n)$ for $b \in VMO(\mathbf{R}^n)$. In fact, suppose that $b \in VMO(\mathbf{R}^n)$, then for any $\eta > 0$ there exists $b^\eta \in C_c^\infty(\mathbf{R}^n)$ such that $\|b - b^\eta\|_* < \eta$. Since

$$\begin{aligned}
&|[b, g_\Omega]f(x) - [b^\eta, g_\Omega]f(x)| \\
&\quad \le \left\{\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}[(b-b^\eta)(x) - (b-b^\eta)(y)]f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2}.
\end{aligned}$$

Then by Theorem D,

$$\|[b, \mu_\Omega] - [b^\eta, g_\Omega]\|_{L^p \mapsto L^p} \le \|[b - b^\eta, \mu_\Omega]\|_{L^p \mapsto L^p} < C\eta. \tag{3.29}$$

Since F is a bounded set in $L^p(\mathbf{R}^n)$, there exists a constant $D > 0$ such that $\|f\|_p \le D$ for every $f \in \mathscr{F}$. If denote $\mathscr{G} = \{[b^\eta, g_\Omega]f : f \in \mathscr{F}\}$, then (3.26)–(3.28) hold for $\mathscr{G}$ by our assumption and Theorem J. We need to show that (3.26)–(3.28) hold also for the set $\tilde{\mathscr{G}} = \{[b, g_\Omega]f : f \in \mathscr{F}\}$. For any $f \in \mathscr{F}$, by (3.26) and (3.29) we get

$$\sup_{f \in \mathscr{F}} \|[b, g_\Omega]f\|_p \le \sup_{f \in \mathscr{F}} \|[b^\eta, g_\Omega]f\|_p + C\eta D < \infty.$$

On the other hand,

$$\begin{aligned}
\lim_{|y|\to 0} \|[b, g_\Omega]f(\cdot + y) - [b, g_\Omega]f(\cdot)\|_p &\le \lim_{|y|\to 0} \|[b^\eta, g_\Omega]f(\cdot + y) - [b^\eta, g_\Omega]f(\cdot)\|_p \\
&\quad + 2\|[b - b^\eta, g_\Omega]f\|_p \\
&\le 2C\eta D \to 0 \quad (\eta \to 0).
\end{aligned}$$

It is obvious to see that the limit above holds uniformly for $\tilde{\mathscr{G}}$. Similarly, we have

$$\begin{aligned}
\lim_{\beta\to\infty} \|[b, g_\Omega]f\chi_{E_\beta}\|_p &\le \lim_{\beta\to\infty} \|[b^\eta, g_\Omega]f\chi_{E_\beta}\|_p + \|[b - b^\eta, g_\Omega]f\|_p \\
&\le C\eta D \to 0 \quad (\eta \to 0).
\end{aligned}$$

Once again, the limit above is uniformly for $\tilde{\mathscr{G}}$. Therefore, by Theorem J we know $\tilde{\mathscr{G}}$ is a strongly pre-compact set in $L^p(\mathbf{R}^n)$.

Thus, to prove Theorem 2 it suffices to prove that $\mathscr{G} = \{[b, g_\Omega]f : f \in \mathscr{F}\}$ is strongly pre-compact in $L^p(\mathbf{R}^n)$ for $b \in C_c^\infty(\mathbf{R}^n)$ and the bounded set $\mathscr{F}$ in $L^p(\mathbf{R}^n)$. By Theorem J, we need only to verify (3.26)–(3.28) hold uniformly in $\mathscr{G}$.

Suppose that $\sup_{f\in\mathscr{F}}\|f\|_p \le D$. Notice that $b \in C_c^\infty(\mathbf{R}^n)$ and applying Theorem D, we have

$$\sup_{f\in\mathscr{F}}\|[b, g_\Omega]f\|_p \le C\|b\|_* \sup_{f\in\mathscr{F}}\|f\|_p \le C'D < \infty. \tag{3.30}$$

On the other hand, for any $\varepsilon > 0$ and $q > 1$, there is an $A > 0$ such that

$$\left(\int_A^\infty \frac{dr}{r^{nq-n+1}}\right)^{1/q} < \varepsilon. \tag{3.31}$$

It is easy to see that (3.28) holds uniformly in $\mathscr{G}$ if we can show that for $\Omega \in L^q(S^{n-1})$

$$\left(\int_{|x|>A} |[b, g_\Omega]f(x)|^p\, dx\right)^{1/p} \le CD\varepsilon. \tag{3.32}$$

Now we verify (3.32) to divide two cases.

The case for $1 < q \le p$. Assume $\operatorname{supp}(b) \subset \{y : |y| < \tau\}$ for some $\tau > 0$. Thus, for any x satisfying $|x| > \max\{2A, 4\tau\}$ and every $f \in \mathscr{F}$, we have

$$\begin{aligned}
|[b, \mu_\Omega]f(x)| &= \left\{\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2}\\
&\le C\int_{|y|<\tau} \frac{|b(y)|\,|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|\left\{\int_{|x-y|\le t}\frac{dt}{t^3}\right\}^{1/2} dy\\
&\le C\int_{|y|<\tau}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\,dy\\
&\le C\left(\int_{|y|<\tau}\frac{|\Omega(x-y)|^q}{|x-y|^{qn}}|f(y)|^q\,dy\right)^{1/q}\\
&\le C\left(\int_{|y|\ge 3|x|/4}\frac{|\Omega(y)|^q}{|y|^{qn}}|f(x-y)|^q\,dy\right)^{1/q}.
\end{aligned}$$

Applying the Minkowski inequality and (3.31), we get

$$\left(\int_{|x|>A} |[b,g_\Omega]f(x)|^p\,dx\right)^{1/p} \le C\|f\|_p\left(\int_{|x|>(3/2)A} \frac{|\Omega(x)|^q}{|x|^{nq}}\,dx\right)^{1/q}$$

$$\le C\|f\|_p\left(\int_{(3/2)A}^{\infty}\int_{S^{n-1}} |\Omega(x')|^q\,d\sigma(x')\frac{dr}{r^{nq-n+1}}\,dr\right)^{1/q}$$

$$\le CD\|\Omega\|_{L^q(S^{n-1})}\varepsilon \le CD\varepsilon.$$

The case for $q>p$. Taking $1<q_0\le p<q$, then $\Omega\in L^{q_0}(S^{n-1})$ with $\|\Omega\|_{L^{q_0}(S^{n-1})}\le C\|\Omega\|_{L^q(S^{n-1})}$. By the conclusion of the above case, we can get

$$\left(\int_{|x|>A} |[b,g_\Omega]f(x)|^p\,dx\right)^{1/p} \le CD\|\Omega\|_{L^{q_0}(S^{n-1})}\varepsilon \le CD\|\Omega\|_{L^q(S^{n-1})}\varepsilon \le CD\varepsilon.$$

Finally, to finish the proof of Theorem 2, it remains to show (3.27) holds uniformly in $\mathscr{G}$. We need to prove that for any $\varepsilon>0$, if $|z|$ is sufficiently small, then for every $f\in\mathscr{F}$,

$$\|[b,\mu_\Omega]f(\cdot)-[b,\mu_\Omega]f(\cdot+z)\|_p \le C\varepsilon. \tag{3.33}$$

To do this, we write

$$\begin{aligned}(3.34)\quad & |[b,g_\Omega]f(x)-[b,g_\Omega]f(x+z)| \\ &\le \left\{\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right.\right. \\ &\qquad \left.\left. -\int_{|x+z-y|\le t}\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}(b(x+z)-b(y))f(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2} \\ &:= \left\{\int_0^\infty |I(x,t)|^2\frac{dt}{t^3}\right\}^{1/2}.\end{aligned}$$

We take ε such that $0<\varepsilon<\frac{1}{2}$. Then for $z\in\mathbf{R}^n$, decompose $I(x,t)$ as

$$\begin{aligned}(3.35)\quad I(x,t) &= \int_{\substack{|x-y|>e^{1/\varepsilon}|z| \\ |x-y|\le t,\,|x+z-y|\ge t}} \frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x+z)-b(y))f(y)\,dy \\ &\quad + \int_{\substack{|x-y|>e^{1/\varepsilon}|z| \\ |x-y|\ge t,\,|x+z-y|\le t}} \frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}(b(y)-b(x+z))f(y)\,dy \\ &\quad + \int_{\substack{|x-y|>e^{1/\varepsilon}|z|, \\ |x-y|\le t,\,|x+z-y|\le t}} \left(\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right)\end{aligned}$$

$$
\begin{aligned}
&\times (b(x+z)-b(y))f(y)\,dy \\
&+\int_{\substack{|x-y|>e^{1/\varepsilon}|z|,\\ |x-y|\le t}} \frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(x+z))f(y)\,dy \\
&+\int_{\substack{|x-y|\le e^{1/\varepsilon}|z|,\\ |x-y|\le t}} \frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy \\
&+\int_{\substack{|x-y|\le e^{1/\varepsilon}|z|,\\ |x+z-y|\le t}} \frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}(b(y)-b(x+z))f(y)\,dy \\
:=\ & J_1(x,t)+J_2(x,t)+J_3(x,t)+J_4(x,t)+J_5(x,t)+J_6(x,t).
\end{aligned}
$$

Note $|b(x+z)-b(y)|<C$ and apply the Minkowski inequality, we have

$$
\begin{aligned}
(3.36)\quad & \left\{\int_0^\infty |J_1(x,t)|^2\frac{dt}{t^3}\right\}^{1/2} \\
&= \left\{\int_0^\infty \left|\int_{\substack{|x-y|>e^{1/\varepsilon}|z|,\\ |x-y|\le t,\,|x+z-y|\ge t}} \frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x+z)-b(y))f(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2} \\
&\le C\int_{|x-y|>e^{1/\varepsilon}|z|} \frac{|b(x+z)-b(y)|\,|f(y)|\,|\Omega(x-y)|}{|x-y|^{n-1}}\left\{\int_{\substack{|x-y|\le t\\ |x+z-y|\ge t}}\frac{dt}{t^3}\right\}^{1/2} dy \\
&\le C\int_{|x-y|>e^{1/\varepsilon}|z|} \frac{|z|^{1/2}|\Omega(x-y)|}{|x-y|^{n+1/2}}|f(y)|\,dy.
\end{aligned}
$$

Then by the Minkowski inequality and $\Omega\in L^1(S^{n-1})$, we get

$$
\begin{aligned}
(3.37)\quad & \left\|\left\{\int_0^\infty |J_1(x,t)|^2\frac{dt}{t^3}\right\}^{1/2}\right\|_p \\
&\le C\left\{\int_{\mathbf{R}^n}\left(\int_{|y|>e^{1/\varepsilon}|z|}\frac{|z|^{1/2}|\Omega(y)|}{|y|^{n+1/2}}|f(x-y)|\,dy\right)^p dx\right\}^{1/p} \\
&\le C\|f\|_p\int_{|y|>e^{1/\varepsilon}|z|}\frac{|z|^{1/2}|\Omega(y)|}{|y|^{n+1/2}}\,dy \\
&\le C\|f\|_p|z|^{1/2}\int_{e^{1/\varepsilon}|z|}^\infty\frac{dr}{r^{1+1/2}}\int_{S^{n-1}}|\Omega(y')|\,d\sigma(y') \\
&\le CD\varepsilon.
\end{aligned}
$$

Similar to the estimate of $J_1(x,t)$, we can get

$$\left\{\int_0^\infty |J_2(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2} \le C\int_{|x-y|>e^{1/\varepsilon}|z|} \frac{|z|^{1/2}|\Omega(x+z-y)|}{|x+z-y|^{n+1/2}}|f(y)|\,dy. \tag{3.38}$$

Then by the Minkowski inequality and $\Omega \in L^1(S^{n-1})$, we get

$$\begin{aligned}
&\left\|\left\{\int_0^\infty |J_2(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2}\right\|_p \\
&\quad \le C\left\{\int_{\mathbf{R}^n}\left(\int_{|y|>(e^{1/\varepsilon}-1)|z|} \frac{|z|^{1/2}|\Omega(y)|}{|y|^{n+1/2}}|f(x+z-y)|\,dy\right)^p dx\right\}^{1/p} \\
&\quad \le C\|f\|_p|z|^{1/2}\int_{|y|>(e^{1/\varepsilon}-1)|z|} \frac{|\Omega(y)|}{|y|^{n+1/2}}\,dy \\
&\quad \le CD\varepsilon.
\end{aligned} \tag{3.39}$$

About J_3. By the Minkowski inequality and $|b(x+z)-b(y)|<C$, we get

$$\begin{aligned}
&\left\{\int_0^\infty |J_3(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2} \\
&\quad = \left\{\int_0^\infty \left|\int_{\substack{|x-y|>e^{1/\varepsilon}|z|,\\ |x-y|\le t,\,|x+z-y|\le t}} \left(\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right)\right.\right. \\
&\qquad\qquad \left.\left. \times (b(x+z)-b(y))f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\
&\quad \le C\int_{|x-y|>e^{1/\varepsilon}|z|} \left|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right| |f(y)| \left\{\int_{\substack{|x-y|\le t,\\ |x+z-y|\le t}} \frac{dt}{t^3}\right\}^{1/2} dy \\
&\quad \le C\int_{|x-y|>e^{1/\varepsilon}|z|} \left|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right| \frac{|f(y)|}{|x-y|}\,dy.
\end{aligned} \tag{3.40}$$

Then by the Minkowski inequality and Lemma 2.7, we get

$$\begin{aligned}
&\left\|\left\{\int_0^\infty |J_3(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2}\right\|_p \\
&\quad \le C\left(\int_{\mathbf{R}^n}\left(\int_{|x-y|>e^{1/\varepsilon}|z|} \left|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right| \frac{|f(y)|}{|x-y|}\,dy\right)^p dx\right)^{1/p}
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
&\le C\|f\|_p \int_{|y|>e^{1/\varepsilon}|z|} \left| \frac{\Omega(y)}{|y|^{n-1}} - \frac{\Omega(y+z)}{|y+z|^{n-1}} \right| \frac{1}{|y|}\, dy \\
&\le C\|f\|_p \sum_{k=0}^{\infty} \int_{2^k e^{1/\varepsilon}|z|<|y|<2^{k+1}e^{1/\varepsilon}|z|} \left| \frac{\Omega(y)}{|y|^{n-1}} - \frac{\Omega(y+z)}{|y+z|^{n-1}} \right| \frac{1}{|y|}\, dy \\
&\le C\|f\|_p \sum_{k=0}^{\infty} \left\{ \frac{|z|}{2^k e^{1/\varepsilon}|z|} + \int_{|z|/2^{k+1}e^{1/\varepsilon}|z|}^{|z|/2^k e^{1/\varepsilon}|z|} \frac{\omega(\delta)}{\delta}\, d\delta \right\} \\
&\le C\|f\|_p \sum_{k=0}^{\infty} \left\{ \frac{1}{2^k e^{1/\varepsilon}} + \frac{1}{1+k+1/\varepsilon} \int_{|z|/2^{k+1}e^{1/\varepsilon}|z|}^{|z|/2^k e^{1/\varepsilon}|z|} \frac{\omega(\delta)}{\delta}(1+|\log\delta|)\, d\delta \right\} \\
&\le C(e^{-1/\varepsilon}+\varepsilon)\|f\|_p \\
&\le CD\varepsilon.
\end{aligned}$$

Before estimating J_4, let us recall the properties of the Hardy-Littlewood maximal operator M_q of order q $(1 \le q < \infty)$. Suppose that $f \in L_{\mathrm{Loc}}(\mathbf{R}^n)$. The maximal operator M_q is defined by

$$M_q f(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y)|^q \, dy \right)^{1/q},$$

where Q is a cube on $\mathbf{R}^n$. Then the maximal operator M_q is of type (p, p) for $1 \le q < p$ (see [28]). We denote simply by M the maximal operator M_1 of order 1, the classical Hardy-Littlewood maximal operator M.

