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ON SEQUENTIALLY COHEN-MACAULAY MODULES

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Abstract

In this paper we introduce the notion of good systems of parameters with respect to certain increasing filtration of submodules of a Noetherian module in order to prove several characterizations of sequentially Cohen-Macaulay modules in terms of good systems of parameters. The sequential Cohen-Macaulayness of Stanley-Reisner rings of small embedding dimension are also examined.

1. Introduction

The concept of sequentially Cohen-Macaulay modules was introduced first by Stanley [11] for graded rings. Similarly one can give the definition of sequentially Cohen-Macaulay modules on local rings (see [4], [10]): Let M be a finitely generated module over a local ring R with $d = \dim M$. M is called a sequentially Cohen-Macaulay module if there exists a filtration of submodules of M

$$\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M$$

such that each D_i/D_{i-1} is Cohen-Macaulay and

$$0 < \dim(D_1/D_0) < \dim(D_2/D_1) < \cdots < \dim(D_t/D_{t-1}) = d.$$

Then, the filtration $\mathscr{D}$ of the sequentially Cohen-Macaulay module M as above is called the Cohen-Macaulay filtration. This filtration is determined uniquely and coincides with the dimension filtration of M (see [4]). A filtration $\mathscr{D}$ of M is said to be the dimension filtration if D_{i-1} is the largest submodule of D_i with $\dim D_{i-1} < \dim D_i$ for all $i = t, t-1, \ldots, 1$ and $D_0 = H^0_{\mathfrak{m}}(M)$ the zeroth local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$. It is clear that an unmixed module M is sequentially Cohen-Macaulay if and only if M is Cohen-Macaulay. Let $t = 1$ in the dimension filtration $\mathscr{D}$ above. Then M is sequentially Cohen-Macaulay if and only if $\ell_R(D_0) < \infty$ and D_1/D_0 is Cohen-Macaulay; and it follows easily from the theory of multiplicity in this case that M is sequentially Cohen-Macaulay if and only if there exists a system of parameters

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$\underline{x} = (x_1, \ldots, x_d)$ of M such that $\ell(M/\underline{x}M) = \ell_R(D_0) + e(\underline{x}; D_1)$. More general, let $\mathscr{D}$ be the dimension filtration of M with $\dim D_i = d_i$ and $\underline{x} = (x_1, \ldots, x_d)$ a system of parameters of M such that $D_i \cap (x_{d_i+1}, \ldots, x_d)M = 0$ for $i = 0, 1, \ldots, t-1$ (that a system of parameters will be called a good system of parameters of M), we proved in [3] that $\ell(M/\underline{x}M) = \sum_{i=0}^{t} e(x_1, \ldots, x_{d_i}; D_i)$ provided M is a sequentially Cohen-Macaulay module. Remember the well-known characterization of Cohen-Macaulay modules in terms of the lenght and the multiplicity of a system of parameters, it raises to the following two natural questions for sequentially Cohen-Macaulay modules:

1. Is it true that M is a sequentially Cohen-Macaulay module if and only if $\ell(M/\underline{x}M) = \sum_{i=0}^{t} e(x_1, \ldots, x_{d_i}; D_i)$ for all good systems of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M?

2. Is it true that M is a sequentially Cohen-Macaulay module if and only if $\ell(M/\underline{x}M) = \sum_{i=0}^{t} e(x_1, \ldots, x_{d_i}; D_i)$ for a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M?

The purpose of this paper is to give a positive answer for the first question (see Theorem 4.2). To the second question we prove that M is sequentially Cohen-Macaulay if and only if there exists a good system of parameters $x = (x_1, \ldots, x_d)$ of M such that $\ell(M/(x_1^2, \ldots, x_d^2)M) = \sum_{i=1}^{t} 2^{d_i} e(x_1, \ldots, x_{d_i}; D_i)$ (see Theorem 4.3). Based on results of Goto in [5] for approximately Cohen-Macaulay rings, we can show that this result is the best possibility. This means that there exist a module M and a good system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M such that $\ell(M/\underline{x}M) = \sum_{i=0}^{t} e(x_1, \ldots, x_{d_i}; D_i)$ but M is not sequentially Cohen-Macaulay (Example 4.7). Therefore the second question is not true in general. It should be mentioned that in order to give answers for the above questions we need some auxiliary results, for examples Lemma 2.7, Proposition 2.9, Lemma 4.1 …, which can be only proved in this paper if we state these results in a more general situation then for the case of the dimension filtration. Therefore we introduce in Section 2 the notions of filtrations satisfying the dimension condition and of good systems of parameters with respect to that a filtration, which are generalizations of the notions of the dimension filtration and of good systems of parameters. Some basic properties of good systems of parameters are given in this section. In Section 3 we restrict our consideration to the case the module is sequentially Cohen-Macaulay. It is shown in this case that a good system of parameters is a special type of strong d-sequences introduced by Goto-Yamagishi in [6]. We then use this result to give the first characterization of sequential Cohen-Macaulayness by means of good systems of parameters. The main results of the paper are stated and proved in Section 4. In the last section, we apply the main theorem to study the sequential Cohen-Macaulayness of Stanley-Reisner rings of small embedding dimension.

2. Good systems of parameters

Throughout this paper, M is a finitely generated module over a commutative Noetherian local ring $(R, \mathfrak{m})$ with $\dim M = d$.

DEFINITION 2.1. (i) We say that a filtration of submodules of M

$$\mathscr{F} : M_0 \subset M_1 \subset \cdots \subset M_t = M,$$

satisfies the dimension condition if $\dim M_{i-1} < \dim M_i$ for $i = 1, \dots, t$.

(ii) A filtration satisfying the dimension condition,

$$\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M,$$

is called the *dimension filtration* of M if the following two conditions are satisfied.

a) $D_0 = H^0_{\mathfrak{m}}(M)$ the zeroth local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$;

b) D_{i-1} is the largest submodule of D_i with $\dim D_{i-1} < \dim D_i$ for all $i = t, t-1, \dots, 1$.

DEFINITION 2.2. Let $\mathscr{F} : M_0 \subset M_1 \subset \cdots \subset M_t = M$ be a filtration satisfying the dimension condition and $d_i = \dim M_i$. A system of parameters $\underline{x} = (x_1, \dots, x_d)$ of M is called a *good system of parameters* with respect to the filtration $\mathscr{F}$ if $M_i \cap (x_{d_i+1}, \dots, x_d)M = 0$ for $i = 0, 1, \dots, t-1$. A good system of parameters with respect to the dimension filtration is simply called a good system of parameters of M.

