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MINIMAL SUBMANIFOLDS WITH SMALL TOTAL SCALAR CURVATURE IN EUCLIDEAN SPACE

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Abstract

Let M be an n-dimensional complete minimal submanifold in $\mathbf{R}^{n+p}$. Lei Ni proved that if M has sufficiently small total scalar curvature, then M has only one end. We improve the upper bound of total scalar curvature. We also prove that if M has the same upper bound of total scalar curvature, there is no nontrivial L^2 harmonic 1-form on M.

1. Introduction and theorems

Let M^n $(n \geq 3)$ be an n-dimensional complete immersed minimal hypersurface in $\mathbf{R}^{n+1}$. Cao, Shen and Zhu [2] proved that if M is stable, then M has only one end. Recall that a minimal submanifold is stable if the second variation of its volume is always nonnegative for any normal variation with compact support. Later Shen and Zhu [8] showed that if M is stable and has finite total scalar curvature, then M is totally geodesic. On the other hand, there are some gap theorems for minimal submanifolds with finite total scalar curvature in $\mathbf{R}^{n+p}$. Recently Lei Ni [6] proved that if M has sufficiently small total scalar curvature then M has only one end. More precisely, he proved the following.

Theorem ([6]). *Let M^n be an n-dimensional complete immersed minimal hypersurface in $\mathbf{R}^{n+p}$, $n \geq 3$. If*

$$\left(\int_M |A|^n \, dv\right)^{1/n} < C_1 = \sqrt{\frac{n}{n-1} C_s^{-1}},$$

then M has only one end. (Here C_s is a Sobolev constant in [4].)

In Section 2 we improve the upper bound C_1 of the total scalar curvature as follows.

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Theorem 1.1. *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$, $n \geq 3$. If*

$$\left(\int_M |A|^n \, dv\right)^{1/n} < \frac{n}{n-1}\sqrt{C_s^{-1}},$$

then M has only one end.

It is well-known that a minimal submanifold with finite total scalar curvature and one end must be an affine n-plane ([1]). Combining this fact, we have

Corollary 1.2. *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$, $n \geq 3$. If*

$$\left(\int_M |A|^n \, dv\right)^{1/n} < \frac{n}{n-1}\sqrt{C_s^{-1}},$$

then M is an affine n-plane.

Moreover, we study L^2 harmonic 1-forms on minimal submanifolds in $\mathbf{R}^{n+p}$. In [7], Palmer proved that if there exists a codimension one cycle C in a complete minimal hypersurface in $\mathbf{R}^{n+1}$, then M is unstable, by using the existence of a nontrivial L^2 harmonic 1-form on such M. Miyaoka [5] showed that if M is a complete stable minimal hypersurface in $\mathbf{R}^{n+1}$, then there are no nontrivial L^2 harmonic 1-forms on M. Recently Yun [10] proved that if M is a complete minimal hypersurface with $(\int_M |A|^n \, dv)^{1/n} < C_2 = \sqrt{C_s^{-1}}$, then there are no nontrivial L^2 harmonic 1-forms on M. We extend Yun's theorem to higher codimensional cases as follows.

Theorem 1.3. *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$, $n \geq 3$. If*

$$\left(\int_M |A|^n \, dv\right)^{1/n} < \frac{n}{n-1}\sqrt{C_s^{-1}},$$

then there are no nontrivial L^2 harmonic 1-forms on M.

2. Proofs of the theorems

Before proving Theorem 1.1, we need some useful facts.

Lemma 2.1 ([4]). *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$, $n \geq 3$. Then for any $\phi \in W_0^{1,2}(M)$ we have*

$$\left(\int_M |\phi|^{2n/(n-2)} \, dv\right)^{(n-2)/n} \leq C_s \int_M |\nabla \phi|^2 \, dv,$$

where C_s depends only on n.