Let us return to the estimate of J_4. Since $b \in C_0^\infty$, we have $|b(x) - b(x+z)| \le C|z|$, then

$$(3.42) \quad \left(\int_0^\infty |J_4(x,t)|^2 \frac{dt}{t^3} \right)^{1/2} \le C|z| \left\{ \int_0^\infty \left| \int_{\substack{|x-y|>e^{1/\varepsilon}|z| \\ |x-y|\le t}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy \right|^2 \frac{dt}{t^3} \right\}^{1/2}$$
$$:= C|z| \mu_{\Omega, e^{1/\varepsilon}|z|} f(x).$$

Let $\eta = e^{1/\varepsilon}|z|$, we claim that

$$(3.43) \qquad \|\mu_{\Omega,\eta} f\|_p \le C\|f\|_p, \quad 1 < p < \infty,$$

where C is independent of η and f. In fact, denote by Q the cube center at $x \in \mathbf{R}^n$ and diameter $\eta/2$. Moreover, set $f_1(y) = f_{\chi_{2Q}}(y)$ and $f_2(y) = f(y) - f_1(y)$. Then

$$\begin{aligned}\mu_{\Omega,\eta}f(x) &= \frac{1}{|Q|}\int_Q |\mu_{\Omega,\eta}f(x)|\,dy\\ &\le \frac{1}{|Q|}\int_Q |g_\Omega f(y)|\,dy + \frac{1}{|Q|}\int_Q |g_\Omega f_1(y)|\,dy\\ &\quad + \frac{1}{|Q|}\int_Q |g_\Omega f_2(y) - \mu_{\Omega,\eta}f(x)|\,dy\\ &\le M(g_\Omega f)(x) + I(f)(x) + II(f)(x).\end{aligned}$$

Since Ω satisfies (1.5), by Remark 3.1 and applying Theorem K and the L^p boundedness of the Hardy-Littlewood maximal operator M, we get

$$\|M(g_\Omega f)\|_p \le C\|f\|_p.$$

By Theorem K again, we know for any $1 < q < \infty$,

$$I(f)(x) \le \frac{C}{|Q|^{1/q}}\|g_\Omega f_1\|_q \le \frac{C}{|Q|^{1/q}}\|f_1\|_q \le C(M(|f|^q)(x))^{1/q}.$$

Taking $1 < q < p$, we get $\|I(f)\|_p \le C\|f\|_p$. Regarding $II(f)(x)$. By the Minkowski inequality, we have

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |g_\Omega f_2(\xi) - \mu_{\Omega,\eta}f(x)|\,d\xi\\ &\le \frac{1}{|Q|}\int_Q\left\{\int_0^\infty\left|\int_{\substack{|\xi-y|\ge t\\ \eta<|x-y|\le t}}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_2(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2}d\xi\\ &\quad+\frac{1}{|Q|}\int_Q\left\{\int_0^\infty\left|\int_{\substack{|\xi-y|\le t\\ |x-y|\ge t}}\frac{\Omega(\xi-y)}{|\xi-y|^{n-1}}f_2(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2}d\xi\\ &\quad+\frac{1}{|Q|}\int_Q\left\{\int_0^\infty\left|\int_{\substack{|\xi-y|\le t\\ \eta<|x-y|\le t}}\left(\frac{\Omega(\xi-y)}{|\xi-y|^{n-1}}-\frac{\Omega(x-y)}{|x-y|^{n-1}}\right)f_2(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2}d\xi\\ &:= H_1(x)+H_2(x)+H_3(x).\end{aligned}$$

Note that $|\xi - y| \sim |x-y|$ by $\xi, x\in Q$ and $y\in(2Q)^c$, we get

$$\begin{aligned}H_1(x) &\le \frac{1}{|Q|}\int_Q\int_{(2Q)^c}\frac{|f(y)|\,|\Omega(x-y)|}{|x-y|^{n-1}}\left\{\int_{|x-y|\le t\le|\xi-y|}\frac{dt}{t^3}\right\}^{1/2}dy\,d\xi\\ &\le C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\frac{\eta^{1/2}|f(y)|\,|\Omega(x-y)|}{|x-y|^{n+1/2}}\,dy\\ &\le CM_\Omega f(x).\end{aligned}$$

Similarly, we have

$$\begin{aligned}
H_2(x) &\leq \frac{1}{|Q|}\int_Q \int_{(2Q)^c} \frac{|f(y)|\,|\Omega(\xi-y)|}{|\xi-y|^{n-1}} \left\{\int_{|\xi-y|\leq t<|x-y|} \frac{dt}{t^3}\right\}^{1/2} dyd\xi \\
&\leq \frac{1}{|Q|}\int_Q \int_{(2Q)^c} \frac{\eta^{1/2}|f(y)|\,|\Omega(\xi-y)|}{|\xi-y|^{n+1/2}}\, dyd\xi \\
&\leq C\frac{1}{|Q|}\int_Q M_\Omega f(\xi)\, d\xi \leq CM(M_\Omega f)(x).
\end{aligned}$$

By Lemma 2.8 and the L^p boundedbess of the Hardy-Littlewood maximal operator, we get

$$\|H_1 f\|_p \leq C\|f\|_p, \quad |H_2 f\|_p \leq C\|f\|_p.$$

Now we discuss H_3. Denote $B(\eta) = \{x : |x| \leq \eta\}$, then we get

$$\begin{aligned}
H_3(x) &\leq \frac{C}{|Q|}\int_Q \int_{\mathbf{R}^n} \left|\frac{\Omega(\xi-y)}{|\xi-y|^{n-1}} - \frac{\Omega(x-y)}{|x-y|^{n-1}}\right| \frac{|f_2(y)|}{|x-y|}\, dyd\xi \\
&= \frac{C}{|B(\eta)|}\int_{B(\eta)} \int_{|y-x|>\eta} \left|\frac{\Omega(x-\xi-y)}{|x-\xi-y|^{n-1}} - \frac{\Omega(x-y)}{|x-y|^{n-1}}\right| \frac{|f(y)|}{|x-y|}\, dyd\xi \\
&= \frac{C}{|B(\eta)|}\int_{B(\eta)} \int_{|y|>\eta} \left|\frac{\Omega(y-\xi)}{|y-\xi|^{n-1}} - \frac{\Omega(y)}{|y|^{n-1}}\right| \frac{|f(x-y)|}{|y|}\, dyd\xi
\end{aligned}$$

Then by the Minkowski inequality and Lemma 2.7, we get

$$\begin{aligned}
\|H_3\|_p &\leq C\|f\|_p \frac{1}{|B(\eta)|}\int_{B(\eta)} \int_{|y|>\eta} \left|\frac{\Omega(y-\xi)}{|y-\xi|^{n-1}} - \frac{\Omega(y)}{|y|^{n-1}}\right| \frac{1}{|y|}\, dyd\xi \\
&= C\|f\|_p \frac{1}{|B(\eta)|}\int_{B(\eta)} \sum_{k=1}^{\infty} \int_{2^{k-1}\eta<|y|<2^k\eta} \left|\frac{\Omega(y-\xi)}{|y-\xi|^{n-1}} - \frac{\Omega(y)}{|y|^{n-1}}\right| \frac{1}{|y|}\, dyd\xi \\
&= C\|f\|_p \frac{1}{|B(\eta)|}\int_{B(\eta)} \sum_{k=1}^{\infty} \left(\frac{|\xi|}{2^{k-1}\eta} + \int_{|\xi|/2^k\eta}^{|\xi|/2^{k-1}\eta} \frac{\omega(\delta)}{\delta}\, d\delta\right) d\xi \\
&\leq C\|f\|_p \frac{1}{|B(\eta)|}\int_{B(\eta)} \left(1 + \int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta\right) d\xi \\
&\leq C\|f\|_p.
\end{aligned}$$

Thus, (3.43) follows from the above estimates. Then by (3.42) and (3.43), we have

$$\left\|\left\{\int_0^\infty |J_4(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2}\right\|_p \leq C|z|\,\|f\|_p \leq CD|z|. \tag{3.44}$$

About J_5, since $|b(x)-b(y)| \le C|x-y|$, by the Minkowski inequality, we get

$$\begin{aligned}\left\{\int_0^\infty |J_5(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2} &\\ &\le C\int_{|x-y|\le e^{1/\varepsilon}|z|} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b(x)-b(y)|\,|f(y)| \left\{\int_{|x-y|\le t} \frac{dt}{t^3}\right\}^{1/2} dy\\ &\le C\int_{|x-y|\le e^{1/\varepsilon}|z|} \frac{|f(y)|\,|\Omega(x-y)|}{|x-y|^{n-1}}\,dy\\ &\le Ce^{1/\varepsilon}|z| M_\Omega f(x).\end{aligned}$$

Then by Lemma 2.8, we get

$$\left\|\left\{\int_0^\infty |J_5(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2}\right\|_p \le CDe^{1/\varepsilon}|z|. \tag{3.45}$$

Similarly, using the estimate $|b(x+z)-b(y)| \le C|x+z-y|$, we have

$$\begin{aligned}\left\{\int_0^\infty |J_6(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2} &\le C\int_{|x-y|\le e^{1/\varepsilon}|z|} \frac{|f(y)|\,|\Omega(x+z-y)|}{|x+z-y|^{n-1}}\,dy\\ &\le C\int_{|x+z-y|\le e^{(1/\varepsilon)|z|}+|z|} \frac{|f(y)|\,|\Omega(x+z-y)|}{|x+z-y|^{n-1}}\,dy\\ &\le C(e^{1/\varepsilon}|z|+|z|) M_\Omega f(x),\end{aligned}$$

and then we get

$$\left\|\left\{\int_0^\infty |J_6(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2}\right\|_p \le CD(e^{1/\varepsilon}|z|+|z|). \tag{3.46}$$

From (3.34), (3.35), (3.37), (3.39), (3.41), (3.44), (3.45) and (3.46) then by taking $|z|$ sufficiently small depending on ε, we can get

$$\lim_{|z|\to 0} \|[b,g_\Omega]f(x) - [b,g_\Omega]f(x+z)\|_p = 0 \quad \text{uniformly in } f \in \mathscr{F}.$$

Thus we complete the proof of Theorem 2.

4. The compactness of the commutator $[b, S^\rho]$

4.1. The proof of Theorem 3: $[b, S^\rho]$ is a compact operator in $L^p \Rightarrow b \in VMO$

The idea of proving Theorem 3 is similar to proving Theorem 1, but it will be more complex than done in §3.1. Since $[b, S^\rho]$ is a compact operator in $L^p(\mathbf{R}^n)$, so $[b, S^\rho]$ is also bounded in $L^p(\mathbf{R}^n)$. By Theorem H, $b \in BMO$ and we may assume $\|b\|_{BMO} = 1$. By Lemma 2.6, to prove that $b \in VMO$, it suffices to show that b satisfies the conditions (i), (ii) and (iii) in Lemma 2.6.

If b does not satisfy (i) of Lemma 2.6. Then there exist a $\zeta > 0$ and a sequence of cubes $\{Q_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} r_j = 0$, where r_j is the diameter of $Q_j := Q_j(z_j)$, such that (3.1) holds for every j. Now we state that $\{[b, S^{\rho}]f_j\}_{j=1}^{\infty}$ is not a strongly pre-compact set in L^p for the function sequence $\{f_j\}_{j=1}^{\infty}$ defined in (3.2).

In the proof below, the constants B_j $(1 \le j \le 13)$ are same constants appearing in the proof of Theorem 1. Moreover, for $i = 14, \dots, 20$, B_i is a positive constant depending only on Ω, n, ζ, γ, ρ and B_k $(1 \le k < i)$.

For $x \in (B_2 Q_j)^c$, we have

$$
\begin{aligned}
(4.1)\quad & |[b, S^{\rho}]f_j(x)| \\
&= \left(\int_0^{\infty}\int_{|x-y|<t}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} \\
&\ge \left(\int_{4|x-z_j|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-z_j|<|y-z_j|<3|x-z_j|}}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\right.\right. \\
&\qquad \left.\left.\times (b(x)-b(z))f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} \\
&\ge \left(\int_{4|x-z_j|}^{\infty}\int_{\substack{|x-y|<t,\,(y-z_j)'\in\Lambda\\ 2|x-z_j|<|y-z_j|<3|x-z_j|}}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\right.\right. \\
&\qquad \left.\left.\times (b(z)-b_{Q_j})f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} \\
&\qquad - |b(x)-b_{Q_j}|\left(\int_{4|x|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-z_j|<|y-z_j|<3|x-z_j|}}\right. \\
&\qquad \left.\times\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} \\
&:= J_1 - J_2,
\end{aligned}
$$

where Λ is defined by (3.7). For J_1, noting that if $|z - z_j| < r_j$, then $|x - z_j| > B_2|z - z_j|$ and $|y - z_j| > 2B_2|z - z_j|$, by Lemma 2.1, we get

$$|(y-z)' - (y-z_j)'| \le 2\frac{|z-z_j|}{|y-z_j|} \le B_1.$$

Then by (3.8) we get $\Omega((y-z)') \ge \dfrac{C_1}{\left(\log \dfrac{1}{B_1}\right)^{\gamma}}$. Since

$$4|x-z_j| > |y-z_j| + |z-z_j| \geq |y-z| \geq |y-z_j| - |z-z_j| > 2|x-z_j| - |x-z_j|/2 = \frac{3|x-z_j|}{2}$$

and

$$4|x-z_j| > |x-y| \geq |y-z_j| - |x-z_j| > |x-z_j|,$$

by (3.3), (3.4), (3.6), and Hölder's inequality, we get

$$\begin{aligned}
J_1 &\geq \frac{C_1}{\left(\log \dfrac{1}{B_1}\right)^{\gamma}} \left\{ \int_{4|x-z_j|}^{\infty} \int_{\substack{|x-y|<t,\,(y-z_j)'\in\Lambda \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \left(\int_{Q_j} (b(z) - b_{Q_j}) \right.\right. \\
&\quad \left.\left. \times f_j(z)|y-z|^{\rho-n}\chi_{\{|y-z|<t\}}\,dz \right)^2 \frac{dydt}{t^{n+1+2\rho}} \right\}^{1/2} \\
&\geq C \int_{4|x-z_j|}^{\infty} \int_{\substack{|x-y|<t,\,(y-z_j)'\in\Lambda \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \int_{Q_j} (b(z) - b_{Q_j}) f_j(z) \\
&\quad \times |y-z|^{\rho-n}\chi_{\{|y-z|)<t\}}\,dz \frac{dydt}{t^{n+1+2\rho}} \\
&\quad \times \left\{ \int_{4|x-z_j|}^{\infty} \int_{\substack{|x-y|<t,\,(y-z_j)'\in\Lambda \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \frac{dydt}{t^{n+1+2\rho}} \right\}^{-1/2} \\
&\geq C|x-z_j|^{2\rho-n} \int_{Q_j} (b(z) - b_{Q_j}) f_j(z) \\
&\quad \times \int_{\substack{(y-z_j)'\in\Lambda \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \int_{\substack{4|x-z_j|<t,\,|x-y|<t \\ |y-z|<t}} \frac{dt}{t^{n+1+2\rho}}\,dydz \\
&\geq C|x-z_j|^{2\rho-n} \int_{Q_j} (b(z) - b_{Q_j}) f_j(z) \\
&\quad \times \int_{\substack{(y-z_j)'\in\Lambda \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \int_{4|x-z_j|<t} \frac{dt}{t^{n+1+2\rho}}\,dydz \\
&\geq C|x-z_j|^{-n} \int_{Q_j} (b(z) - b_{Q_j}) f_j(z)\,dz \\
&= C|x-z_j|^{-n}|Q_j|^{-1/p} \int_{Q_j} |b(z) - b_{Q_j}|\,dz \\
&\geq C\zeta |Q_j|^{1/p'}|x-z_j|^{-n}.
\end{aligned} \tag{4.2}$$