It should be noted that in [10] Schenzel defined the dimension filtration of M as an increasing family of submodules $\{M_i\}_{0 \leqslant i \leqslant d}$ where M_i is the largest submodule of M with $\dim M_i \leqslant i$. Reduce this family to a set of submodules and renumber the indicators of this set, then we get a filtration of submodules which is just the dimension filtration in the sense of our definition. Also, a good system of parameters of M is always a distinguished system of parameters defined by Schenzel in [10]. But, in general, the converse does not hold.

Remark 2.3. i) Because of the Noetherian property of M, there always exists the dimension filtration $\mathscr{D}$ of M and it is unique. Moreover, let $\bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} N(\mathfrak{p}) = 0$ be a reduced primary decomposition of the zero module of M. Then $D_i = \bigcap_{\dim(R/\mathfrak{p}) \geqslant d_{i+1}} N(\mathfrak{p})$, where $d_i = \dim D_i$.

ii) Let N be a submodule of M and $\dim N < \dim M$. From the definition of the dimension filtration, there exists a D_i such that $N \subseteq D_i$ and $\dim N = \dim D_i$. Therefore, if $\mathscr{F} : M_0 \subset M_1 \subset \cdots \subset M_{t'} = M$ is a filtration satisfying the dimension condition, for each M_j there exists a D_i such that $M_j \subseteq D_i$ and $\dim M_j = \dim D_i$.

iii) If a system of parameters $\underline{x} = (x_1, \dots, x_d)$ is good with respect to a filtration $\mathscr{F}$, so is $(x_1^{n_1}, \dots, x_d^{n_d})$ for any positive integers $n_1, \dots, n_d$.

iv) A good system of parameters of M is also a good system of parameters with respect to any filtration satisfying the dimension condition.

LEMMA 2.4. *Let $\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M$ be the dimension filtration and $\underline{x} = (x_1, \dots, x_d)$ a good system of parameters of M. Put $d_i = \dim D_i$. Then*

$(x_1, \ldots, x_{d_i})$ is a good system of parameters of D_i and $D_i = 0 :_M x_j$ for all $d_i < j \leqslant d_{i+1}$, $i = 0, 1, \ldots, t-1$.

Proof. Since $D_i \cap (x_{d_i+1}, \ldots, x_d)M = 0$, $(x_1, \ldots, x_{d_i})$ is a system of parameters of D_i and we have $D_i \subseteq 0 :_M x_j$ for all $j > d_i$, $i = 0, 1, \ldots, t$. Note that D_i has the dimension filtration $D_0 \subset D_1 \subset \cdots \subset D_i$. Then it is obvious that $(x_1, \ldots, x_{d_i})$ is a good system of parameters of D_i. For the second conclusion, it suffices to prove that $0 :_M x_j \subseteq D_i$ for any $d_i < j \leqslant d_{i+1}$. Assume that $0 :_M x_j \nsubseteq D_i$. Let s be the greatest integer such that $0 :_M x_j \nsubseteq D_{s-1}$. Then $t \geqslant s > i$ and $0 :_M x_j \subseteq D_s$. We have $0 :_M x_j = 0 :_{D_s} x_j$. Since $d_s \geqslant d_{i+1} \geqslant j$, x_j is a parameter element of D_s and $\dim 0 :_M x_j < d_s$. Hence $0 :_M x_j \subseteq D_{s-1}$ by the maximality of D_{s-1}. This contradicts to the choise of s. Therefore, $0 :_M x_j \subseteq D_i$ for any $d_i < j \leqslant d_{i+1}$. □

The next result is about the existence of good system of parameters.

LEMMA 2.5. *There always exists a good system of parameters of M.*

Proof. Let $\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M$ be the dimension filtration of M with $d_i = \dim D_i$. By Remark 2.3, (i), $D_i = \bigcap_{\dim(R/\mathfrak{p}) \geqslant d_{i+1}} N(\mathfrak{p})$ where $\bigcap_{\mathfrak{p} \in \operatorname{Ass} M} N(\mathfrak{p}) = 0$ is a reduced primary decomposition of 0 of M. Put $N_i = \bigcap_{\dim(R/\mathfrak{p}) \leq d_i} N(\mathfrak{p})$. Then $D_i \cap N_i = 0$ and $\dim(M/N_i) = d_i$. By the Prime Avoidance Theorem there exists a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ such that $x_{d_i+1}, \ldots, x_d \in \operatorname{Ann}(M/N_i)$. Therefore $(x_{d_i+1}, \ldots, x_d)M \cap D_i \subseteq N_i \cap D_i = 0$ as required. □

The following result is an immediate consequence of Lemma 2.5 and Remark 2.3, (ii).

COROLLARY 2.6. *Let $\mathscr{F} : M_0 \subset M_1 \subset \cdots \subset M_t = M$ be a filtration satisfying the dimension condition. There always exists a good system of parameters with respect to $\mathscr{F}$.*

Let $\mathscr{F} : M_0 \subset M_1 \subset \cdots \subset M_t = M$ be a filtration of M satisfying the dimension condition with $d_i = \dim M_i$ and $\underline{x} = (x_1, \ldots, x_d)$ a good system of parameters with respect to $\mathscr{F}$. It is clear that $(x_1, \ldots, x_{d_i})$ is a system of parameters of M_i. Therefore the following difference is well defined

$$I_{\mathscr{F},M}(\underline{x}) = \ell(M/\underline{x}M) - \sum_{i=0}^{t} e(x_1, \ldots, x_{d_i}; M_i),$$

where $e(x_1, \ldots, x_{d_i}; M_i)$ is the Serre multiplicity and we set $e(x_1, \ldots, x_{d_0}; M_0) = \ell(M_0)$ if M_0 is of finite length.

Lemma 2.7. *Let $\mathcal{F}$ be a filtration of M satisfying the dimension condition and $\underline{x} = (x_1, \dots, x_d)$ a good system of parameters with respect to $\mathcal{F}$. Then $I_{\mathcal{F},M}(\underline{x}) \geqslant 0$.*

Proof. Denote

$$\frac{\mathcal{F}}{x_d\mathcal{F}} : \frac{M_0 + x_dM}{x_dM} \subset \frac{M_1 + x_dM}{x_dM} \subset \cdots \subset \frac{M_s + x_dM}{x_dM} \subset \frac{M}{x_dM},$$

where $s = t-1$ if $d_{t-1} < d-1$ and $s = t-2$ if $d_{t-1} = d-1$. Put $\underline{x}' = (x_1, \dots, x_{d-1})$. Since $(M_i + x_dM)/x_dM \simeq M_i$ for $i \leqslant s$, the filtration $\mathcal{F}/x_d\mathcal{F}$ satisfies the dimension condition and it is easy to prove that $\underline{x}'$ is a good system of parameters of M/x_dM with respect to $\mathcal{F}/x_d\mathcal{F}$. On the other hand, we have

$$I_{\mathcal{F}/x_d\mathcal{F}, M/x_dM}(\underline{x}') = \ell(M/\underline{x}M) - e(\underline{x}'; 0 :_M x_d) - e(\underline{x}; M) - \sum_{i=0}^{s} e(x_1, \dots, x_{d_i}; M_i).$$