Lemma 2.2 ([3]). *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$. Then the Ricci curvature of M satisfies*

$$\mathrm{Ric}(M) \geq -\frac{n-1}{n}|A|^2.$$

Now let u be a harmonic function on M. Using normal coordinate system $\{x^i\}$ at $p \in M$, we have Bochner formula

$$\frac{1}{2}\Delta(|\nabla u|^2) = \sum u_{ij}^2 + \mathrm{Ric}(\nabla u, \nabla u).$$

Then Lemma 2.2 gives

$$\frac{1}{2}\Delta(|\nabla u|^2) \geq \sum u_{ij}^2 - \frac{n-1}{n}|A|^2|\nabla u|^2.$$

We may choose the normal coordinates at p such that $u_1(p) = |\nabla u|(p)$, $u_i(p) = 0$ for $i \geq 2$. Then we have

$$\nabla_j|\nabla u| = \nabla_j\left(\sqrt{\sum u_i^2}\right) = \frac{\sum u_i u_{ij}}{|\nabla u|} = u_{1j}.$$

Therefore we obtain $|\nabla|\nabla u||^2 = \sum u_{1j}^2$. On the other hand, we know

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla u|\Delta|\nabla u| + |\nabla|\nabla u||^2.$$

Then we have

$$\sum u_{ij}^2 - \frac{n-1}{n}|A|^2|\nabla u|^2 \leq |\nabla u|\Delta|\nabla u| + \sum u_{1j}^2.$$

Hence we get

$$\begin{aligned}
|\nabla u|\Delta|\nabla u| + \frac{n-1}{n}|A|^2|\nabla u|^2 &\geq \sum u_{ij}^2 - \sum u_{1j}^2 \\
&\geq \sum_{i\neq 1} u_{i1}^2 + \sum_{i\neq 1} u_{ii}^2 \\
&\geq \sum_{i\neq 1} u_{i1}^2 + \frac{1}{n-1}\left(\sum_{i\neq 1} u_{ii}\right)^2 \\
&\geq \frac{1}{n-1}\sum_{i\neq 1} u_{i1}^2 = \frac{1}{n-1}|\nabla|\nabla u||^2,
\end{aligned}$$

where we used $\Delta u = \sum u_{ii} = 0$ in the last inequality. Therefore we get

$$|\nabla u|\Delta|\nabla u| + \frac{n-1}{n}|A|^2|\nabla u|^2 - \frac{1}{n-1}|\nabla|\nabla u||^2 \geq 0. \tag{2.1}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that M has at least two ends. First we note that if M has more than one end then there exists a nontrivial bounded harmonic function $u(x)$ on M which has finite total energy ([2] and [6]). Let $f = |\nabla u|$. From (2.1) we have

$$f\Delta f + \frac{n-1}{n}|A|^2 f^2 \geq \frac{1}{n-1}|\nabla f|^2.$$

Fix a point $p \in M$ and for $R > 0$ choose a cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_p(R)$, $\varphi = 0$ on $M \backslash B_p(2R)$, and $|\nabla \varphi| \leq \dfrac{1}{R}$. Multiplying both sides by φ^2 and integrating over M, we have

$$\int_M \varphi^2 f \Delta f\, dv + \frac{n-1}{n}\int_M \varphi^2 |A|^2 f^2\, dv \geq \frac{1}{n-1}\int_M \varphi^2 |\nabla f|^2\, dv.$$

Using integration by parts, we get

$$\begin{aligned} -\int_M |\nabla f|^2 \varphi^2\, dv - 2\int_M f\varphi\langle \nabla f, \nabla \varphi\rangle\, dv + \frac{n-1}{n}\int_M \varphi^2 |A|^2 f^2\, dv \\ \geq \frac{1}{n-1}\int_M \varphi^2 |\nabla f|^2\, dv. \end{aligned}$$

Applying Schwarz inequality, for any positive number $a > 0$, we obtain

$$\frac{n-1}{n}\int_M \varphi^2 |A|^2 f^2\, dv + \frac{1}{a}\int_M f^2 |\nabla \varphi|^2\, dv \geq \left(\frac{n}{n-1} - a\right)\int_M \varphi^2 |\nabla f|^2\, dv. \tag{2.2}$$