By (3.3) and (3.5), we have

$$J_2 = |b(x) - b_{Q_j}| \left\{ \int_{4|x-z_j|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \left| \int_{\mathbf{R}^n} \left(\frac{\Omega(y-z)}{|y-z|^{n-\rho}} \chi_{\{|y-z|<t\}} - \frac{\Omega(y-z_j)}{|y-z_j|^{n-\rho}} \chi_{\{|y-z_j|<t\}} \right) f_j(z)\, dz \right|^2 \frac{dydt}{t^{n+1+2\rho}} \right\}^{1/2}$$

$$\leq |b(x) - b_{Q_j}| \left\{ \int_{4|x-z_j|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \left| \int_{\substack{|y-z|<t \\ |y-z_j|<t}} \left(\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-z_j)}{|y-z_j|^{n-\rho}} \right) f_j(z)\, dz \right|^2 \frac{dydt}{t^{n+1+2\rho}} \right\}^{1/2}$$

$$+ |b(x) - b_{Q_j}| \left(\int_{4|x-z_j|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \times \left| \int_{\substack{|y-z|<t \\ |y-z_j|\geq t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f_j(z)\, dz \right|^2 \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2}$$

$$+ |b(x) - b_{Q_j}| \left(\int_{4|x-z_j|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-z_j|<|y-z_j|<3|x-z_j|}} \times \left| \int_{\substack{|y-z|\geq t \\ |y-z_j|<t}} \frac{\Omega(y-z_j)}{|y-z_j|^{n-\rho}} f_j(z)\, dz \right|^2 \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2}$$

$$:= J_2^1 + J_2^2 + J_2^3.$$

Actually, $J_2^2 = J_2^3 = 0$ because $t \leq |y - z_j| < 3|x - z_j|$ and $t > 4|x - z_j|$ in J_2^2, and $t \leq |y - z| < 4|x - z_j|$ and $t > 4|x - z_j|$ in J_2^3. So, we need only to estimate J_2^1. By the Minkowski inequality, Lemma 2.2 for $|y - z_j| > 2|z - z_j|$, (3.3) and (3.6) we get

$$J_2^1 \le C|b(x) - b_{Q_j}| \int_{Q_j} |f_j(z)| \left(\int_{2|x-z_j|<|y-z_j|<3|x-z_j|} \int_{\substack{4|x-z_j| \le t, |y-z|<t \\ |y-z_j|<t, |x-y| \le t}} \times \frac{1}{|y-z_j|^{2(n-\rho)}} \frac{1}{\left(\log \frac{|y-z_j|}{r_j}\right)^{2\gamma}} \frac{dtdy}{t^{n+1+2\rho}} \right)^{1/2} dz \tag{4.3}$$

$$\le C|Q_j|^{1/p'}|b(x) - b_{Q_j}|\,|x-z_j|^{-n}\left(\log \frac{|x-z_j|}{r_j}\right)^{-\gamma}.$$

By (4.1)–(4.3), we get

$$|[b, S^\rho]f_j(x)| \ge C\zeta \frac{|Q_j|^{1/p'}}{|x-z_j|^n} - C\frac{|Q_j|^{1/p'}|b(x) - b_{Q_j}|}{|x-z_j|^n\left(\log \frac{|x-z_j|}{r_j}\right)^{\gamma}}. \tag{4.4}$$

On the other hand. Since $2t > |y-z| + |x-y| \ge |x-z|$, so $t > 1/2|x-z|$ and $|x-z_j| > 2|z-z_j|$, we get $|x-z| \ge |x-z_j| - |z-z_j| > 1/2|x-z_j|$, then $t > |x-z_j|/4$. Note that $n - \rho < n/2$, then by the Minkowski inequality and $\Omega \in L^\infty(S^{n-1})$, we get

$$|S^\rho((b - b_{Q_j})f_j)(x)| \tag{4.5}$$

$$\le C\int_{Q_j} |b(z) - b_{Q_j}|\,|f_j(z)| \left(\int_0^\infty \int_{|y-z|<t, |x-y|<t} \frac{1}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} dz$$

$$\le C\int_{Q_j} |b(z) - b_{Q_j}|\,|f_j(z)| \left(\int_{|x-z_j|/4}^\infty \int_{|y-z|<t} \frac{1}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} dz$$

$$\le C|x-z_j|^{-n}\int_{Q_j} |b(z) - b_{Q_j}|\,|f_j(z)|\, dz$$

$$\le C|x-z_j|^{-n}|Q_j|^{1/p'}\left(\frac{1}{|Q_j|}\int_{Q_j} |b(z) - b_{Q_j}|^{p'}\, dz\right)^{1/p'}\left(\int_{Q_j} |f_j(z)|^p\, dz\right)^{1/p}$$

$$\le C|Q_j|^{1/p'}|x-z_j|^{-n}.$$

Similar to the estimate of (4.5), we get

$$|(b(x) - b_{Q_j})S^\rho(f_j)(x)| \le C|x-z_j|^{-n}|b(x) - b_{Q_j}| \int_{Q_j} |f_j(z)|\, dz \tag{4.6}$$

$$\le C|x-z_j|^{-n}|b(x) - b_{Q_j}|\,|Q_j|^{1/p'}.$$

From (4.5) and (4.6), we get

$$|[b, S^{\rho}]f_j(x)| \le C|Q_j|^{1/p'}|x - z_j|^{-n} + C|x - z_j|^{-n}|b(x) - b_{Q_j}|\,|Q_j|^{1/p'}. \tag{4.7}$$

Then for $\eta > \nu > B_2$, using (4.10) and (3.12), we get

$$\begin{aligned}
&\left(\int_{\{\nu r_j < |x - z_j| < \eta r_j\}} |[b, S^{\rho}]f_j(x)|^p\,dx\right)^{1/p} \\
&\quad \ge C\zeta|Q_j|^{1/p'}\left(\int_{\{\nu r_j < |x - z_j| < \eta r_j\}} \frac{1}{|x - z_j|^{np}}\,dx\right)^{1/p} \\
&\qquad - C|Q_j|^{1/p'}\left(\int_{|x - z_j| > \nu r_j} \frac{|b(x) - b_{Q_j}|^p}{|x - z_j|^{np}\left(\log \dfrac{|x - z_j|}{r_j}\right)^{\gamma p}}\,dx\right)^{1/p} \\
&\quad \ge B_{14}\zeta(\nu^{-np+n} - \eta^{-np+n})^{1/p} - B_{15}(\log \nu)^{1-\gamma}\nu^{-n+n/p}.
\end{aligned} \tag{4.8}$$

By (4.7), we have

$$\begin{aligned}
&\left(\int_{|x - z_j| > \eta r_j} |[b, S^{\rho}]f_j(x)|^p\,dx\right)^{1/p} \\
&\quad \le C|Q_j|^{1/p'}\left(\int_{|x - z_j| \ge \eta r_j} |x - z_j|^{-np}\,dx\right)^{1/p} \\
&\qquad + C|Q_j|^{1/p'}\left(\int_{|x - z_j| \ge \eta r_j} |x - z_j|^{-np}(b(x) - b_{Q_j})\,dx\right)^{1/p} \\
&\quad \le B_{16}\eta^{-n+n/p}.
\end{aligned} \tag{4.9}$$

By (4.8) and (4.9), there exist B_{17}, B_{18} and B_{19} satisfying $B_2 < B_{17} < B_{18}$ and

$$\left(\int_{B_{17}r_j < |x - y_j| < B_{18}r_j} |[b, S^{\rho}]f_j|^p\,dy\right)^{1/p} \ge B_{19} \tag{4.10}$$

and

$$\left(\int_{|x - y_j| > B_{18}r_j} |[b, S^{\rho}]f_j|^p\,dy\right)^{1/p} \le B_{19}/4. \tag{4.11}$$

Let $E \subset \{x : B_{17}r_j < |x - y_j| < B_{18}r_j\}$ be an arbitrary measurable set. Then by (4.7) and the Minkowski inequality

$$(4.12)\quad \left(\int_E |[b,S^\rho]f_j(x)|^p\,dx\right)^{1/p}$$
$$\le C|Q_j|^{1/p'}\left(\int_E |x-y_j|^{-pn}\,dx\right)^{1/p} + C|Q_j|^{1/p'}\left(\int_E \frac{|b(x)-b_{Q_j}|^p}{|x-y_j|^{np}}\,dx\right)^{1/p}$$
$$\le C\left\{\frac{|E|^{1/p}}{|Q_j|^{1/p}} + \left(\frac{1}{|Q_j|}\int_E |b(x)-b_{Q_j}|^p\,dx\right)^{1/p}\right\}.$$

Apply the same method estimating (3.20), we may get

$$\left(\int_E |[b,S^\rho]f_j(y)|^p\,dy\right)^{1/p} \le B_{19}/4$$

for every measurable set E satisfying

$$E \subset \{x : B_{17}r_j < |x-y_j| < B_{18}r_j\} \quad \text{and} \quad |E|/|Q_j| < B_{20}^n.$$

By the above estimate and using same idea in the proof of Theorem 1, we may show $[b,S^\rho]$ is not a compact operator in $L^p(\mathbf{R}^n)$. This contradiction shows that b satisfies the condition (i) of Lemma 2.6. Similar proof states also b should satisfy the conditions (ii) and (iii) in Lemma 2.6. Thus, $b \in VMO(\mathbf{R}^n)$. Here we omit the details of the last part proof.

4.2. The proof of Theorem 4: $b \in VMO \Rightarrow [b,S^\rho]$ is a compact operator in L^p

We first recall the $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ boundedness of S^ρ and $g_\lambda^{*,\rho}$, which will be used in the proofs of Theorem 4 and 6.

THEOREM L ([16]). *Suppose that $\Omega \in L^2(S^{n-1})$ satisfying* (1.1), (1.2) *and* (1.6). *Then foe $\rho > n/2$ and $\lambda > 2$, S^ρ and $g_\lambda^{*,\rho}$ are both bounded in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Let us now return to the proof of Theorem 4. Suppose that $b \in VMO(\mathbf{R}^n)$. Then by Theorem G, it suffices to prove that for any bounded set $\mathscr{F}$ in $L^p(\mathbf{R}^n)$, $\mathscr{G} = \{[b,S^\rho]f : f \in \mathscr{F}\}$ is strongly pre-compact in $L^p(\mathbf{R}^n)$. As done in proving Theorem 2, it is reduced to to verify (3.26)–(3.28) hold uniformly in $\mathscr{G}$ for $b \in C_0^\infty(\mathbf{R}^n)$.

Denote $D = \sup_{f\in\mathscr{F}}\|f\|_p$, then Theorem G tells us that (3.26) holds for $\mathscr{G}$. We first discuss (3.28). Assume $\mathrm{supp}(b) \subset \{z : |z| < r\}$ for some $r > 0$. Thus, for any x satisfying $|x| > \max\{2A, 4r\}$, where the constant A is fixed in (3.31), and every $f \in F$,

$$(4.13)\quad |[b,S^\rho]f(x)|$$
$$= \left(\int_0^\infty \int_{|x-y|<t}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f(z)\,dz\right|^2\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}$$

$$= \left(\int_0^\infty \int_{|x-y|<t} \left| \int_{\substack{|y-z|<t \\ |z|<r}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z) f(z)\, dz \right|^2 \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2}$$

$$:= U.$$

For U, since $2(n-\rho) < n$ and $1/2|x| < |x| - |z| < |x-z| < |x-y| + |y-z| < 2t$. By the Minkowski inequality and $\Omega \in L^2(S^{n-1})$, we have

$$\begin{aligned}
U &\le C \int_{|z|<r} |b(z)|\, |f(z)| \left(\int_0^\infty \int_{\substack{|y-z|<t \\ |x-y|<t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz \\
&\le C \int_{|z|<r} |b(z)|\, |f(z)| \left(\int_{|x|/4}^\infty \int_{|y-z|<t} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz \\
&\le C|x|^{-n} \int_{|z|<r} |b(z)|\, |f(z)|\, dz \\
&\le C|x|^{-n} \left(\int_{|z|<r} |b(z)|^{p'}\, dy \right)^{1/p'} \|f\|_p \\
&\le CD|x|^{-n}.
\end{aligned}$$

Applying (3.31), we have

$$\left(\int_{|x|>2A} |[b, S^\rho] f(x)|^p\, dx \right)^{1/p} \le CD \left(\int_{|x|>2A} |x|^{-np}\, dx \right)^{1/p} \le CD\varepsilon. \tag{4.14}$$

(4.14) shows that (3.28) holds uniformly in $\mathscr{G}$. Finally, to finish the proof of Theorem 4, it remains to show (3.27) holds uniformly in $\mathscr{G}$. We need to prove that for any $\varepsilon > 0$, if $|z|$ is sufficiently small, then for every $f \in \mathscr{F}$,

$$\|[b, S^\rho] f(\cdot) - [b, S^\rho] f(\cdot + z)\|_p \le C\varepsilon. \tag{4.15}$$