If $d_{t-1} < d-1$, $I_{\mathcal{F},M}(\underline{x}) - I_{\mathcal{F}/x_d\mathcal{F}, M/x_dM}(\underline{x}') = e(\underline{x}'; 0 :_M x_d) \geqslant 0$. If $d_{t-1} = d-1$, then $M_{t-1} \subseteq 0 :_M x_d$ since $M_{t-1} \cap x_dM = 0$. Hence $I_{\mathcal{F},M}(\underline{x}) - I_{\mathcal{F}/x_d\mathcal{F}, M/x_dM}(\underline{x}') = e(\underline{x}'; 0 :_M x_d) - e(\underline{x}'; M_{t-1}) \geqslant 0$. Therefore $I_{\mathcal{F},M}(\underline{x}) \geqslant I_{\mathcal{F}/x_d\mathcal{F}, M/x_dM}(\underline{x}')$ and the lemma follows immediately by induction on d. □

Lemma 2.7 leads to some consequences which are useful in the sequel.

Corollary 2.8. *Let $\underline{x} = (x_1, \dots, x_d)$ be a good system of parameters with respect to the filtration $\mathcal{F}$. Then $I_{\mathcal{F},M}(\underline{x}) \geqslant I_{\mathcal{F}/x_d\mathcal{F}, M/x_dM}(x_1, \dots, x_{d-1})$.*

Set $\underline{x}(\underline{n}) = (x_1^{n_1}, \dots, x_d^{n_d})$ for any d-tuple of positive integers $\underline{n} = (n_1, \dots, n_d)$. We consider the difference $I_{\mathcal{F},M}(\underline{x}(\underline{n}))$ as a function in $n_1, \dots, n_d$.

Proposition 2.9. *Let $\mathcal{F}$ be a filtration satisfying the dimension condition and $\underline{x} = (x_1, \dots, x_d)$ a good system of parameters with respect to $\mathcal{F}$. Then the function $I_{\mathcal{F},M}(\underline{x}(\underline{n}))$ is non-decreasing, i.e., $I_{\mathcal{F},M}(\underline{x}(\underline{n})) \leqslant I_{\mathcal{F},M}(\underline{x}(\underline{m}))$ for all $n_i \leqslant m_i$, $i = 1, \dots, d$.*

Proof. We need only prove that the function $I_{\mathcal{F},M}(x_1, \dots, x_{r-1}, x_r^n, x_{r+1}, \dots, x_d)$ is non-decreasing in n for each $r \in \{1, 2, \dots, d\}$. Put $\underline{x}(n) = (x_1, \dots, x_{r-1}, x_r^n, x_{r+1}, \dots, x_d)$. We have

$$\begin{aligned} I_{\mathcal{F},M}(\underline{x}(n+1)) - I_{\mathcal{F},M}(\underline{x}(n)) &= \ell(M/\underline{x}(n+1)M) - \ell(M/\underline{x}(n)M) \\ &\quad - \sum_{d_i \geqslant r} e(x_1, \dots, x_{d_i}; M_i) \\ &\geqslant e(x_r; M/(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_d)M) \\ &\quad - \sum_{d_i \geqslant r} e(x_1, \dots, x_{d_i}; M_i). \end{aligned}$$

Applying Lech's formula and Lemma 2.7 we obtain

$$
\begin{aligned}
&e(x_r; M/(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_d)M) \\
&\quad = \lim_n \frac{1}{n} \ell(M/(x_1, \ldots, x_{r-1}, x_r^n, x_{r+1}, \ldots, x_d)M) \\
&\quad \geqslant \lim_n \frac{1}{n} \sum_{i=0}^{t} e(x_1, \ldots, x_{r-1}, x_r^n, x_{r+1}, \ldots, x_{d_i}; M_i) \\
&\quad = \sum_{d_i \geqslant r} e(x_1, \ldots, x_{d_i}; M_i)
\end{aligned}
$$

Therefore, $I_{\mathscr{F},M}(\underline{x}(n+1)) \geqslant I_{\mathscr{F},M}(\underline{x}(n))$. □

3. Sequentially Cohen-Macaulay modules

In this section we shall investigate some properties of sequentially Cohen-Macaulay modules. We first have the following definition.

Definition 3.1. A module M is called a *sequentially Cohen-Macaulay module* if for the dimension filtration $\mathscr{D}: D_0 \subset D_1 \subset \cdots \subset D_t = M$, each module D_i/D_{i-1} is Cohen-Macaulay for $i = 1, 2, \ldots, t$.

Note that the notion of sequentially Cohen-Macaulay module was introduced first by Stanley [11] for graded case (see also Herzog-Sbara [7], Cuong-Nhan [4], Schenzel [10]). In our study of sequentially Cohen-Macaulay modules, the notion of dd-sequences defined in [3] is used frequently. For convenience, we recall briefly the definition and some basic results of dd-sequences presented in [3]. We first recall the definition of d-sequence and of strong d-sequence due to Huneke [8] and Goto-Yamaghishi [6] respectively. A sequence $(x_1, x_2, \ldots, x_s)$ of elements of $\mathfrak{m}$ is called a d-*sequence* on M if $(x_1, \ldots, x_{i-1})M : x_j = (x_1, \ldots, x_{i-1})M : x_i x_j$ for $i = 1, 2, \ldots, s$ and $j \geqslant i$. Then $(x_1, x_2, \ldots, x_s)$ is a strong d-sequence on M if $(x_1^{n_1}, x_2^{n_2}, \ldots, x_s^{n_s})$ is a d-sequence for all positive integers $n_1, \ldots, n_s$.

Definition 3.2 [3, Definition 3.2]. A sequence $(x_1, \ldots, x_s)$ of elements of $\mathfrak{m}$ is called a dd-*sequence* on M if $(x_1, \ldots, x_i)$ is a strong d-sequence on the module $M/(x_{i+1}^{n_{i+1}}, \ldots, x_s^{n_s})M$ for all positive integers $n_i, \ldots, n_s$ and $i = 1, 2, \ldots, s$.

A dd-sequence has many good properties, especially when $\underline{x}$ is a system of parameters we have the following characterization.

Lemma 3.3 [6, Corollary 3.6]. *Let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters of M. Then $\underline{x}$ is a dd-sequence on M if and only if there exist integers $a_0, a_1, \ldots, a_d$ such that*

$$\ell(M/\underline{x}(\underline{n})M) = \sum_{i=0}^{d} a_i n_1 \cdots n_i$$

for all positive integers $n_1, \ldots, n_d$. *In this case,*

$$a_i = e(x_1, \ldots, x_i; (x_{i+2}, \ldots, x_d)M : x_{i+1}/(x_{i+2}, \ldots, x_d)M).$$

The next proposition gives some properties of sequentially Cohen-Macaulay modules which is used later on.