On the other hand, applying Sobolev inequality (Lemma 2.1), we have

$$\int_M |\nabla (f\varphi)|^2\, dv \geq C_s^{-1}\left(\int_M (f\varphi)^{2n/(n-2)}\, dv\right)^{(n-2)/n}.$$

Thus applying Schwarz inequality again, we have for any positive number $b > 0$,

$$\begin{aligned} (1+b)\int_M \varphi^2 |\nabla f|^2\, dv \geq C_s^{-1}\left(\int_M (f\varphi)^{2n/(n-2)}\, dv\right)^{(n-2)/n} \\ - \left(1 + \frac{1}{b}\right)\int_M f^2 |\nabla \varphi|^2\, dv. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$\begin{aligned} \frac{n-1}{n}\int_M \varphi^2 |A|^2 f^2\, dv \geq \frac{\left(\frac{n}{n-1} - a\right)}{b+1}\left\{C_s^{-1}\left(\int_M (f\varphi)^{2n/(n-2)}\, dv\right)^{(n-2)/n}\right\} \\ - \left(\frac{1}{a} + \frac{\frac{n}{n-1} - a}{b}\right)\int_M f^2 |\nabla \varphi|^2\, dv. \end{aligned}$$

Using Hölder inequality, we have

$$\int_M \varphi^2 |A|^2 f^2 \, dv \le \left(\int_M |A|^n\right)^{2/n} \left(\int_M (f\varphi)^{2n/(n-2)} \, dv\right)^{(n-2)/n}.$$

Hence we have

$$\left(\frac{1}{a} + \frac{\frac{n}{n-1} - a}{b}\right) \int_M f^2 |\nabla\varphi|^2 \, dv$$
$$\ge \left\{ \frac{\left(\frac{n}{n-1} - a\right) C_s^{-1}}{b+1} - \frac{n-1}{n} \left(\int_M |A|^n \, dv\right)^{2/n} \right\} \left(\int_M (f\varphi)^{2n/(n-2)} \, dv\right)^{(n-2)/n}.$$

By assumption, we choose a and b small enough such that

$$\left\{ \frac{\left(\frac{n}{n-1} - a\right) C_s^{-1}}{b+1} - \frac{n-1}{n} \left(\int_M |A|^n \, dv\right)^{2/n} \right\} \ge \varepsilon > 0.$$

Then letting $R \to \infty$, we have $f \equiv 0$, i.e., $|\nabla u| \equiv 0$. Therefore u is constant. This contradicts the assumption that u is a nontrivial harmonic function. □

Proof of Theorem 1.3. Let ω be an L^2 harmonic 1-form on minimal submanifold M in $\mathbf{R}^{n+p}$. We recall that such ω means

$$\Delta\omega = 0 \quad \text{and} \quad \int_M |\omega|^2 \, dv < \infty.$$

We will use confused notation for a harmonic 1-form ω and its dual harmonic vector field $\omega^{\#}$. From Bochner formula we have

$$\Delta|\omega|^2 = 2(|\nabla\omega|^2 + \mathrm{Ric}(\omega, \omega)).$$

We also have

$$\Delta|\omega|^2 = 2(|\omega|\Delta|\omega| + |\nabla|\omega||^2).$$

Since $|\nabla\omega|^2 \ge \frac{n}{n-1} |\nabla|\omega||^2$ by [9], it follows that

$$|\omega|\Delta|\omega| - \mathrm{Ric}(\omega, \omega) = |\nabla\omega|^2 - |\nabla|\omega||^2 \ge \frac{1}{n-1} |\nabla|\omega||^2.$$

By Lemma 2.2, we have

$$|\omega|\Delta|\omega| - \frac{1}{n-1}|\nabla|\omega||^2 \geq \mathrm{Ric}(\omega,\omega) \geq -\frac{n-1}{n}|A|^2|\omega|^2.$$

Therefore we get

$$|\omega|\Delta|\omega| + \frac{n-1}{n}|A|^2|\omega|^2 - \frac{1}{n-1}|\nabla|\omega||^2 \geq 0.$$