To do this, for any $v \in \mathbf{R}^n$, by the Minkowski's inequality, we have

$$\begin{aligned}
&|[b, S^\rho] f(x) - [b, S^\rho] f(x+v)| \qquad (4.16)\\
&= \Bigg| \left(\int_0^\infty \int_{|x-y|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x) - b(z)) f(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \\
&\quad - \Bigg(\int_0^\infty \int_{|x+v-y|<t} \Bigg| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\
&\quad \times (b(x+v) - b(z)) f(z)\, dz \Bigg|^2 \frac{dydt}{t^{n+2\rho+1}} \Bigg)^{1/2} \Bigg|
\end{aligned}$$

$$\leq \left(\int_0^\infty \int_{|x-y|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x)-b(z)) f(z)\, dz \right.\right.$$

$$\left. - \int_{|y+v-z|<t} \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} (b(x+v)-b(z)) f(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}$$

$$:= \left(\int_0^\infty \int_{|x-y|<t} |I(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2}.$$

For any $0<\varepsilon<\frac{1}{3}$ and $v \in \mathbf{R}^n$, write $I(x,v,y,t)$ as

$$\begin{aligned}
(4.17)\qquad & I(x,v,y,t) \\
&= \int_{\substack{|x-z|>2^{1/\varepsilon}|v| \\ |y-z|<t,\, |y+v-z|\geq t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x+v)-b(z)) f(z)\, dz \\
&\quad + \int_{\substack{|x-z|>2^{1/\varepsilon}|v| \\ |y-z|\geq t,\, |y+v-z|<t}} \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} (b(z)-b(x+v)) f(z)\, dz \\
&\quad + \int_{\substack{|x-z|>2^{1/\varepsilon}|v|, \\ |y-z|<t,\, |y+v-z|<t}} \left(\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} \right) \\
&\quad \times (b(x+v)-b(z)) f(z)\, dz \\
&\quad + \int_{\substack{|x-z|>2^{1/\varepsilon}|v|, \\ |y-z|<t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x)-b(x+v)) f(z)\, dz \\
&\quad + \int_{\substack{|x-z|\leq 2^{1/\varepsilon}|v|, \\ |y-z|<t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x)-b(z)) f(z)\, dz \\
&\quad + \int_{\substack{|x-z|\leq 2^{1/\varepsilon}|v|, \\ |y+v-z|<t}} \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} (b(z)-b(x+v)) f(z)\, dz \\
&:= \sum_{i=1}^{6} J_i(x,v,y,t).
\end{aligned}$$

First, we give the estimate for J_1. By $|b(x+v)-b(z)|<C$ and the Minkowski inequality, we have

$$\left\{ \int_0^\infty \int_{|x-y|<t} |J_1(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2}$$

$$\leq C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|<t \\ |y-z|<t,\, |y+v-z|\geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$

$$\leq C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|<t,\,|y-z|\leq(1/3)|x-z|\\ |y-z|<t,\,|y+v-z|\geq t}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$

$$+\,C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|<t,\,|y-z|>(1/3)|x-z|\\ |y-z|<t,\,|y+v-z|\geq t}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$

$$:= J_1^1 + J_1^2.$$

For J_1^1. Since $|y-z|\leq\frac{1}{3}|x-z|$ and

$$|y-z|+|v|\geq|y+v-z|\geq t>|x-y|\geq|x-z|-|y-z|\geq\frac{2}{3}|x-z|,$$

so $|v|\geq|x-z|/3$. However, $|x-z|>2^{1/\varepsilon}|v|>8|v|$. Hence $J_1^1=0$. For J_1^2. Since

$$|y-z|>\frac{1}{3}|x-z|>\frac{2^{1/\varepsilon}}{3}|v|>2|v|,$$

then by $\Omega\in L^2(S^{n-1})$, we get

$$J_1^2\leq C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_{|y-z|>(1/3)|x-z|}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_{\substack{|y-z|<t,\\ |y+v-z|\geq t}}\frac{dt}{t^{2\rho+n+1}}\,dy\right\}^{1/2}|f(z)|\,dz$$

$$\leq C\int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|v|^{1/2}}{|x-z|^{n+1/2}}|f(z)|\,dz.$$

Using the Minkowski inequality again, we get

$$\left\|\left\{\int_0^\infty\int_{|x-y|<t}|J_1(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2}\right\|_p \tag{4.18}$$

$$\leq C\left\{\int_{\mathbf{R}^n}\left(\int_{|y|>2^{1/\varepsilon}|v|}\frac{|v|^{1/2}}{|y|^{n+1/2}}|f(x-y)|\,dy\right)^p dx\right\}^{1/p}$$

$$\leq C\|f\|_p\int_{|y|>2^{1/\varepsilon}|v|}\frac{|v|^{1/2}}{|y|^{n+1/2}}\,dy$$

$$\leq C2^{-1/\varepsilon}\|f\|_p$$

$$\leq CD\varepsilon.$$

Similarly, we can get

$$\left\| \left\{ \int_0^\infty \int_{|x-y|<t} |J_2(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} \right\|_p \leq CD\varepsilon. \tag{4.19}$$

Now let us consider J_3. By the Minkowski inequality and $|b(x+v)-b(z)|<C$, we have

$$\begin{aligned}
&\left\{ \int_0^\infty \int_{|x-y|<t} |J_3(x,v,y,t)|^2 \frac{dtdy}{t^{2\rho+n+1}} \right\}^{1/2} \\
&\quad \leq C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|<t \\ |y-z|<t,\,|y+v-z|<t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\
&\qquad \left. \left. - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} \right|^2 \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\,dz \\
&\quad \leq C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|<t,\,|y-z|\leq 8|v| \\ |y-z|<t,\,|y+v-z|<t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\
&\qquad \left. \left. - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} \right|^2 \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\,dz \\
&\qquad + C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|<t,\,|y-z|>8|v| \\ |y-z|<t,\,|y+v-z|<t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\
&\qquad \left. \left. - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} \right|^2 \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\,dz \\
&\quad := J_3^1 + J_3^2.
\end{aligned}$$

For J_3^1, since $2n-2\rho<n$, $2t>|x-y|+|y-z|>|x-z|$, and $|y+v-z| \leq |y-z|+v \leq 9|v|$. Thus by $\Omega \in L^2(S^{n-1})$, we get

$$\begin{aligned}
J_3^1 &\leq C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_{1/2|x-z|}^\infty \int_{\substack{|y-z|<8|v|, \\ |y+v-z|<9|v|}} \right. \\
&\quad \left. \times \left(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} + \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}} \right) \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\,dz \\
&\leq C|v|^{\rho-n/2} \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|f(z)|}{|x-z|^{n/2+\rho}}\,dz.
\end{aligned}$$

Let us turn to J_3^2. Note that $t > |z-x|/2$. Since $|y-z| > 8|v|$ and $2\rho - n > 0$, by Lemma 2.2 in [13] we get

$$(4.20)\qquad \int_{|y-z|}^{\infty} \frac{\left(\log \frac{t}{|v|}\right)^{4+2\theta}}{t^{2\rho-n+1}}\, dt \le C \frac{\left[\log\left(\frac{|y-z|}{|v|}\right)\right]^{4+2\theta}}{|y-z|^{2\rho-n}} \quad \text{for}$$

$$\theta < \min\{1, (\lambda-2)n, 2\rho-n, \sigma-2\}.$$

By (4.20) and using Lemma 2.7, we have

$$\begin{aligned}
(4.21)\quad J_3^2 &\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|f(z)|}{|z-x|^n \left(\log \frac{|z-x|}{|v|}\right)^{2+\theta}} \\
&\quad \times \left(\int_{|y-z|>8|v|} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(v+y-z)}{|v+y-z|^{n-\rho}} \right|^2 \right. \\
&\quad \left. \times \int_{|y-z|<t}^{\infty} \frac{\left(\log \frac{t}{|v|}\right)^{4+2\theta} dt}{t^{2\rho-n+1}}\, dy \right)^{1/2} dz \\
&\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|f(z)|}{|z-x|^n \left(\log \frac{|z-x|}{|v|}\right)^{2+\theta}} \\
&\quad \times \left(\int_{|y-z|>8|v|} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(v+y-z)}{|v+y-z|^{n-\rho}} \right|^2 \frac{\left(\log \frac{|y-z|}{|v|}\right)^{4+2\theta}}{|y-z|^{2\rho-n}}\, dy \right)^{1/2} dz \\
&\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|f(z)|}{|z-x|^n \left(\log \frac{|z-x|}{|v|}\right)^{2+\theta}} \\
&\quad \times \left(\sum_{j=3}^{\infty} \int_{2^j|v| \le |y-z| < 2^{j+1}|v|} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(v+y-z)}{|v+y-z|^{n-\rho}} \right|^2 \right. \\
&\quad \left. \times \frac{\left(\log \frac{|y-z|}{|v|}\right)^{4+2\theta}}{|y-z|^{2\rho-n}}\, dy \right)^{1/2} dz
\end{aligned}$$

$$\leq C\int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|f(z)|}{|z-x|^n\left(\log\frac{|z-x|}{|v|}\right)^{2+\theta}}\sum_{j=3}^{\infty}\frac{\left(\log\frac{2^{j+1}|v|}{|v|}\right)^{2+\theta}}{(2^j|v|)^{\rho-n/2}}$$

$$\times\left(\int_{2^j|v|\leq|y-z|<2^{j+1}|v|}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v+y-z)}{|v+y-z|^{n-\rho}}\right|^2dy\right)^{1/2}dz$$

$$\leq C\int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|f(z)|}{|z-x|^n\left(\log\frac{|z-x|}{|v|}\right)^{2+\theta}}\sum_{j=3}^{\infty}\frac{(j+1)^{2+\theta}}{(2^j|v|)^{\rho-n/2}}(2^j|v|)^{n/2-(n-\rho)}$$

$$\times\left\{\frac{|v|}{2^j|v|}+\int_{|v|/2^{j+1}|v|}^{|v|/2^j|v|}\frac{\omega_2(\delta)}{\delta}\,d\delta\right\}dz$$

$$\leq C\int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|f(z)|}{|z-x|^n\left(\log\frac{|z-x|}{|v|}\right)^{2+\theta}}\,dz.$$

Combining with the estimates of J_3^1 and J_3^2, and note that $\rho-n/2<0$, apply the Minkowski inequality we get

$$\left\|\left\{\int_0^\infty\int_{|x-y|<t}|J_3(x,v,y,t)|^2\frac{dtdy}{t}\right\}^{1/2}\right\|_p \tag{4.22}$$

$$\leq C\left(\int_{\mathbf{R}^n}\left|\int_{|y|>2^{1/\varepsilon}|v|}\frac{|f(x-y)|}{|y|^n\left(\log\frac{|y|}{|v|}\right)^{2+\theta}}\,dy\right|^p dx\right)^{1/p}$$

$$+C|v|^{\rho-n/2}\left(\int_{\mathbf{R}^n}\left|\int_{|y|>2^{1/\varepsilon}|v|}\frac{|f(x-y)|}{|y|^{n/2+\rho}}\,dy\right|^p dx\right)^{1/p}$$

$$\leq C\|f\|_p\left(\int_{|y|>2^{1/\varepsilon}|v|}\frac{1}{|y|^n\left(\log\frac{|y|}{|v|}\right)^{2+\theta}}\,dy\right.$$

$$\left.+|v|^{\rho-n/2}\int_{|y|>2^{1/\varepsilon}|v|}\frac{1}{|y|^{n/2+\rho}}\,dy\right)$$

$$\leq C\|f\|_p\left(\int_{2^{1/\varepsilon}}^{\infty} \frac{1}{r(\log r)^{2+\theta}}\,dr + 2^{(\rho-n/2)/\varepsilon}\right)$$
$$\leq CD\varepsilon.$$

Now we give the estimate of J_4.

$$\left(\int_0^\infty \int_{|x-y|<t} |J_4(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}$$
$$\leq |b(x)-b(x+v)|\left\{\int_0^\infty \int_{|x-y|<t} \left|\int_{\substack{|x-z|>2^{1/\varepsilon}|v|,\\ |y-z|<t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2}$$
$$:= |b(x)-b(x+v)|S^{\rho}_{2^{1/\varepsilon}|v|}f(x).$$

We claim that

$$S^{\rho}_{2^{1/\varepsilon}|v|}f(x) \leq C(M(\mu^{\rho}_S f)(x) + (M(|f|^q)(x))^{1/q} + Mf(x)), \quad 1<q<\infty, \tag{4.23}$$

where C is independent of v and ε. In fact, let Q denote the cube center at x and diameter $r = 2^{1/\varepsilon}|v|/8$. Moreover, $f_1(x) = f_{\chi_{8Q}}(x)$ and $f_2(x) = f(x) - f_1(x)$. Then

$$\begin{aligned} S^{\rho}_{2^{1/\varepsilon}|v|}f(x) &= \frac{1}{|Q|}\int_Q |S^{\rho}_{2^{1/\varepsilon}|v|}f(x)|\,d\xi \\ &\leq \frac{1}{|Q|}\int_Q |S^{\rho}f(\xi)|\,d\xi + \frac{1}{|Q|}\int_Q |S^{\rho}f_1(\xi)|\,d\xi \\ &\quad + \frac{1}{|Q|}\int_Q |S^{\rho}f_2(\xi) - S^{\rho}_{2^{1/\varepsilon}|v|}f(x)|\,d\xi \\ &\leq M(S^{\rho}f)(x) + I(f)(x) + II(f)(x). \end{aligned}$$

By Theorem L, we know

$$I(f)(x) \leq \frac{C}{|Q|^{1/q}}\|S^{\rho}f_1\|_q \leq \frac{C}{|Q|^{1/q}}\|f_1\|_q \leq C(M(|f|^q)(x))^{1/q}.$$

Finally, let us give the estimate of II. Let $\xi \in Q$, by the Minkowski inequality, we have

$$|S^{\rho}f_2(\xi) - S^{\rho}_{2^{1/\varepsilon}|v|}f(x)| \tag{4.24}$$
$$= \left|\left\{\int_0^\infty \int_{|x-y|<t} \left|\int_{|\xi-x+y-z|<t} \frac{\Omega(\xi-x+y-z)}{|\xi-x+y-z|^{n-\rho}} f_2(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2}\right.$$

$$
\begin{aligned}
&\quad - \left\{\int_0^\infty \int_{|x-y|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f_2(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2} \Bigg| \\
&\le \left\{\int_0^\infty \int_{|x-y|<t} \left| \int_{\substack{|y-z|<t,\\ |\xi-x+y-z|\ge t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f_2(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2} \\
&\quad + \left\{\int_0^\infty \int_{|x-y|<t} \left| \int_{\substack{|\xi-x+y-z|<t,\\ |y-z|\ge t}} \frac{\Omega(\xi-x+y-z)}{|\xi-x+y-z|^{n-\rho}} f_2(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2} \\
&\quad + \left\{\int_0^\infty \left\{\int_0^\infty \int_{|x-y|<t} \left| \int_{\substack{|y-z|<t,\\ |\xi-x+y-z|<t}} \left(\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(\xi-x+y-z)}{|\xi-x+y-z|^{n-\rho}}\right)\right.\right.\right. \\
&\quad \left. \times f_2(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}\Bigg\}^{1/2} \\
&:= G_1 + G_2 + G_3.
\end{aligned}
$$