PROPOSITION 3.4. *Let* M *be a sequentially Cohen-Macaulay module. Let* $\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M$ *be the dimension filtration with* $d_i = \dim D_i$ *and* $\underline{x} = (x_1, \ldots, x_d)$ *a good system of parameters of* M. *Then*

i) $I_{\mathscr{D},M}(\underline{x}(\underline{n})) = 0$ *for all positive integers* $n_1, \ldots, n_d$. *In particular,* $\underline{x}$ *is a dd-sequence on* M.

ii) $(x_1, \ldots, x_{d_i})$ *is a regular sequence on* M/D_{i-1}, *for* $i = 1, \ldots, t$.

iii) $\operatorname{depth}(M/D_{i-1}) = d_i$, $i = 1, \ldots, t$.

iv) $(x_1, \ldots, x_i)M : x_j^2 = (x_1, \ldots, x_i)M + 0 :_M x_j$, *for all* $0 \leqslant i < j \leqslant d$.

v) $(x_1^{n_1}, \ldots, x_i^{n_i})M : x_{i+1}^{n_{i+1}} = (x_1^{n_1}, \ldots, x_i^{n_i})M + 0 :_M x_{i+1}$ *for all positive integers* $n_1, \ldots, n_d$ *and* $i = 0, 1, \ldots, d-1$.

Proof. i) is proved in [3, Theorem 1.5].

ii) By Lemma 2.4 $(x_1, \ldots, x_{d_i})$ is a system of parameters of D_i, so is of D_i/D_{i-1}. Then since D_i/D_{i-1} is Cohen-Macaulay, $(x_1, \ldots, x_{d_i})$ is a regular sequence on D_i/D_{i-1}, $i = 1, \ldots, t$. We prove the assertion by decreasing induction on i. Suppose $(x_1, \ldots, x_{d_{i+1}})$ is a regular sequence on M/D_i. From the exact sequence of Koszul homology modules

$$\begin{aligned} \cdots \to H_j(x_1, \ldots, x_{d_i}; D_i/D_{i-1}) &\to H_j(x_1, \ldots, x_{d_i}; M/D_{i-1}) \\ &\to H_j(x_1, \ldots, x_{d_i}; M/D_i) \to \cdots \end{aligned}$$

we imply $H_j(x_1, \ldots, x_{d_i}; M/D_{i-1}) = 0$ for all $j > 0$, since $H_j(x_1, \ldots, x_{d_i}; D_i/D_{i-1}) = 0$ and $H_j(x_1, \ldots, x_{d_i}; M/D_i) = 0$. Therefore, $(x_1, \ldots, x_{d_i})$ is a regular sequence on M/D_{i-1}.

iii) By (ii), we have $\operatorname{depth}(M/D_{i-1}) \geqslant d_i$, for $i = 1, \ldots, t$. Thus from the long exact sequence of local cohomology modules

$$\cdots \to H_{\mathfrak{m}}^{j-1}(M/D_i) \to H_{\mathfrak{m}}^j(D_i/D_{i-1}) \to H_{\mathfrak{m}}^j(M/D_{i-1}) \to H_{\mathfrak{m}}^j(M/D_i) \to \cdots$$

we get that $H_{\mathfrak{m}}^j(D_i/D_{i-1}) \simeq H_{\mathfrak{m}}^j(M/D_{i-1})$ for all $j < d_{i+1}$. In particular,

$$H_{\mathfrak{m}}^{d_i}(D_i/D_{i-1}) \simeq H_{\mathfrak{m}}^{d_i}(M/D_{i-1})$$

which is always non-zero. Hence, $\operatorname{depth}(M/D_{i-1}) = d_i$.

iv) Assume that $d_s < j \leqslant d_{s+1}$. We have

$$(x_1, \ldots, x_i)(M/D_s) : x_j^2 = (x_1, \ldots, x_i)(M/D_s)$$

since $(x_1, \dots, x_{d_{s+1}})$ is a regular sequence on M/D_s. This implies that

$$[(x_1, \dots, x_i)M + D_s] : x_j^2 = (x_1, \dots, x_i)M + D_s.$$

By Lemma 2.4, $D_s = 0 :_M x_j$. Then by the fact

$$[(x_1, \dots, x_i)M + D_s] : x_j^2 \supseteq (x_1, \dots, x_i)M : x_j^2 \supseteq (x_1, \dots, x_i)M + 0 :_M x_j$$

we obtain $(x_1, \dots, x_i)M : x_j^2 = (x_1, \dots, x_i)M + 0 :_M x_j$.

v) From (iv) we have

$$(x_1, \dots, x_i)M : x_{i+1} = (x_1, \dots, x_i)M + 0 :_M x_{i+1},$$

for $i = 0, 1, \dots, d-1$. Then the statement follows from the fact that $(x_1^{n_1}, \dots, x_d^{n_d})$ is also a good system of parameters of M for all positive integers $n_1, \dots, n_d$. □

Lemma 3.5. *Let $\underline{x} = (x_1, \dots, x_d)$ be a system of parameters and $\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M$ the dimension filtration of M with* $\dim D_i = d_i$. *Suppose that $\underline{x}$ is a dd-sequence on M. Then $D_i = 0 :_M x_{d_i+1}$ for $i = 0, 1, \dots, t-1$.*

Proof. We prove the lemma by induction on the dimension of M. The case $d = 1$ is clear. Assume that $d > 1$. From the hypothesis $\underline{x}$ is a dd-sequence, we have $D_{t-1} = 0 :_M x_d$ by [3, Lemma 6.3], and therefore $D_i \cap x_d^{n_d} M \subseteq D_{t-1} \cap x_d^{n_d} M = 0$ for $i = 0, \dots, t-1$ and all positive integers n_d. Then $\dim(D_i + x_d^{n_d} M)/x_d^{n_d} M = d_i$ for all $n_d \geqslant 1$, since

$$(D_i + x_d^{n_d} M)/x_d^{n_d} M \simeq D_i/x_d^{n_d} M \cap D_i = D_i.$$

Assume now that either $d_{t-1} < d-1$ or $i < t-1$. From Remark 2.3, (ii), there exists an R-module D in the dimension filtration of $M/x_d^{n_d}M$ such that $(D_i + x_d^{n_d} M)/x_d^{n_d} M \subseteq D$ and $\dim D = d_i$. Since the system of parameters $\underline{x}' = (x_1, \dots, x_{d-1})$ is also a dd-sequence on the module $M/x_d^{n_d}M$ and $\dim D = d_i$, it follows from the induction assumption that $D = (0 : x_{d_i+1})_{M/x_d^{n_d}M}$. Thus $D_i + x_d^{n_d} M \subseteq x_d^{n_d} M : x_{d_i+1}$ for all $n_d > 0$ and therefore $D_i \subseteq 0 :_M x_{d_i+1}$ by the Krull Intersection Theorem. On the other hand, since $\underline{x}$ is a d-sequence,

$$(x_{d_i+1}, \dots, x_d)(0 :_M x_{d_i+1}) = 0.$$

Thus $\dim(0 :_M x_{d_i+1}) \leqslant d_i$. This implies by the maximality of D_i that $D_i = 0 :_M x_{d_i+1}$. □