Multiplying both sides by φ^2 as in the proof of Theorem 1.1 and integrating over M, we have from integration by parts that

$$\begin{aligned} (2.4)\qquad 0 &\leq \int_M \varphi^2|\omega|\Delta|\omega| + \frac{n-1}{n}\varphi^2|A|^2|\omega|^2 - \frac{1}{n-1}\varphi^2|\nabla|\omega||^2\, dv \\ &= -2\int_M \varphi|\omega|\langle\nabla\varphi, \nabla|\omega|\rangle\, dv - \frac{n}{n-1}\int_M \varphi^2|\nabla|\omega||^2\, dv \\ &\quad + \frac{n-1}{n}\int_M |A|^2|\omega|^2\varphi^2\, dv. \end{aligned}$$

On the other hand, we get the following from Hölder inequality and Sobolev inequality (Lemma 2.1)

$$\begin{aligned} \int_M |A|^2|\omega|^2\varphi^2\, dv &\leq \left(\int_M |A|^n\, dv\right)^{2/n}\left(\int_M (\varphi|\omega|)^{2n/(n-2)}\, dv\right)^{(n-2)/n} \\ &\leq C_s\left(\int_M |A|^n\, dv\right)^{2/n}\int_M |\nabla(\varphi|\omega|)|^2\, dv \\ &= C_s\left(\int_M |A|^n\, dv\right)^{2/n} \\ &\quad \times\left(\int_M |\omega|^2|\nabla\varphi|^2 + |\varphi|^2|\nabla|\omega||^2 + 2\varphi|\omega|\langle\nabla\varphi, \nabla|\omega|\rangle\, dv\right). \end{aligned}$$

Then (2.4) becomes

$$\begin{aligned} (2.5)\qquad 0 &\leq -2\int_M \varphi|\omega|\langle\nabla\varphi, \nabla|\omega|\rangle\, dv - \frac{n}{n-1}\int_M \varphi^2|\nabla|\omega||^2\, dv \\ &\quad + \frac{n-1}{n}C_s\left(\int_M |A|^n\, dv\right)^{2/n} \\ &\quad \times\left(\int_M |\omega|^2|\nabla\varphi|^2 + \varphi^2|\nabla|\omega||^2 + 2\varphi|\omega|\langle\nabla\varphi, \nabla|\omega|\rangle\, dv\right). \end{aligned}$$

Using the following inequality for $\varepsilon > 0$,

$$2\left|\int_M \varphi|\omega|\langle\nabla\varphi, \nabla|\omega|\rangle\, dv\right| \leq \frac{\varepsilon}{2}\int_M \varphi^2|\nabla|\omega||^2\, dv + \frac{2}{\varepsilon}\int_M |\omega|^2|\nabla\varphi|^2\, dv,$$

we have from (2.5)

$$\left\{\frac{n}{n-1}-\frac{n-1}{n}C_s\left(\int_M |A|^n\,dv\right)^{2/n}-\frac{\varepsilon}{2}\left(1+\frac{n-1}{n}C_s\left(\int_M |A|^n\,dv\right)^{2/n}\right)\right\}$$
$$\times\int_M \varphi^2|\nabla|\omega||^2\,dv$$
$$\leq\left\{\frac{2}{\varepsilon}\left(1+\frac{n-1}{n}\left(\int_M |A|^n\,dv\right)^{2/n}\right)+\frac{n-1}{n}C_s\left(\int_M |A|^n\,dv\right)^{2/n}\right\}$$
$$\times\int_M |\omega|^2|\nabla\varphi|^2\,dv.$$

Since $(\int_M |A|^n\,dv)^{1/n}<\dfrac{n}{n-1}\sqrt{C_s^{-1}}$ by assumption, choosing $\varepsilon>0$ sufficiently small and letting $R\to\infty$, we obtain $\nabla|\omega|\equiv 0$, i.e., $|\omega|$ is constant. However, since $\int_M |\omega|^2\,dv<\infty$ and the volume of M is infinite, we get $\omega\equiv 0$. $\square$

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