Similar to the proof of J_1^1 and J_1^2, we get

$$
G_1 \le C \int_{\mathbf{R}^n} \frac{r^{1/2}}{|x-z|^{n+1/2}} f_2(z)\, dz \le C r^{1/2} \sum_{k=3}^\infty \int_{2^{k+1}Q\setminus 2^k Q} \frac{|f(z)|}{|x-z|^{n+1/2}}\, dy \le CMf(x).
$$

Similarly, $G_2 \le CMf(x)$. For G_3 we have

$$
\begin{aligned}
G_3 &\le C r^{\rho-n/2} \int_{\mathbf{R}^n} \frac{|f_2(z)|}{|x-z|^{n/2+\rho}}\, dz + C \int_{\mathbf{R}^n} \frac{|f_2(z)|}{|z-x|^n \left(\log \frac{|z-x|}{r}\right)^{2+\theta}}\, dz \\
&\le C r^{\rho-n/2} \sum_{k=3}^\infty \int_{2^{k+1}Q\setminus 2^k Q} \frac{|f(z)|}{|x-z|^{n/2+\rho}}\, dz \\
&\quad + \sum_{k=3}^\infty \int_{2^{k+1}Q\setminus 2^k Q} \frac{|f(z)|}{|x-z|^n \left(\log \frac{|z-x|}{r}\right)^{2+\theta}}\, dz \\
&\le CMf(x) + \sum_{k=3}^\infty \frac{1}{k^{2+\theta}} \int_{2^{k+1}Q\setminus 2^k Q} \frac{|f(z)|}{|x-z|^n}\, dz \\
&\le CMf(x).
\end{aligned}
$$

By the the estimates of G_1, G_2 and G_3 above, we get $II \le CMf(x)$. Summing up I, II, we get (4.23). Now we may give the estimate of J_4. Since $b \in C_0^\infty$, we have $|b(x) - b(x+v)| \le C|v|$. Then apply (4.23) for $1 < q < p$, Theorem L and the L^p $(p > 1)$ boundedness of M and M_q, we get

$$\left\| \left(\int_0^\infty \int_{|x-y|<t} |J_4(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \right\|_p \le C|v| \, \|f\|_p \le CD|v|. \tag{4.25}$$

About J_5, since $|b(x) - b(z)| \le C|x-z|$, $t > \dfrac{|x-z|}{2}$, $2n - 2\rho < n$ and $\Omega \in L^2(S^{n-1})$, by the Minkowski inequality, we get

$$\begin{aligned}
&\left(\int_0^\infty \int_{|x-y|<t} |J_5(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \\
&\quad \le C \int_{|x-z| \le 2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{|x-y|<t,\, |y-z|<t} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |x-z| \, |f(z)| \, dz \\
&\quad \le \int_{|x-z| \le 2^{1/\varepsilon}|v|} \left\{ \int_{(1/2)|x-z|}^\infty \int_{|y-z|<t} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |x-z| \, |f(z)| \, dz \\
&\quad \le \int_{|x-z| \le 2^{1/\varepsilon}|v|} \left\{ \int_{(1/2)|x-z|}^\infty \frac{dt}{t^{2n+1}} \right\}^{1/2} |x-z| \, |f(z)| \, dz \\
&\quad \le \int_{|x-z| \le 2^{1/\varepsilon}|v|} \frac{|f(z)|}{|x-z|^{n-1}} \, dz.
\end{aligned}$$

Then we get

$$\left\| \left(\int_0^\infty \int_{|x-y|<t} |J_5(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \right\|_p \le CD2^{1/\varepsilon}|v|. \tag{4.26}$$

Similar to the estimate of J_5, using the estimate $|b(x+v) - b(z)| \le C|x+v-z|$, and $2t > |x-y| + |y+v-z| \ge |x+v-z|$ we have

$$\left(\int_0^\infty \int_{|x-y|<t} |J_6(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \le C \int_{|x-z| \le 2^{1/\varepsilon}|v|} \frac{|f(z)|}{|x+v-z|^{n-1}} \, dz.$$

Then we get

$$\left\| \left(\int_0^\infty \int_{|x-y|<t} |J_6(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \right\|_p \le CD(2^{1/\varepsilon}|v| + |v|). \tag{4.27}$$

In (4.16)–(4.19), (4.22), (4.25)–(4.27), taking $|v|$ sufficiently small depending on ε, we can get

$$\lim_{|v|\to 0}\|[b,S^{\rho}]f(x)-[b,S^{\rho}]f(x+v)\|_p=0 \quad \text{uniformly in } f\in\mathscr{F}.$$

Thus we complete the proof of Theorem 4.

5. The compactness of commutator $[b, g_{\lambda}^{*,\rho}]$

5.1. The proof of Theorem 5: $[b, g_{\lambda}^{*,\rho}]$ is a compact operator in $L^p \Rightarrow b \in VMO$

By Theorem H, $[b, g_{\lambda}^{*,\rho}]$ is a compact operator implies $b\in BMO(\mathbf{R}^n)$. We need to verify b to satisfy the conditions (i), (ii) and (iii) in Lemma 2.6. Suppose that $\|b\|_{BMO}=1$ and b does not satisfy (i) of Lemma 2.6. Then there exist a $\zeta>0$ and a sequence of cubes $\{Q_j\}_{j=1}^{\infty}$, where $Q_j=Q_j(z_j,r_j)$, such that $\lim_{j\to\infty} r_j=0$ and (3.1) holds for every j. As done above, we will show $\{[b,g_{\lambda}^{*,\rho}]f_j\}_{j=1}^{\infty}$ is not a compact set in $L^p(\mathbf{R}^n)$ with the sequence $\{f_j\}_{j=1}^{\infty}$ defined in (3.2). With some estimates given above, we give the main idea of the proof here.

Note that $[b,S^{\rho}]f_j(x)\le 2^{\lambda n}[b,g_{\lambda}^{*,\rho}]f_j(x)$, hence for $x\in(B_2Q_j)^c$ where $B_2=3B_1^{-1}+1$, by (4.4), we get

$$|[b,g_{\lambda}^{*,\rho}]f_j(x)|\ge C\zeta\frac{|Q_j|^{1/p'}}{|x-z_j|^n}-C\frac{|Q_j|^{1/p'}|b(x)-b_{Q_j}|}{|x-z_j|^n\left(\log\dfrac{|x-z_j|}{r_j}\right)^{\gamma}}. \tag{5.1}$$

On the other hand, for $x\in(B_2Q_j)^c$, we have

$$\begin{aligned}
&g_{\lambda}^{*,\rho}((b-b_{Q_j})f_j)(x) \\
&\le\left(\int_0^{\infty}\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^{\rho}}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\right.\right. \\
&\qquad\left.\left.\times(b(z)-b_{Q_j})f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&\quad+\left(\int_0^{\infty}\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^{\rho}}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\right.\right. \\
&\qquad\left.\left.\times(b(z)-b_{Q_j})f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&:=K_1+K_2.
\end{aligned} \tag{5.2}$$

Since $\left(\frac{t}{t+|x-y|}\right)^{\lambda n} \le 1$, then $K_1 \le S^{\rho}((b-b_{Q_j})f_j)(x)$, and by (4.5), we get

$$K_1 \le C|Q_j|^{1/p'}|x-z_j|^{-n}. \tag{5.3}$$

Since $\Omega \in L^{\infty}(S^{n-1})$, the Minkowsky inequality yields

$$\begin{aligned}
K_2 \le{}& \left(\int_0^{\infty}\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^{\rho}}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\right.\right.\\
&\left.\left.\times (b(z)-b_{Q_j})f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
\le{}& C\int_{Q_j}|b(z)-b_{Q_j}|\,|f_j(z)|\\
&\times\left(\int_0^{\infty}\int_{|y-z|<t,\,|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{1}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}dz\\
\le{}& C\int_{Q_j}|b(z)-b_{Q_j}|\,|f_j(z)|\\
&\times\left(\int_0^{\infty}\int_{\substack{|y-z|<t,\,|x-y|\ge t\\ |y-z|\ge(1/2)|x-z|}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{1}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}dz\\
&+C\int_{Q_j}|b(z)-b_{Q_j}|\,|f_j(z)|\\
&\times\left(\int_0^{\infty}\int_{\substack{|y-z|<t,\,|x-y|\ge t\\ |y-z|<(1/2)|x-z|}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{1}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}dz\\
:={}& K_2^1+K_2^2
\end{aligned}\tag{5.4}$$

For K_2^1. Since $|x-z_j|>2|z-z_j|$, we get $|x-z|\ge|x-z_j|-|z-z_j|>\frac{1}{2}|x-z_j|$. Thus

$$t>|y-z|>\frac{1}{2}|x-z|>\frac{1}{4}|x-z_j|.$$

Note that $\left(\frac{t}{t+|x-y|}\right)^{\lambda n}<1$, we get

$$K_2^1 \le \int_{Q_j} |b(z)-b_{Q_j}|\,|f_j(z)| \times \left(\int_{(1/4)|x-z_j|}^{\infty} \int_{\substack{|y-z|<t \\ |y-z|\ge(1/4)|x-z_j|}} \frac{1}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz \tag{5.5}$$

$$\le C|x-z_j|^{-n+\rho} \int_{Q_j} |b(z)-b_{Q_j}|\,|f_j(z)| \left(\int_{(1/4)|x-z_j|}^{\infty} \frac{dt}{t^{1+2\rho}} \right)^{1/2} dz$$

$$\le C|x-z_j|^{-n}|Q_j|^{1/p'} \left(\frac{1}{|Q_j|} \int_{Q_j} |b(z)-b_{Q_j}|^{p'}\, dz \right)^{1/p'} \left(\int_{Q_j} |f_j(z)|^p\, dz \right)^{1/p}$$

$$\le C|Q_j|^{1/p'}|x-z_j|^{-n}.$$

Let us turn to K_2^2. Since $|y-z| < \frac{1}{2}|z-x|$, we get

$$|x-y| \ge |x-z| - |y-z| > \frac{1}{2}|x-z| \quad \text{and}$$

$$t \le |x-y| \le |x-z| + |y-z| < \frac{3}{2}|x-z|.$$

But $\frac{1}{2}|x-z_j| \le |x-z| \le \frac{3}{2}|x-z_j|$, then

$$t < \frac{9}{4}|x-z_j|, \quad |x-y| \ge \frac{1}{4}|x-z_j| \quad \text{and} \quad |y-z| < \frac{3}{4}|x-z_j|.$$

Then we get

$$K_2^2 \le \int_{Q_j} |b(z)-b_{Q_j}|\,|f_j(z)| \left(\int_0^{(9/4)|x-z_j|} \int_{\substack{|y-z|<(3/4)|x-z_j|,\,|y-z|<t \\ |x-y|\ge(1/4)|x-z_j|}} \frac{t^{2n+\theta}}{|x-y|^{2n+\theta}} \times \frac{1}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz \tag{5.6}$$

$$\le C|x-z_j|^{-n-\theta/2} \int_{Q_j} |b(z)-b_{Q_j}|\,|f_j(z)| \times \left(\int_0^{(9/4)|x-z_j|} \int_{|y-z|<(3/4)|x-z_j|} \frac{1}{|y-z|^{(n-\theta/2)}} \frac{dydt}{t^{1-\theta/2}} \right)^{1/2} dz$$

$$\le C|x-z_j|^{-n}|Q_j|^{1/p'} \left(\frac{1}{|Q_j|} \int_{Q_j} |b(z)-b_{Q_j}|^{p'}\, dz \right)^{1/p'} \left(\int_{Q_j} |f_j(z)|^p\, dz \right)^{1/p}$$

$$\le C|Q_j|^{1/p'}|x-z_j|^{-n},$$

where θ is defined in (4.20). By (5.3), (5.4), (5.5) and (5.6), we get

$$|g_\lambda^{*,\rho}((b-b_{Q_j})f_j)(x)| \le C|Q_j|^{1/p'}|x-z_j|^{-n}. \tag{5.7}$$

Since

$$\begin{aligned}
|b(x)-b_{Q_j}|g_\lambda^{*,\rho}(f_j)(x) &\le |b(x)-b_{Q_j}|\left(\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.\\
&\quad\left.\times\left|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\quad+|b(x)-b_{Q_j}|\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.\\
&\quad\left.\times\left|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_j(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&:= F_1+F_2.
\end{aligned} \tag{5.8}$$

Since $\left(\frac{t}{t+|x-y|}\right)^{\lambda n} \le 1$, then $F_1 \le |b(x)-b_{Q_j}|S^\rho(f_j)(x)$, and by (4.6), we get

$$F_1 \le C|Q_j|^{1/p'}|b(x)-b_{Q_j}|\,|x-z_j|^{-n}. \tag{5.9}$$

Similar to the estimate of K_2,

$$F_2 \le C|x-z_j|^{-n}|b(x)-b_{Q_j}|\int_{Q_j}|f_j(z)|\,dz \le C|x-z_j|^{-n}|b(x)-b_{Q_j}|\,|Q_j|^{1/p'}. \tag{5.10}$$

By (5.8), (5.9) and (5.10), we have

$$|(b(x)-b_{Q_j})g_\lambda^{*,\rho}f_j(x)| \le C|x-z_j|^{-n}|b(x)-b_{Q_j}|\,|Q_j|^{1/p'}. \tag{5.11}$$

By (5.7) and (5.11), we get

$$|[b,g_\lambda^{*,\rho}]f_j(x)| \le C|Q_j|^{1/p'}|x-z_j|^{-n}+C|x-z_j|^{-n}|b(x)-b_{Q_j}|\,|Q_j|^{1/p'}. \tag{5.12}$$

Thus, by the estimates of (5.1) and (5.12) and using the method of proving Theorem 4, we can show $\{[b, g_\lambda^{*,\rho}]f_j\}_{j=1}^\infty$ is not a compact set in $L^p(\mathbf{R}^n)$ with $\{f_j\}_{j=1}^\infty$ chosen in (3.2). So, b must satisfy Lemma 2.6 (i). Similarly, we can state that b satisfies also (ii) and (iii) of Lemma 2.6. Hence $b \in VMO(\mathbf{R}^n)$.