Lemma 3.6. *Let $(x_1, \dots, x_s)$ be a d-sequence on M. Then, for $i = 1, \dots, s$,*

$$(0 :_M x_i) \cap (x_1, \dots, x_s)M = (0 :_M x_i) \cap (x_1, \dots, x_{i-1})M.$$

In particular, $(0 :_M x_2) \cap (x_1, \dots, x_s)M = x_1(0 :_M x_2)$.

Proof. For the case $i = s$, we need to prove that

$$(0 :_M x_s) \cap (x_1, \dots, x_s)M \subseteq (0 :_M x_s) \cap (x_1, \dots, x_{s-1})M.$$

Let $a \in (0 :_M x_s) \cap (x_1, \dots, x_s)M$, then $ax_s = 0$ and $a = x_1a_1 + \cdots + x_sa_s$ where $a_1, \dots, a_s \in M$. Thus $a_s \in (x_1, \dots, x_{s-1})M : x_s^2 = (x_1, \dots, x_{s-1})M : x_s$ and so $a \in (0 :_M x_s) \cap (x_1, \dots, x_{s-1})M$. The case $i < s$ is proved by decreasing induction. Assume that $(0 :_M x_{i+1}) \cap (x_1, \dots, x_s)M = (0 :_M x_{i+1}) \cap (x_1, \dots, x_i)M$. Since $0 :_M x_i \subseteq 0 :_M x_{i+1}$, we have $(0 :_M x_i) \cap (x_1, \dots, x_s)M = (0 :_M x_i) \cap (x_1, \dots, x_i)M$. Note that $(x_1, \dots, x_i)$ is also a d-sequence, then by our proof above for $i = s$, we have $(0 :_M x_i) \cap (x_1, \dots, x_s)M = (0 :_M x_i) \cap (x_1, \dots, x_{i-1})M$. □

COROLLARY 3.7. *Let $\underline{x} = (x_1, \dots, x_d)$ be a system of parameters of M. Suppose $\underline{x}$ is a dd-sequence on M. Then $\underline{x}$ is a good system of parameters of M.*

Proof. Let $\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M$ be the dimension filtration of M with $\dim D_i = d_i$. From Lemma 3.5, $D_i = 0 :_M x_{d_i+1}$ for $i = 0, 1, \dots, t-1$. Since $\underline{x}$ is a dd-sequence, it is a d-sequence on M. Thus so is $(x_{d_i+1}, \dots, x_d)$. By Lemma 3.6 we have for all $0 \leqslant i < t$,

$$D_i \cap (x_{d_i+1}, \dots, x_d)M = 0.$$ □

COROLLARY 3.8. *Let M be a sequentially Cohen-Macaulay module and $\underline{x} = (x_1, \dots, x_d)$ a system of parameters of M. Then $\underline{x}$ is a good system of parameters of M if and only if $\underline{x}$ is a dd-sequence on M.*

Proof. The conclusion is immediate from Proposition 3.4, (i) and Corollary 3.7. □

We have the first characterization of sequentially Cohen-Macaulay modules.

THEOREM 3.9. *Let $\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M$ be the dimension filtration of M with $\dim D_i = d_i$ and $\underline{x} = (x_1, \dots, x_d)$ a good system of parameters of M. The following statements are equivalent:*

i) *M is a sequentially Cohen-Macaulay module.*
ii) *$(x_1, \dots, x_{d_i})$ is a regular sequence on M/D_{i-1} for $i = 1, \dots, t$.*
iii) *$(x_1^{n_1}, \dots, x_i^{n_i})M : x_{i+1}^{n_{i+1}} = (x_1^{n_1}, \dots, x_i^{n_i})M + 0 :_M x_{i+1}$ for all positive integers $n_1, \dots, n_d$ and $i = 0, 1, \dots, d-1$.*
iv) *$(x_1, \dots, x_i)M : x_j^2 = (x_1, \dots, x_i)M + 0 :_M x_j$, for all $0 \leqslant i < j \leqslant d$.*
v) depth $M/D_{i-1} = d_i$ *for* $i = 1, \dots, t$.

Proof. i) ⇒ ii) and i) ⇒ v) are proved in Proposition 3.4.

ii) ⇒ iii): Assume that $(x_1, \dots, x_{d_s})$ is a regular sequence on M/D_{s-1}, $s = 1, \dots, t$ and $d_{s-1} \leqslant i < d_s$. By Lemma 2.4 we have $D_{s-1} = 0 :_M x_{i+1}$. Then

$$(x_1^{n_1}, \dots, x_i^{n_i})(M/0 :_M x_{i+1}) : x_{i+1}^{n_{i+1}} = (x_1^{n_1}, \dots, x_i^{n_i})(M/0 :_M x_{i+1}),$$

for all positive integers $n_1, \dots, n_d$. Thus

$$[(x_1^{n_1}, \dots, x_i^{n_i})M + 0 :_M x_{i+1}] : x_{i+1}^{n_{i+1}} = (x_1^{n_1}, \dots, x_i^{n_i})M + 0 :_M x_{i+1}.$$

Then by the fact

$$\begin{aligned}[(x_1^{n_1},\dots,x_i^{n_i})M+D_{s-1}]:x_{i+1}^{n_{i+1}} &\supseteq (x_1^{n_1},\dots,x_i^{n_i})M:x_{i+1}^{n_{i+1}}\\ &\supseteq (x_1^{n_1},\dots,x_i^{n_i})M+0:_M x_{i+1}\end{aligned}$$

we obtain

$$(x_1^{n_1},\dots,x_i^{n_i})M:x_{i+1}^{n_{i+1}}=(x_1^{n_1},\dots,x_i^{n_i})M+0:_M x_{i+1}.$$

iii) $\Rightarrow$ iv): For $0\leqslant i<j\leqslant d$, by taking $n_j=2$ and using Krull's Intersection Theorem we have