5.2. The proof of Theorem 6: $b \in VMO \Rightarrow [b, g_\lambda^{*,\rho}]$ is a compact operator in L^p

Suppose that $b \in VMO(\mathbf{R}^n)$, then by Theorem G, the commutator $[b, g_\lambda^{*,\rho}]$ is bounded on $L^p(\mathbf{R}^n)$. We need to prove that for any bounded set $\mathscr{F}$ in $L^p(\mathbf{R}^n)$, $\mathscr{G} = \{[b, g_\lambda^{*,\rho}]f : f \in \mathscr{F}\}$ is strongly pre-compact in $L^p(\mathbf{R}^n)$. Notice that

$$\begin{aligned}(5.13)\quad &|[b, g_\lambda^{*,\rho}]f(x) - [b^\varepsilon, g_\lambda^{*,\rho}]f(x)| \\ &\le \left(\int_0^\infty \int_{\mathbf{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right.\right. \\ &\qquad \left.\left. \times [(b(x)-b^\varepsilon(x)) - (b(z)-b^\varepsilon(z))]f(z)\,dz \right|^2 \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}.\end{aligned}$$

Thus, if $b^\varepsilon \in C_0^\infty$ such that $\|b - b^\varepsilon\|_* < \varepsilon$, then

$$(5.14)\qquad \|[b, g_\lambda^{*,\rho}] - [b^\varepsilon, g_\lambda^{*,\rho}]\|_{L^p \mapsto L^p} \le \|[b - b^\varepsilon, g_\lambda^{*,\rho}]\|_{L^p \mapsto L^p} < C\varepsilon$$

by Theorem G. Hence, to prove Theorem 6 it suffices to state that $\mathscr{G}$ is strongly pre-compact in $L^p(\mathbf{R}^n)$ for $b \in C_0^\infty$. By Theorem I in §3.2, we need only to verify (3.26)–(3.28) hold uniformly in G.

Denote $\sup_{f \in \mathscr{F}} \|f\|_p = D$, then (3.26) can be obtained from Theorem G. We now discuss (3.28). Assume $\mathrm{supp}(b) \subset \{z : |z| < r\}$ for some $r > 0$, for any x satisfying $|x| > \max\{2A, 4r\}$ and every $f \in \mathscr{F}$, where the constant A is fixed by (3.31), we have

$$\begin{aligned}|[b, g_\lambda^{*,\rho}]f(x)| &= \left(\int_0^\infty \int_{\mathbf{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{\substack{|y-z|<t\\|z|<r}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z) f(z)\,dz \right|^2 \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} \\ &= \left(\int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right. \\ &\qquad \left. \times \left| \int_{\substack{|y-z|<t\\|z|<r}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z) f(z)\,dz \right|^2 \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}\end{aligned}$$

$$+\left(\int_0^\infty \int_{|x-y|\ge t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.$$

$$\left.\times \left|\int_{\substack{|y-z|<t\\ |z|<r}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z)f(z)\,dz\right|^2 \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}$$

$$:= P_1 + P_2$$

Since $\left(\dfrac{t}{t+|x-y|}\right)^{\lambda n} \le 1$, then $P_1 \le U$, where U is defined in (4.13). We therefore get

$$P_1 \le C|x|^{-n}\|f\|_p \le CD|x|^{-n}. \tag{5.15}$$

On the other hand, then by the Minkowski inequality, we get

$$\begin{aligned}
P_2 &\le C\int_{|z|<r} |b(z)|\,|f(z)| \\
&\quad \times \left(\int_0^\infty \int_{|y-z|<t,\,|x-y|\ge t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} dz \\
&\le C\int_{|z|<r} |b(z)|\,|f(z)| \\
&\quad \times \left(\int_0^\infty \int_{\substack{|y-z|<t,\,|x-y|\ge t\\ |y-z|\ge(1/2)|x-z|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} dz \\
&\quad + C\int_{|z|<r} |b(z)|\,|f(z)| \\
&\quad \times \left(\int_0^\infty \int_{\substack{|y-z|<t,\,|x-y|\ge t\\ |y-z|<(1/2)|x-z|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} dz \\
&:= P_2^1 + P_2^2
\end{aligned} \tag{5.16}$$

For P_2^1. Since $|x| > 2|z|$, we get $|x-z| \ge |x| - |z| > 1/2|x|$. Thus $t > |y-z| > \frac{1}{2}|x-z| > \frac{1}{4}|x|$. Note that $\left(\dfrac{t}{t+|x-y|}\right)^{\lambda n} < 1$, and $\Omega \in L^2(S^{n-1})$, we have

$$(5.17)\quad P_2^1 \le \int_{|z|<r} |b(z)|\,|f(z)| \left(\int_{(1/4)|x|}^{\infty} \int_{\substack{|y-z|<t \\ |y-z|\ge(1/4)|x|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz$$

$$\le C|x|^{-n+\rho} \int_{|z|<r} |b(z)|\,|f(z)| \left(\int_{(1/4)|x|}^{\infty} \int_{|y-z|<t} |\Omega(y-z)|^2 \, dy \frac{dt}{t^{n+1+2\rho}} \right)^{1/2} dz$$

$$\le C|x|^{-n+\rho} \int_{|z|<r} |b(z)|\,|f(z)| \left(\int_{(1/4)|x|}^{\infty} \frac{dt}{t^{1+2\rho}} \right)^{1/2} dz$$

$$\le C|x|^{-n} \left(\int_{|z|<r} |b(z)|^{p'} \, dz \right)^{1/p'} \|f\|_p$$

$$\le CD|x|^{-n}.$$

Let us turn to P_2^2. Since $|y-z| < \frac{1}{2}|z-x|$, we get

$$|x-y| \ge |x-z| - |y-z| > \frac{1}{2}|x-z| \quad \text{and}$$

$$t \le |x-y| \le |x-z| + |y-z| < \frac{3}{2}|x-z|.$$

By $\frac{1}{2}|x| \le |x-z| \le \frac{3}{2}|x|$, then

$$t < \frac{4}{9}|x|, \quad |x-y| \ge \frac{1}{4}|x| \quad \text{and} \quad |y-z| < \frac{3}{4}|x|.$$

Then by $\Omega \in L^2(S^{n-1})$, we get

$$(5.18)\quad P_2^2 \le \int_{|z|<r} |b(z)|\,|f(z)|$$

$$\times \left(\int_0^{(9/4)|x|} \int_{\substack{|y-z|<(3/4)|x|,\,|y-z|<t \\ |x-y|\ge(1/4)|x|}} \frac{t^{2n+\theta}}{|x-y|^{2n+\theta}} \frac{|\Omega(y-z)|}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz$$

$$\le C|x|^{-n-\theta/2} \int_{|z|<r} |b(z)|\,|f(z)|$$

$$\times \left(\int_0^{(9/4)|x|} \int_{|y-z|<(3/4)|x|} \frac{|\Omega(y-z)|}{|y-z|^{(n-\theta/2)}} \frac{dydt}{t^{1-\theta/2}} \right)^{1/2} dz$$

$$\le CD|x|^{-n}.$$

where θ is defined in (4.20). From (5.15)–(5.18) and applying (3.31), we obtain

$$\left(\int_{|x|>B} |[b, g_\lambda^{*,\rho}]f(x)|^p \, dx\right)^{1/p} \le CD\varepsilon. \tag{5.19}$$

(5.19) shows that (3.28) holds uniformly in $\mathscr{G}$. Finally, let us to show (3.27) holds uniformly in $\mathscr{G}$. We need to prove that for any $\varepsilon > 0$, if $|z|$ is sufficiently small, then for every $f \in \mathscr{F}$,

$$\|[b, g_\lambda^{*,\rho}]f(\cdot) - [b, g_\lambda^{*,\rho}]f(\cdot + z)\|_p \le C\varepsilon.$$

To do this, for any $v \in \mathbf{R}^n$, by the Minkowski's inequality, we have

$$\begin{aligned}
&|[b, g_\lambda^{*,\rho}]f(x) - [b, g_\lambda^{*,\rho}]f(x+v)| \\
&= \Bigg| \left(\int_0^\infty \int_{\mathbf{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right.\right. \\
&\qquad \left.\left. \times (b(x) - b(z)) f(z) \, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \\
&\qquad - \left(\int_0^\infty \int_{\mathbf{R}^n} \left(\frac{t}{t+|x+v-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right.\right. \\
&\qquad \left.\left. \times (b(x+v) - b(z)) f(z) \, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \Bigg| \\
&\le \left(\int_0^\infty \int_{\mathbf{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x) - b(z)) f(z) \, dz \right.\right. \\
&\qquad \left.\left. - \int_{|y+v-z|<t} \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} (b(x+v) - b(z)) f(z) \, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \\
&= \left(\int_0^\infty \int_{\mathbf{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |I(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \\
&\le \left(\int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |I(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \\
&\qquad + \left(\int_0^\infty \int_{|x-y|\ge t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |I(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \\
&:= T_1 + T_2.
\end{aligned} \tag{5.20}$$

where $I(x, v, y, t)$ is defined in (4.16). Since $\left(\frac{t}{t+|x-y|}\right)^{\lambda n} \le 1$, then

$$T_1 \le \left(\int_0^\infty \int_{|x-y|<t} |I(x, v, y, t)|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}.$$

From (4.17) to (4.26), we know that

$$\|T_1\|_p \le \left\| \left(\int_0^\infty \int_{|x-y|<t} |I(x, v, y, t)|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \right\|_p \le CD\varepsilon. \tag{5.21}$$

Now we estimate T_2. Decompose $I(x, v, y, t)$ as $I(x, v, y, t) := \sum_{i=1}^6 J_i(x, v, y, t)$, where J_i, $i = 1, \dots, 6$ is defined in (4.17). Thus

$$\begin{aligned} \|T_2\|_p &\le \sum_{j=1}^6 \left\| \left\{\int_0^\infty \int_{|x-y|\ge t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |J_j(x, v, y, t)|^2 \frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2} \right\|_p \\ &:= \sum_{j=1}^6 \|T_2^j\|_p. \end{aligned} \tag{5.22}$$

Below we give the estimates of T_2^j for $1 \le j \le 6$. Since $|b(x+v) - b(z)| < C$ and the Minkowski inequality, we have

$$\begin{aligned} T_2^1 &\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|\ge t \\ |y-z|<t,\, |y+v-z|\ge t}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \right. \\ &\qquad \left. \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz \\ &\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|\ge t,\, |y-z|\le 2|v| \\ |y-z|<t,\, |y+v-z|\ge t}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \right. \\ &\qquad \left. \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz \end{aligned} \tag{5.23}$$

$$+ C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t,\,|y-z|>2|v|\\ |y-z|<t,\,|y+v-z|\ge t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.$$

$$\left.\times\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$

$$:= O_1 + O_2.$$

For O_1. Since $|y-z| \le 2|v|$, then

$$t \le |y-z+v| \le |y-z| + |v| \le 3\cdot 2^{-1/\varepsilon}|x-z| \le |x-z|,$$

and $|x-y| > |x-z| - |y-z| > \frac{|x-z|}{2}$, then by $\Omega \in L^2(S^{n-1})$, we get

$$O_1 \le \int_{|x-z|>2^{1/\varepsilon}|v|}|f(z)| \tag{5.24}$$

$$\times\left(\int_0^{|x-z|}\int_{\substack{|y-z|<2|v|,\,|y-z|<t\\ |x-y|\ge(1/2)|x-z|}}\frac{t^{2n+\theta}}{|x-y|^{2n+\theta}}\frac{|\Omega(y-z)|}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}dz$$

$$\le C\int_{|x-z|>2^{1/\varepsilon}|v|}|x-z|^{-n-\theta/2}|f(z)|$$

$$\times\left(\int_0^{|x-z|}\int_{|y-z|<2|v|}\frac{|\Omega(y-z)|^2}{|y-z|^{(n-\theta/2)}}\frac{dydt}{t^{1-\theta/2}}\right)^{1/2}dz$$

$$\le C\int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|v|^{\theta/4}}{|x-z|^{n+\theta/4}}|f(z)|\,dz,$$

where θ is defined in (4.20). On the other hand, if denote

$$E_1 = \left\{y\in\mathbf{R}^n : |x-y|\ge t, |y-z|>2|v|, |y-z|\le\frac{1}{2}|x-z|, |y-z|<t,\right.$$
$$\left.|y+v-z|\ge t\right\}$$

and

$$E_2 = \left\{y\in\mathbf{R}^n : |x-y|\ge t, |y-z|>2|v|, |y-z|>\frac{1}{2}|x-z|, |y-z|<t,\right.$$
$$\left.|y+v-z|\ge t\right\},$$

then

$$O_2 \le \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{E_1} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$

$$+ \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{E_2} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$

$$:= O_{2,1} + O_{2,2}.$$

For $O_{2,1}$. Since $|y-z| > 2|v|$ and $|y-z| \le \frac{1}{2}|x-z|$, then

$$t < |y-z+v| \le |y-z| + |v| \le \frac{3|y-z|}{2} \quad \text{and}$$

$$|x-y| > |x-z| - |y-z| > \frac{1}{2}|x-z|.$$

Then by $\Omega \in L^2(S^{n-1})$ and the choice of θ in (4.20), we get

$$(5.25) \quad O_{2,1} \le \int_{|x-z|>2^{1/\varepsilon}|v|} |f(z)| \left(\iint_{\substack{t\le|y-z+v|,\,|y-z|<t,\,|y-z|>2|v|,\,t\le 3|y-z|/2 \\ |x-y|\ge(1/2)|x-z|,\,|y-z|<(1/2)|x-z|}} \frac{t^{2n+\theta}}{|x-y|^{2n+\theta}} \right.$$

$$\left. \times \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} |x-z|^{-n-\theta/2} |f(z)| \left(\int_{2|v|<|y-z|<(1/2)|x-z|} \frac{|\Omega(y-z)|^2}{|y-z|^{(-\theta-2\rho)}} \right.$$

$$\left. \times \int_{\substack{t>|y-z| \\ t\le|y-z+v|}} \frac{dt}{t^{n+1+2\rho}}\, dy \right)^{1/2} dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} |x-z|^{-n-\theta/2} |f(z)|$$

$$\times \left(\int_{2|v|<|y-z|<(1/2)|x-z|} \frac{|\Omega(y-z)|^2 |v|}{|y-z|^{(n+1-\theta)}}\, dy \right)^{1/2} dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} |x-z|^{-n-\theta/2} |f(z)|\, |v|^{\theta/4}$$

$$\times \left(\int_{|y-z|<(1/2)|x-z|} \frac{|\Omega(y-z)|^2}{|y-z|^{(n-\theta/2)}}\, dy \right)^{1/2} dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|v|^{\theta/4}}{|x-z|^{n+\theta/4}} |f(z)|\, dz.$$

For $O_{2,2}$. Since $|y-z| > \frac{1}{2}|x-z|$ and $\left(\frac{t}{t+|x-y|}\right)^{\lambda n} \le 1$, we have

$$(5.26) \quad O_{2,2} \le \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_{|y-z|>(1/2)|x-z|} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \times \int_{\substack{|y-z|>2|v|,\\ |y-z|<t,\, |y+v-z|\ge t}} \frac{dt}{t^{n+2\rho+1}}\, dy \right\}^{1/2} |f(z)|\, dz$$

$$\le \int_{|x-z|>2^{1/\varepsilon}|v|} |v| \left\{ \int_{|y-z|>(1/2)|x-z|} \frac{|\Omega(y-z)|^2}{|y-z|^{3n+1}}\, dy \right\}^{1/2} |f(z)|\, dz$$

$$\le \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|v|^{1/2}}{|x-z|^{n+1/2}} |f(z)|\, dz.$$

From the estimates of O_1, $O_{2,1}$ and $O_{2,2}$, we obtain

$$(5.27) \quad \|T_2^1\|_p \le C \left\{ \int_{\mathbf{R}^n} \left(\int_{|y|>2^{1/\varepsilon}|v|} \frac{|v|^{1/2}}{|y|^{n+1/2}} |f(x-y)|\, dy \right)^p dx \right\}^{1/p} + \left\{ \int_{\mathbf{R}^n} \left(\int_{|y|>2^{1/\varepsilon}|v|} \frac{|v|^{\theta/4}}{|y|^{n+\theta/4}} |f(x-y)|\, dy \right)^p dx \right\}^{1/p}$$

$$\le C\|f\|_p \left(\int_{|y|>2^{1/\varepsilon}|v|} \frac{|v|^{1/2}}{|y|^{n+1/2}}\, dy + \int_{|y|>2^{1/\varepsilon}|v|} \frac{|v|^{\theta/4}}{|y|^{n+\theta/4}}\, dy \right)$$

$$\le CD\varepsilon.$$

Regarding T_2^2. Since $|b(x+v)-b(z)| < C$ and by the Minkowski inequality, we have

$$T_2^2 \le C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|\ge t\\ |y-z|\ge t,\, |y+v-z|<t}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \times \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|\ge t,\, |y-z|\le 2|v|\\ |y-z|\ge t,\, |y+v-z|<t}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \times \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$

$$+ C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t,\,|y-z|>2|v|\\|y-z|\ge t,\,|y+v-z|<t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.$$

$$\left.\times\frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$

$$:= R_1 + R_2.$$

For R_1. Since $|y-z|\le 2|v|$, then

$$t\le|y-z|\le 2\cdot 2^{-1/\varepsilon}|x-z|\le|x-z|\quad\text{and}\quad|x-y|>|x-z|-|y-z|>\frac{|x-z|}{2}.$$

By $\Omega\in L^2(S^{n-1})$, we get

$$\begin{aligned}(5.28)\quad R_1 &\le \int_{|x-z|>2^{1/\varepsilon}|v|}|f(z)|\\ &\quad\times\left(\int_0^{|x-z|}\int_{\substack{|y-z|<2|v|,\,|y+v-z|<t\\|x-y|\ge(1/2)|x-z|}}\frac{t^{2n+\theta}}{|x-y|^{2n+\theta}}\frac{|\Omega(y+v-z)|}{|y+v-z|^{2(n-\rho)}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}dz\\ &\le C\int_{|x-z|>2^{1/\varepsilon}|v|}|x-z|^{-n-\theta/2}|f(z)|\\ &\quad\times\left(\int_0^{|x-z|}\int_{|y+v-z|<3|v|}\frac{|\Omega(y+v-z)|^2}{|y+v-z|^{(n-\theta/2)}}\frac{dydt}{t^{1-\theta/2}}\right)^{1/2}dz\\ &\le C\int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|v|^{\theta/4}}{|x-z|^{n+\theta/4}}|f(z)|\,dz.\end{aligned}$$

On the other hand for R_2, denote

$$F_1=\left\{y\in\mathbf{R}^n:|x-y|\ge t,|y-z|>2|v|,|y+v-z|\le\frac{1}{2}|x-z|,|y-z|\ge t,\right.$$
$$\left.|y+v-z|<t\right\}$$

and

$$F_2=\left\{y\in\mathbf{R}^n:|x-y|\ge t,|y-z|>2|v|,|y+v-z|>\frac{1}{2}|x-z|,|y-z|\ge t,\right.$$
$$\left.|y+v-z|<t\right\}.$$

Then

$$R_2 \le \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{F_1} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$

$$+ \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{F_2} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$

$$:= R_{2,1} + R_{2,2}.$$

For $R_{2,1}$, Since $|y-z| > 2|v|$ and $|y+v-z| \le \frac{1}{2}|x-z|$, then

$$|y-z| < |y+v-z| + |v| < \frac{3}{4}|x-z| \quad \text{and} \quad |x-y| > |x-z| - |y-z| > \frac{1}{4}|x-z|.$$

By $\Omega \in L^2(S^{n-1})$, we get

$$R_{2,1} \le \int_{|x-z|>2^{1/\varepsilon}|v|} |f(z)| \tag{5.29}$$

$$\times \left(\iint_{\substack{|y-z+v|<t,\, |y-z|\ge t,\, |y-z|>2|v| \\ |x-y|\ge(1/4)|x-z|,\, |y+v-z|<(1/2)|x-z|,\, t<(3/2)|y-z+v|}} \frac{t^{2n+\theta}}{|x-y|^{2n+\theta}} \right.$$

$$\left. \times \frac{|\Omega(y+v-z)|}{|y+v-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} |x-z|^{-n-\theta/2} |f(z)|$$

$$\times \left(\int_{|v|<|y+v-z|<(1/2)|x-z|} \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{(-\theta-2\rho)}} \int_{\substack{t<|y-z| \\ t\ge|y-z+v|}} \frac{dt}{t^{n+1+2\rho}}\, dy \right)^{1/2} dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} |x-z|^{-n-\theta/2} |f(z)|$$

$$\times \left(\int_{|v|<|y+v-z|<(1/2)|x-z|} \frac{|\Omega(y+v-z)|^2 |v|}{|y+v-z|^{(n+1-\theta)}}\, dy \right)^{1/2} dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} |x-z|^{-n-\theta/2} |f(z)|\, |v|^{\theta/4}$$

$$\times \left(\int_{|y+v-z|<(1/2)|x-z|} \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{(n-\theta/2)}}\, dy \right)^{1/2} dz$$

$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|v|^{\theta/4}}{|x-z|^{n+\theta/4}} |f(z)|\, dz.$$

For $R_{2,2}$. Since $|y+v-z|>\frac{1}{2}|x-z|$, and $\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\le 1$, we get

$$R_{2,2}\le\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_{|y+v-z|>(1/2)|x-z|}\frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}}\right.$$
$$\left.\times\int_{\substack{|y-z|>2|v|,\\ |y-z|\ge t,\,|y+v-z|<t}}\frac{dt}{t^{n+2\rho+1}}\,dy\right\}^{1/2}|f(z)|\,dz$$
$$\le\int_{|x-z|>2^{1/\varepsilon}|v|}|v|\left\{\int_{|y+v-z|>(1/2)|x-z|}\frac{|\Omega(y+v-z)|^2}{|y+v-z|^{3n+1}}\,dy\right\}^{1/2}|f(z)|\,dz$$
$$\le\int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|v|^{1/2}}{|x-z|^{n+1/2}}|f(z)|\,dz. \tag{5.30}$$

Similar to (5.30), we get from the estimates of R_1, $R_{2,1}$ and $R_{2,2}$,

$$\|T_2^2\|_p=\left\|\left\{\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}|J_2(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2}\right\|_p\le CD\varepsilon. \tag{5.31}$$

About T_2^3, by the Minkowski inequality and $|b(x+v)-b(z)|<C$, we have

$$T_2^3\le C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t\\ |y-z|<t,\,|y+v-z|<t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.$$
$$\left.\times\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2\frac{dtdy}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$
$$\le C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t,\,|y-z|\le 8|v|\\ |y-z|<t,\,|y+v-z|<t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.$$
$$\left.\times\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2\frac{dtdy}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$
$$+C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t,\,|y-z|>8|v|\\ |y-z|<t,\,|y+v-z|<t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.$$
$$\left.\times\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2\frac{dtdy}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$
$$:=W_1+W_2.$$

For W_1, since $|y-z| \le 8|v|$, then

$$t \le |x-y| \le |x-z| + |y-z| \le 2|x-z| \quad \text{and}$$
$$|x-y| > |x-z| - |y-z| > \frac{|x-z|}{2}.$$

By $\Omega \in L^2(S^{n-1})$, we get

$$W_1 \le C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \iint_{\substack{|y-z|<8|v|,\, |y+v-z|<9|v|,\, |y-z|<t \\ |x-y|\ge|x-z|/2,\, |y+v-z|<t}} \frac{t^{2n+\theta}}{|x-y|^{2n+\theta}} \right. \tag{5.32}$$
$$\left. \times \left(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} + \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}} \right) \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$
$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} |x-z|^{-n-\theta/2}$$
$$\times \left\{ \int_{|y-z|<8|v|,\, |y+v-z|<9|v|} \left(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} + \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2n-2\rho}} \right) \right.$$
$$\left. \times \int_{|y-z|<t,\, |y+v-z|<t} \frac{dt}{t^{2\rho-n+1-\theta}}\, dy \right\}^{1/2} |f(z)|\, dz$$
$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} |x-z|^{-n-\theta/2}$$
$$\times \left\{ \int_{\substack{|y-z|<8|v|, \\ |y+v-z|<9|v|}} \frac{|\Omega(y-z)|^2}{|y-z|^{n-\theta}} + \frac{|\Omega(y+v-z)|^2}{|y+v-z|^{n-\theta}}\, dy \right\}^{1/2} |f(z)|\, dz$$
$$\le C|v|^{\theta/2} \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|f(z)|}{|x-z|^{n+\theta/2}}\, dz.$$

On the other hand, for W_2 we get

$$W_2 \le C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|\ge t,\, |y-z|>8|v| \\ |y-z|>|x-z|/2,\, |y-z|<t,\, |y+v-z|<t}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \right.$$
$$\left. \times \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}} \right|^2 \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$

$$\begin{aligned}
&+ C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t,\,|y-z|>8|v|\\ |y-z|\le|x-z|/2,\,|y-z|<t,\,|y+v-z|<t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.\\
&\quad\left.\times\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2\frac{dtdy}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz\\
&:= W_{2,1}+W_{2,2}.
\end{aligned}$$

Now for $W_{2,1}$. Since $|y-z|>\frac{1}{2}|x-z|$, then $t>\frac{1}{2}|x-z|$ and $\left(\dfrac{t}{t+|x-y|}\right)^{\lambda n}\le 1.$ Thus

$$\begin{aligned}
W_{2,1} &\le C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_{|y-z|>8|v|}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2\right.\\
&\quad\left.\times\int_{1/2|x-z|<t,\,|y-z|<t}\frac{dt}{t^{n+2\rho+1}}\right\}^{1/2} dy|f(z)|\,dz\\
&\le \int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|f(z)|}{|z-x|^n\left(\log\dfrac{|z-x|}{|v|}\right)^{2+\theta}}\\
&\quad\times\left\{\int_{|y-z|>8|v|}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2\int_{|y-z|<t}\frac{\left(\log\dfrac{t}{|v|}\right)^{4+2\theta}dt}{t^{2\rho-n+1}}\right\}^{1/2} dydz.
\end{aligned}$$

Then by the process of estimating J_3^2 in (4.21), we get

$$W_{2,1}\le\int_{|x-z|>2^{1/\varepsilon}|v|}\frac{|f(z)|}{|z-x|^n\left(\log\dfrac{|z-x|}{|v|}\right)^{2+\theta}}\,dz. \tag{5.33}$$

As for $W_{2,2}$, we have

$$\begin{aligned}
W_{2,2} &\le C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t,\,\gamma|v|>|y-z|>8|v|\\ |y-z|\le|x-z|/2,\,|y-z|<t,\,|y+v-z|<t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.\\
&\quad\left.\times\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2\frac{dtdy}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz
\end{aligned}$$

$$+ C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t,\,|y-z|>\gamma|v|\\|y-z|\le|x-z|/2,\,|y-z|<t,\,|y+v-z|<t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.$$

$$\left.\times\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2\frac{dtdy}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$

$$= W^1_{2,2}+W^2_{2,2},$$

where $\gamma = e^{[(4+2\theta)/(\lambda n-2n)]}$. For $W^1_{2,2}$, since $|y-z|\le|x-z|/2$, then

$$|y-z+v|\le|y-z|+|v|\le 3|x-z|/4 \quad\text{and}$$
$$|x-y|>|x-z|-|y-z|>|x-z|/2.$$

Thus, we get

$$W^1_{2,2}\le C\int_{|x-z|>2^{1/\varepsilon}|v|}\left\{\int_0^\infty\int_{\substack{|x-y|\ge t,\,\gamma|v|>|y-z|>8|v|\\|y-z|\le|x-z|/2,\,|y-z|<t,\,|y+v-z|<t}}\frac{t^{2n+\theta}}{|x-y|^{2n+\theta}}\right.$$

$$\left.\times\left(\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}+\frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2(n-\rho)}}\right)\frac{dtdy}{t^{n+2\rho+1}}\right\}^{1/2}|f(z)|\,dz$$

$$\le C\int_{|x-z|>2^{1/\varepsilon}|v|}|x-z|^{-n-\theta/2}\left\{\int_{\substack{\gamma|v|>|y-z|>8|v|\\|y-z|\le|x-z|/2,\,|y-z+v|\le3|x-z|/4}}\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}\right.$$

$$\left.+\frac{|\Omega(y+v-z)|^2}{|y+v-z|^{2(n-\rho)}}\int_{|y-z|<t,\,|y+v-z|<t}\frac{dt}{t^{2\rho-n+1-\theta}}\,dy\right\}^{1/2}|f(z)|\,dz$$

$$\le C\int_{|x-z|>2^{1/\varepsilon}|v|}|x-z|^{-n-\theta/2}(\gamma|v|)^{\theta/4}\left\{\int_{\substack{|y-z|\le|x-z|/2,\\|y-z+v|\le3|x-z|/4}}\frac{|\Omega(y-z)|^2}{|y-z|^{n-\theta/2}}\right.$$

$$\left.+\frac{|\Omega(y+v-z)|^2}{|y+v-z|^{n-\theta/2}}\,dy\right\}^{1/2}|f(z)|\,dz$$

$$\le C|v|^{\theta/4}\int_{|x-z|>2^{1/\varepsilon}|v|}|x-z|^{-n-\theta/4}|f(z)|\,dz.$$