$$\begin{aligned}(x_1,\dots,x_i)M:x_j^2 &= \bigcap_{n_{i+1},\dots,n_{j-1}} (x_1,\dots,x_i,x_{i+1}^{n_{i+1}},\dots,x_{j-1}^{n_{j-1}})M:x_j^2\\ &= \bigcap_{n_{i+1},\dots,n_{j-1}} [(x_1,\dots,x_i,x_{i+1}^{n_{i+1}},\dots,x_{j-1}^{n_{j-1}})M+0:_M x_j]\\ &= (x_1,\dots,x_i)M+0:_M x_j.\end{aligned}$$

iv) $\Rightarrow$ ii): It suffices to prove that $[(x_1,\dots,x_i)M+D_s]:x_{i+1}=(x_1,\dots,x_i)M+D_s$ for $s=0,1,\dots,t-1$, $i<d_{s+1}$. We have $(x_1,\dots,x_i)M+0:_M x_{i+1}\subseteq (x_1,\dots,x_i)M:x_{i+1}\subseteq (x_1,\dots,x_i)M:x_{i+1}^2$. Hence

$$(x_1,\dots,x_i)M:x_{i+1}=(x_1,\dots,x_i)M+0:_M x_{i+1}.$$

By Lemma 2.4, $D_s=0:_M x_{d_{s+1}}\supseteq 0:_M x_{i+1}$. Therefore,

$$\begin{aligned}[(x_1,\dots,x_i)M+D_s]:x_{i+1} &\subseteq (x_1,\dots,x_i)M:x_{d_{s+1}}x_{i+1}\\ &= [(x_1,\dots,x_i)M+0:_M x_{i+1}]:x_{d_{s+1}}\\ &\subseteq (x_1,\dots,x_i)M:_M x_{d_{s+1}}^2\\ &= (x_1,\dots,x_i)M+D_s.\end{aligned}$$

ii) $\Rightarrow$ i) and v) $\Rightarrow$ i): Both (ii) and (v) imply that depth $M/D_{i-1}\geqslant d_i$. Then the assertion follows by the short exact sequence

$$0\to D_i/D_{i-1}\to M/D_{i-1}\to M/D_i\to 0. \qquad \square$$

It is well-known that $\operatorname{depth}(M/D_{i-1})=\min\{j: H^j_{\mathfrak m}(M/D_{i-1})\neq 0\}$. Then the following result is an immediate consequence of Theorem 3.9.

COROLLARY 3.10. *M is a sequentially Cohen-Macaulay module if and only if $H^j_{\mathfrak m}(M/D_{i-1})=0$ for all $i=1,\dots,t$ and $j<d_i$.*

4. Parametric characterizations

In this section, we shall give answers for the questions raised in the first section. We first begin with an auxiliary lemma.

LEMMA 4.1. *Let $\mathscr{F} : M_0 \subset M_1 \subset \cdots \subset M_t = M$ be a filtration satisfying the dimension condition and $\underline{x} = (x_1, \ldots, x_d)$ a good system of parameters with respect to $\mathscr{F}$. Assume that $I_{\mathscr{F},M}(\underline{x}(\underline{n})) = 0$ for all positive integers $n_1, \ldots, n_d$. Then $(x_1, \ldots, x_i)M : x_j^2 = (x_1, \ldots, x_i)M + 0 :_M x_j$ for all $0 \leqslant i < j \leqslant d$.*

Proof. The lemma is proved by induction on d. The case $d = 1$ is trivial. Let $d \geqslant 2$. By Lemma 3.3, $\underline{x}$ is a dd-sequence on M, thus $(x_1, \ldots, x_i)M : x_j^2 = (x_1, \ldots, x_i)M : x_j$ for all $0 \leqslant i < j \leqslant d$. It is sufficient to prove that

$$(x_1, \ldots, x_i)M : x_j = (x_1, \ldots, x_i)M + 0 :_M x_j$$

for all $1 \leqslant i < j \leqslant d$. Assume first that $\dim M_1 > 1$. For any positive integer n_1 we consider the following filtration

$$\frac{\mathscr{F}}{x_1^{n_1}\mathscr{F}} : \frac{M_0 + x_1^{n_1}M}{x_1^{n_1}M} \subset \frac{M_1 + x_1^{n_1}M}{x_1^{n_1}M} \subset \cdots \subset \frac{M_{t-1} + x_1^{n_1}M}{x_1^{n_1}M} \subset \frac{M}{x_1^{n_1}M}$$

where $\dim(M_i + x_1^{n_1}M)/x_1^{n_1}M = d_i - 1$ for all $i > 0$. Thus the filtration $\mathscr{F}/x_1^{n_1}\mathscr{F}$ satisfies the dimension condition and it can be proved easily that $(x_2, \ldots, x_d)$ is a good system of parameters of $M/x_1^{n_1}M$ with respect to $\mathscr{F}/x_1^{n_1}\mathscr{F}$. Moreover, by the same argument as in the proof of Lemma 2.7, we get that $I_{\mathscr{F}/x_1^{n_1}\mathscr{F}, M/x_1^{n_1}M}(x_2^{n_2}, \ldots, x_d^{n_d}) \leqslant I_{\mathscr{F},M}(\underline{x}(\underline{n}))$, which implies $I_{\mathscr{F}/x_1^{n_1}\mathscr{F}, M/x_1^{n_1}M}(x_2^{n_2}, \ldots, x_d^{n_d}) = 0$ for all positive integers $n_1, \ldots, n_d$. By the induction assumption we have for all $1 \leqslant i < j \leqslant d$,

$$(x_2, \ldots, x_i)(M/x_1^{n_1}M) : x_j = (x_2, \ldots, x_i)(M/x_1^{n_1}M) + x_1^{n_1}M : x_j/x_1^{n_1}M,$$

thus

$$(x_1^{n_1}, x_2, \ldots, x_i)M : x_j = (x_1^{n_1}, x_2, \ldots, x_i)M + x_1^{n_1}M : x_j.$$

Take $n_1 = 1$, it remains to prove that $x_1M : x_j = x_1M + 0 :_M x_j$. Since $\dim M_1 \geqslant 2$, we can permute x_1 and x_2 in the sequence $x_1, x_2, \ldots, x_d$ without loss of generality. Hence we have

$$(x_2^{n_2}, x_1, x_3, \ldots, x_i)M : x_j = (x_2^{n_2}, x_1, x_3, \ldots, x_i)M + x_2^{n_2}M : x_j.$$

Let $i = 1$. By applying Krull Intersection Theorem we have

$$x_1M : x_j = \bigcap_{n_2}(x_1, x_2^{n_2})M : x_j = \bigcap_{n_2}((x_1, x_2^{n_2})M + x_2^{n_2}M : x_j) = x_1M + 0 :_M x_j.$$