Other the other hand, for $W^2_{2,2}$, since $t>|y-z|>\gamma|v|$ and $|y-z|\le|x-z|/2$, we get

$$|x-y|>|x-z|-|y-z|>|x-z|/2.$$

Hence

$$W_{2,2}^2 \le C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|\ge t,\, |y-z|>\gamma|v| \\ |y-z|\le|x-z|/2,\, |y-z|<t,\, |y+v-z|<t}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \right.$$
$$\left. \times \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2 \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$
$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \left\{ \int_0^\infty \int_{\substack{|x-y|\ge t,\, |y-z|>\gamma|v| \\ |y-z|\le|x-z|/2,\, |y-z|<t,\, |y+v-z|<t}} \frac{t^{\lambda n}}{|x-z|^{2n}\left(\log \frac{|z-x|}{|v|}\right)^{4+2\theta}} \right.$$
$$\left. \times \frac{\left(\log \frac{|y-x|+t}{|v|}\right)^{4+2\theta}}{(t+|x-y|)^{\lambda n-2n}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2 \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} |f(z)|\, dz$$
$$\le C \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|f(z)|}{|x-z|^{n}\left(\log \frac{|z-x|}{|v|}\right)^{2+\theta}} \left\{ \int_0^\infty \int_{\substack{|x-y|\ge t,\, |y-z|>\gamma|v| \\ |y-z|\le|x-z|/2,\, |y-z|<t,\, |y+v-z|<t}} \right.$$
$$\left. \times\, t^{\lambda n} \frac{\left(\log \frac{|y-x|+t}{|v|}\right)^{4+2\theta}}{(t+|x-y|)^{\lambda n-2n}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2 \frac{dtdy}{t^{n+2\rho+1}} \right\}^{1/2} dz.$$

Note that the function $N(s) = \dfrac{(\log s)^{4+2\theta}}{s^{\lambda n-2n}}$ is decreasing when $s > e^{[(4+2\theta)/(\lambda n-2n)]}$, where θ is defined in (4.20). Then

$$\text{(5.34)} \qquad \frac{\left(\log \frac{|y-x|+t}{|v|}\right)^{4+2\theta}}{(t+|x-y|)^{\lambda n-2n}} \le \frac{\left(\log \frac{t}{|v|}\right)^{4+2\theta}}{t^{\lambda n-2n}}, \quad \text{for } t > \gamma|v|.$$

So by (5.34), we have

$$W_{2,2}^2 \le \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|f(z)|}{|z-x|^{n}\left(\log \frac{|z-x|}{|v|}\right)^{2+\theta}} \left\{ \int_{|y-z|>\gamma|v|} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right|^2 \right.$$
$$\left. \times \int_{|y-z|<t} \frac{\left(\log \frac{t}{|v|}\right)^{4+2\theta}}{t^{2\rho-n+1}}\, dt \right\}^{1/2} dydz.$$

Then by the estimate of (4.21), we get

$$W_{2,2}^2 \leq \int_{|x-z|>2^{1/\varepsilon}|v|} \frac{|f(z)|}{|z-x|^n \left(\log \dfrac{|z-x|}{|v|}\right)^{2+\theta}} \, dz. \tag{5.35}$$

Combining the estimates of W_1, $W_{2,1}$, $W_{2,2}^1$ and $W_{2,2}^2$, we get

$$\begin{aligned}
\|T_2^3\|_p &\leq C\left(\int_{\mathbf{R}^n}\left|\int_{|y|>2^{1/\varepsilon}|v|} \frac{|f(x-y)|}{|y|^n\left(\log \dfrac{|y|}{|v|}\right)^{2+\theta}}\, dy\right|^p dx\right)^{1/p} \\
&\quad + C|v|^{\theta/2}\left(\int_{\mathbf{R}^n}\left|\int_{|y|>2^{1/\varepsilon}|v|} \frac{|f(x-y)|}{|y|^{n+\theta/2}}\, dy\right|^p dx\right)^{1/p} \\
&\quad + C|v|^{\theta/4}\left(\int_{\mathbf{R}^n}\left|\int_{|y|>2^{1/\varepsilon}|v|} \frac{|f(x-y)|}{|y|^{n+\theta/4}}\, dy\right|^p dx\right)^{1/p} \\
&\leq C\|f\|_p\left(\int_{|y|>2^{1/\varepsilon}|v|} \frac{1}{|y|^n\left(\log \dfrac{|y|}{|v|}\right)^{2+\theta}}\, dy\right. \\
&\quad \left. + |v|^{\theta/2}\int_{|y|>2^{1/\varepsilon}|v|} \frac{1}{|y|^{n+\theta/2}}\, dy + |v|^{\theta/4}\int_{|y|>2^{1/\varepsilon}|v|} \frac{1}{|y|^{n+\theta/4}}\, dy\right) \\
&\leq C(\varepsilon^{1+\theta} + 2^{-\theta/4\varepsilon} + 2^{-\theta/2\varepsilon})\|f\|_p \leq CD\varepsilon.
\end{aligned} \tag{5.36}$$

Now we give the estimate of T_2^4.

$$\begin{aligned}
T_2^4 &= \left(\int_0^\infty \int_{|x-y|\geq t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |J_4(x,v,y,t)|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \\
&\leq |b(x)-b(x+v)| \\
&\quad \times \left\{\int_0^\infty \int_{\mathbf{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|x-z|>2^{1/\varepsilon}|v|,\, |y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\, dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2} \\
&= |b(x)-b(x+v)|\mu_{\lambda,2^{1/\varepsilon}|v|}^{*,\rho} f(x).
\end{aligned}$$

We claim the following fact:

(5.37) $\mu^{*,\rho}_{\lambda,2^{1/\varepsilon}|v|}f(x) \le C(M(g^{*,\rho}_\lambda f)(x) + (M(|f|^q)(x))^{1/q} + Mf(x)), \quad 1 < q < \infty,$

where C is independent of v, ε. In fact, let Q denote the cube center at x and of diameter $r = 2^{1/\varepsilon}|v|/8$. Moreover, $f_1(x) = f_{\chi_{8Q}}(x)$ and $f_2(x) = f(x) - f_1(x)$. Then

$$\begin{aligned}\mu^{*,\rho}_{\lambda,2r}f(x) &\le \frac{1}{|Q|}\int_Q |g^{*,\rho}_\lambda f(\xi)|\,d\xi + \frac{1}{|Q|}\int_Q |g^{*,\rho}_\lambda f_1(\xi)|\,d\xi \\ &\quad + \frac{1}{|Q|}\int_Q |g^{*,\rho}_\lambda f_2(\xi) - \mu^{*,\rho}_{\lambda,8r}f(x)|\,d\xi \\ &\le M(g^{*,\rho}_\lambda f)(x) + I + II.\end{aligned}$$

By Theorem L in §4.2, we know

$$I \le \frac{C}{|Q|^{1/q}}\|g^{*,\rho}_\lambda f_1\|_q \le \frac{C}{|Q|^{1/q}}\|f_1\|_q \le C(M(|f|^q)(x))^{1/q}.$$

Let $\xi \in Q$, by the Minkowski inequality, we have

$$\begin{aligned}&|g^{*,\rho}_\lambda f_2(\xi) - \mu^{*,\rho}_{\lambda,8r}f(x)| \\ &\le \left\{\int_0^\infty \int_{\mathbf{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|\xi-x+y-z|<t}\frac{\Omega(\xi-x+y-z)}{|\xi-x+y-z|^{n-\rho}}f_2(z)\,dz\right.\right. \\ &\quad \left.\left. - \int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_2(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2} \\ &\le \left\{\int_0^\infty \int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|\xi-x+y-z|<t}\frac{\Omega(\xi-x+y-z)}{|\xi-x+y-z|^{n-\rho}}f_2(z)\,dz\right.\right. \\ &\quad \left.\left. - \int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_2(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2} \\ &\quad + \left\{\int_0^\infty \int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|\xi-x+y-z|<t}\frac{\Omega(\xi-x+y-z)}{|\xi-x+y-z|^{n-\rho}}f_2(z)\,dz\right.\right. \\ &\quad \left.\left. - \int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_2(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right\}^{1/2} \\ &:= U_1 + U_2.\end{aligned}$$

Since $\left(\frac{t}{t+|x-y|}\right)^{\lambda n} \le 1$, we know

$$U_1 \le G_1 + G_2 + G_3 \le Mf(x).$$

where G_i, $i = 1, 2, 3$ are defined in (4.24). As for U_2, we have

$$\begin{aligned}
U_2 \le{} & \left\{ \int_0^\infty \int_{|x-y|\ge t} \left| \int_{\substack{|y-z|<t,\\ |\xi-x+y-z|\ge t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f_2(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} \\
& + \left\{ \int_0^\infty \int_{|x-y|\ge t} \left| \int_{\substack{|\xi-x+y-z|<t,\\ |y-z|\ge t}} \frac{\Omega(\xi-x+y-z)}{|\xi-x+y-z|^{n-\rho}} f_2(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} \\
& + \left\{ \int_0^\infty \left\{ \int_0^\infty \int_{|x-y|\ge t} \left| \int_{\substack{|y-z|<t,\\ |\xi-x+y-z|<t}} \right. \right. \right. \\
& \times \left. \left. \left(\frac{\Omega(\xi-x+y-z)}{|\xi-x+y-z|^{n-\rho}} - \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right) f_2(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right\}^{1/2} \\
& := U_{2,1} + U_{2,2} + U_{2,3}.
\end{aligned}$$

Similar to the estimates of O_1 and O_2 in (5.24)–(5.26), we get

$$\begin{aligned}
U_{2,1} &\le C \int_{\mathbf{R}^n} \frac{r^{1/2}}{|x-z|^{n+1/2}} |f_2(z)|\, dz + \int_{\mathbf{R}^n} \frac{r^{\theta/4}}{|x-z|^{n+\theta/4}} |f_2(z)|\, dz \\
&\le Cr^{1/2} \int_{(8Q)^c} \frac{|f(z)|}{|x-z|^{n+1/2}}\, dz + \int_{(8Q)^c} \frac{r^{\theta/4}}{|x-z|^{n+\theta/4}} |f(z)|\, dz \\
&\le CMf(x).
\end{aligned}$$

Using the same way of estimating R_1 and R_2 in (5.28)–(5.30), we can get

$$U_{2,2} \le CMf(x).$$

Finally, similar to the estimates of W_1 and W_2 in (5.32)–(5.35), we have

$$\begin{aligned}
U_{2,3} \le{} & Cr^{n+\theta/2} \int_{\mathbf{R}^n} \frac{|f_2(z)|}{|x-z|^{n+\theta/2}}\, dz + C \int_{\mathbf{R}^n} \frac{|f_2(z)|}{|z-x|^n \left(\log \frac{|z-x|}{r}\right)^{2+\theta}}\, dz \\
& + Cr^{n+\theta/4} \int_{\mathbf{R}^n} \frac{|f_2(z)|}{|x-z|^{n+\theta/4}}\, dz \\
\le{} & CMf(x).
\end{aligned}$$

Thus $II \le CMf(x)$ and (5.37) follows. Since $b \in C_0^\infty$, we have $|b(x) - b(x+v)| \le C|v|$. Using (5.37) for $1 < q < p$, Theorem L and the L^p $(p > 1)$ boundedness of M, we get

$$\|T_2^4\|_p = \left\| \left(\int_0^\infty \int_{|x-y| \ge t} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} |J_4(x, v, y, t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \right\|_p \le C|v| \, \|f\|_p \le CD|v|. \tag{5.38}$$

About T_2^5, since $|b(x) - b(z)| \le C|x - z|$, by the Minkowski inequality

$$\begin{aligned} T_2^5 &= \left(\int_0^\infty \int_{|x-y| \ge t} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} |J_5(x, v, y, t)|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} \\ &\le \int_{|x-z| \le 2^{1/\varepsilon}|v|} |x-z| \, |f(z)| \\ &\quad \times \left(\int_0^\infty \int_{\substack{|y-z|<t, |x-y| \ge t \\ |y-z| \ge |x-z|/2}} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz \\ &\quad + \int_{|x-z| \le 2^{1/\varepsilon}|v|} |x-z| \, |f(z)| \\ &\quad \times \left(\int_0^\infty \int_{\substack{|y-z|<t, |x-y| \ge t \\ |y-z| < |x-z|/2}} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz \\ &:= Y_1 + Y_2. \end{aligned}$$

For Y_1. Since $t > |y - z| > 1/2|x - z|$ and $\left(\frac{t}{t + |x-y|} \right)^{\lambda n} < 1$, we get

$$\begin{aligned} Y_1 &\le \int_{|x-z| \le 2^{1/\varepsilon}|v|} |x-z| \, |f(z)| \\ &\quad \times \left(\int_{|x-z|/2}^\infty \int_{\substack{|y-z|<t \\ |y-z| \ge |x-z|/2}} \frac{|\Omega(y-z)|}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} dz \\ &\le C \int_{|x-z| \le 2^{1/\varepsilon}|v|} |x-z|^{1-n+\rho} |f(z)| \left(\int_{|x-z|/2}^\infty \frac{dt}{t^{1+2\rho}} \right)^{1/2} dz \\ &\le C \int_{|x-z| \le 2^{1/\varepsilon}|v|} |x-z|^{1-n} |f(z)| \, dz. \end{aligned} \tag{5.39}$$

Let us turn to Y_2. Since $|y - z| < 1/2|z - x|$, then

$$|x-y| \ge |x-z| - |y-z| > |x-z|/2 \quad \text{and}$$
$$t \le |x-y| \le |x-z| + |y-z| < 3|x-z|/2.$$

Hence

$$(5.40)\quad Y_2 \le \int_{|x-z|\le 2^{1/\varepsilon}|v|} |x-z|\,|f(z)| \times \left(\int_0^{3/2|x-z|}\int_{\substack{|y-z|<|x-z|/2,\,|y-z|<t\\ |x-y|\ge|x-z|/2}} \frac{t^{2n+\theta}}{|x-y|^{2n+\theta}}\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2} dz$$

$$\le C\int_{|x-z|\le 2^{1/\varepsilon}|v|} |f(z)|\,|x-z|^{1-n-\theta/2} \times \left(\int_0^{3|x-z|/2}\int_{|y-z|<|x-z|/2} \frac{|\Omega(y-z)|}{|y-z|^{(n-\theta/2)}}\frac{dydt}{t^{1-\theta/2}}\right)^{1/2} dz$$

$$\le C\int_{|x-z|\le 2^{1/\varepsilon}|v|} |x-z|^{1-n}|f(z)|\,dz$$

Then we get

$$(5.41)\qquad \|T_2^5\|_p = \left\|\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}|J_5(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}\right\|_p \le CD2^{1/\varepsilon}|v|.$$

Notice that $|b(x+v)-b(z)|\le C|x+v-z|$, similar to the estimate of T_2^5, we may get

$$T_2^6 = \left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}|J_6(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2} \le C\int_{|x-z|\le 2^{1/\varepsilon}|v|}\frac{|f(z)|}{|x+v-z|^{n-1}}\,dz.$$

Then we get

$$(5.42)\qquad \|T_2^6\|_p = \left\|\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}|J_6(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}\right\|_p \le CD(2^{1/\varepsilon}|v|+|v|).$$

From (5.20), (5.21) and the estimates of T_2^j, we get

$$\lim_{|v|\to 0}\|[b,g_\lambda^{*,\rho}]f(x)-[b,g_\lambda^{*,\rho}]f(x+v)\|_p = 0 \quad \text{uniformly in } f\in\mathscr{F}.$$

Thus we show that (3.28) holds and complete the proof of Theorem 6.

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