Now assume that $\dim M_1 = 1$. Denote $N = 0 :_M x_2$, then $M_1 \subseteq N$ and $\dim N = 1$. Put $\overline{M} = M/N$. We have an exact sequence

$$N/\underline{x}(\underline{n}) \xrightarrow{\varphi} M/\underline{x}(\underline{n})M \to \overline{M}/\underline{x}(\underline{n})\overline{M} \to 0,$$

where $\operatorname{Ker}\varphi = N \cap \underline{x}(\underline{n})M = x_1^{n_1}N$ by Lemma 3.6. Hence

$$\ell(M/\underline{x}(\underline{n})M) = \ell(N/x_1^{n_1}N) + \ell(\overline{M}/\underline{x}(\underline{n})\overline{M}).$$

The module N has a filtration satisfying the dimension condition $\mathcal{F}_1 : M_0 \subset N$ and the module $\overline{M}$ has a filtration satisfying the dimension condition

$$\mathcal{F}_2 : 0 \subset (M_2 + N)/N \subset (M_3 + N)/N \subset \cdots \subset (M_{t-1} + N)/N \subset M/N.$$

Since $\dim N = 1$, $\dim(M_i + N)/N = d_i$ and $e(x_1, \ldots, x_{d_i}; M_i + N/N) = e(x_1, \ldots, x_{d_i}; M_i)$ for all $i > 1$. Hence

$$I_{\mathcal{F}_2, \overline{M}}(\underline{x}(\underline{n})) + I_{\mathcal{F}_1, N}(x_1^{n_1}) \leqslant I_{\mathcal{F}, M}(\underline{x}(\underline{n})) = 0$$

which implies $I_{\mathcal{F}_2, \overline{M}}(\underline{x}(\underline{n})) = 0$. Note that $\dim(M_2 + N)/N \geqslant 2$, then by applying the first part of the proof to the module $\overline{M}$ and the filtration $\mathcal{F}_2$ we have

$$(x_1, \ldots, x_i)\overline{M} : x_j = (x_1, \ldots, x_i)\overline{M} + 0 :_{\overline{M}} x_j,$$

for all $1 \leqslant i < j \leqslant d$. Thus

$$[(x_1, \ldots, x_i)M + N] : x_j = (x_1, \ldots, x_i)M + N :_M x_j.$$

Keep in mind that $\underline{x}$ is a dd-sequence, then $N :_M x_j = 0 :_M x_2 x_j = 0 :_M x_j$. By the fact that

$$[(x_1, \ldots, x_i)M + N] : x_j \supseteq (x_1, \ldots, x_i)M : x_j \supseteq (x_1, \ldots, x_i)M + 0 :_M x_j,$$

we obtain

$$(x_1, \ldots, x_i)M : x_j = (x_1, \ldots, x_i)M + 0 :_M x_j$$

for all $1 \leqslant i < j \leqslant d$ as required. □

The following result is a complete answer for the first question mentioned in the introduction.

THEOREM 4.2. *Let M be a finitely generated R-module of dimension d and $\mathcal{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M$ the dimension filtration of M. The following statements are equivalent*:

i) *M is a sequentially Cohen-Macaulay module.*
ii) *$I_{\mathcal{D}, M}(\underline{x}) = 0$ for all good systems of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M.*
iii) *There exists a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M such that $\underline{x}$ is a dd-sequence on M and $I_{\mathcal{D}, M}(\underline{x}) = 0$.*
iv) *There exists a good system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M such that $I_{\mathcal{D}, M}(\underline{x}(\underline{n})) = 0$ for all positive integers $n_1, \ldots, n_d$.*

Proof. (i $\Rightarrow$ ii) follows from Proposition 3.4.

(ii $\Rightarrow$ iii): Let $\underline{x} = (x_1, \ldots, x_d)$ be a good system of parameters of M. By Remark 2.3, (iii), $\underline{x}(\underline{n})$ is also a good system of parameters for all positive integers $n_1, \ldots, n_d$. From the assumption, $I_{\mathcal{D}, M}(\underline{x}(\underline{n})) = 0$. Hence, $\underline{x}$ is a dd-sequence on M by Lemma 3.3.

(iv $\Rightarrow$ i) follows from Theorem 3.9 and Lemma 4.1.

(iii $\Rightarrow$ iv): Assume that $\underline{x}$ is a dd-sequence and $I_{\mathcal{D}, M}(\underline{x}) = 0$. By Lemma 3.3 we have for all positive integers $n_1, \ldots, n_d$,

$$\ell(M/\underline{x}(\underline{n})M) = \sum_{i=0}^{d} a_i n_1 \cdots n_i,$$

where $a_i = e(x_1, \ldots, x_i; (x_{i+2}, \ldots, x_d)M : x_{i+1}/(x_{i+2}, \ldots, x_d)M)$. Hence

$$I_{\mathscr{D},M}(\underline{x}(\underline{n})) = \sum_{i=0}^{d} b_i n_1 \cdots n_i$$

where $b_i = a_i - e(x_1, \ldots, x_i; D_j)$ if $i = \dim D_j$ for some j and $b_i = a_i$ otherwise. We first claim that $b_i \geqslant 0$ for $i = 0, 1, \ldots, d$. It suffices to treat the case $i = \dim D_j$ for some j. Since $\underline{x}$ is a good system of parameters, $D_j = 0 :_M x_{i+1}$ by Lemma 2.4 and

$$D_j \cong (x_{i+1}, \ldots, x_d)M + D_j/(x_{i+1}, \ldots, x_d)M.$$

This implies

$$\begin{aligned} b_i &= e(x_1, \ldots, x_i; (x_{i+2}, \ldots, x_d)M : x_{i+1}/(x_{i+2}, \ldots, x_d)M) \\ &\quad - e(x_1, \ldots, x_i; (x_{i+2}, \ldots, x_d)M + 0 :_M x_{i+1}/(x_{i+2}, \ldots, x_d)M) \\ &= e(x_1, \ldots, x_i; (x_{i+2}, \ldots, x_d)M : x_{i+1}/(x_{i+2}, \ldots, x_d)M + 0 :_M x_{i+1}) \end{aligned}$$

which is non-negative. This proves the claim.

We have $I_{\mathscr{D},M}(\underline{x}) = \sum_{i=0}^{d} b_i = 0$. Hence $b_i = 0$ for all $i = 0, 1, \ldots, d$. Therefore $I_{\mathscr{D},M}(\underline{x}(\underline{n})) = 0$ for all positive integers $n_1, \ldots, n_d$. □

The next theorem is a finite criterion for the sequentially Cohen-Macaulay property, which gives a partial answer of our second question for a more general situation as follows.

THEOREM 4.3. *Let $\underline{x} = (x_1, \ldots, x_d)$ be a good system of parameters of M. Then M is a sequentially Cohen-Macaulay module if and only if there is a filtration $\mathscr{F}$ satisfying the dimension condition such that $I_{\mathscr{F},M}(x_1^2, \ldots, x_d^2) = 0$.*

Proof. The sufficient condition is proved by virtue of Lemma 2.5 and Proposition 3.4. We prove the necessary condition. By Remark 2.3, (ii) and Lemmas 2.7, 2.9, we have $I_{\mathscr{F},M}(x_1^2, \ldots, x_d^2) \geqslant I_{\mathscr{F},M}(\underline{x}(\underline{n})) \geqslant I_{\mathscr{D},M}(\underline{x}) \geqslant 0$ for all $n_1, \ldots, n_d \in \{1, 2\}$. Hence $I_{\mathscr{F},M}(\underline{x}(\underline{n})) = I_{\mathscr{D},M}(\underline{x}) = 0$ for all $n_1, \ldots, n_d \in \{1, 2\}$. By Theorem 4.2, we need only to prove that if $I_{\mathscr{F},M}(x_1^2, \ldots, x_d^2) = 0$ then $\underline{x}$ is a dd-sequence on M. First we prove that $I_{\mathscr{F},M}(\underline{x}(\underline{n})) = 0$ for all $n_1, \ldots, n_{d-1} \in \{1, 2\}$ and arbitrary positive integer n_d. In fact, applying Corollary 4.3 of [1] to the sequence $(x_d^{n_d}, x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}})$ we have

$$\begin{aligned} \ell(M/\underline{x}(\underline{n})M) - e(\underline{x}(\underline{n}); M) = n_d \sum_{i=0}^{d-2} e(x_d, x_1^{n_1}, \ldots, x_i^{n_i}; (0 : x_{i+1}^{n_{i+1}})_{M/(x_{i+2}^{n_{i+2}}, \ldots, x_{d-1}^{n_{d-1}})M}) \\ + \ell((0 : x_d^{n_d})_{M/(x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}})M}), \end{aligned}$$

for all positive integers $n_1, \ldots, n_d$. Moreover, for $n_1, \ldots, n_d \in \{1, 2\}$,

$$\ell(M/\underline{x}(\underline{n})M) - e(\underline{x}(\underline{n}); M) = I_{\mathscr{F},M}(\underline{x}(\underline{n})) + \sum_{i=0}^{t-1} n_1 \cdots n_{d_i} e(x_1, \ldots, x_{d_i}; M_i),$$

which is independent of n_d. Hence

$$\sum_{i=0}^{d-2} e(x_d, x_1^{n_1}, \ldots, x_i^{n_i}; (0 : x_{i+1}^{n_{i+1}})_{M/(x_{i+2}^{n_{i+2}}, \ldots, x_{d-1}^{n_{d-1}})M}) = 0$$

and

$$(0 : x_d^2)_{M/(x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}})M} = (0 : x_d)_{M/(x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}})M}.$$

Thus for all $n_1, \ldots, n_{d-1} \in \{1, 2\}$, $n_d \geqslant 1$,

$$\begin{aligned}\ell(M/\underline{x}(\underline{n})M) - e(\underline{x}(\underline{n}); M) &= \ell((0 : x_d^{n_d})_{M/(x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}})M}) \\ &= \ell((0 : x_d)_{M/(x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}})M}) \\ &= \sum_{i=0}^{t-1} n_1 \cdots n_{d_i} e(x_1, \ldots, x_{d_i}; M_i).\end{aligned}$$

Therefore, $I_{\mathscr{F},M}(\underline{x}(\underline{n})) = 0$ for all $n_1, \ldots, n_{d-1} \in \{1, 2\}$ and all $n_d \geqslant 1$. Next, we prove the statement by induction on the dimension of M. The case $d = 1$ is done by the proof above. Assume $d > 1$. For an arbitrary positive integer n_d we denote

$$\frac{\mathscr{F}}{x_d^{n_d}\mathscr{F}} : \frac{M_0 + x_d^{n_d}M}{x_d^{n_d}M} \subset \frac{M_1 + x_d^{n_d}M}{x_d^{n_d}M} \subset \cdots \subset \frac{M_s + x_d^{n_d}M}{x_d^{n_d}M} \subset \frac{M}{x_d^{n_d}M},$$

where $s = t - 1$ if $d_{t-1} < d - 1$ and $s = t - 2$ if $d_{t-1} = d - 1$. It follows from Corollary 2.8 and the hypothesis that $I_{\mathscr{F}/x_d^{n_d}\mathscr{F}, M/x_d^{n_d}M}(x_1^2, \ldots, x_{d-1}^2) = 0$. Hence, by the induction assumption, $I_{\mathscr{F}/x_d^{n_d}\mathscr{F}, M/x_d^{n_d}M}(x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}}) = 0$, for all positive integers $n_1, \ldots, n_{d-1}$. Therefore

$$\begin{aligned}\ell(M/\underline{x}(\underline{n})M) &= n_1 \cdots n_{d-1} e(x_1, \ldots, x_{d-1}; M/x_d^{n_d}M) \\ &\quad + \sum_{i=0}^{s} n_1 \cdots n_{d_i} e(x_1, \ldots, x_{d_i}; (M_i + x_d^{n_d}M)/x_d^{n_d}M) \\ &= n_1 \cdots n_d e(\underline{x}; M) + n_1 \cdots n_{d-1} e(x_1, \ldots, x_{d-1}; 0 :_M x_d) \\ &\quad + \sum_{i=0}^{s} n_1 \cdots n_{d_i} e(x_1, \ldots, x_{d_i}; M_i)\end{aligned}$$

for all positive integers $n_1, \ldots, n_d$. Hence $\underline{x}$ is a dd-sequence on M by Lemma 3.3. □

In [5], Goto gave the notion of approximately Cohen-Macaulay rings. A local ring $(R, \mathfrak{m})$ is called approximately Cohen-Macaulay if R is not a Cohen-Macaulay ring and there exists an element $a \in \mathfrak{m}$ such that $R/a^n R$ is a Cohen-Macaulay ring of dimension $d-1$ for every $n > 0$. Similarly, we can define the notion of approximately Cohen-Macaulay modules.

Definition 4.4. A non Cohen-Macaulay module M is called an *approximately Cohen-Macaulay module* if there exists an element $a \in \mathfrak{m}$ such that $M/a^n M$ is Cohen-Macaulay of dimension $d-1$ for every $n > 0$.

Then, the following characterization of approximately Cohen-Macaulay modules are easily derived by Theorem 4.3. It should be mentioned that the equivalence of (i) and (ii) was proved in [5, Theorem 1] for local rings.

Proposition 4.5. *Let M be a non Cohen-Macaulay R-module of dimension d. The following statements are equivalent.*

i) *M is an approximately Cohen-Macaulay module.*
ii) *There exists an element $a \in \mathfrak{m}$ such that $0 :_M a = 0 :_M a^2$ and M/a^2M is a Cohen-Macaulay module of dimension $d-1$.*
iii) *M is a sequentially Cohen-Macaulay module with the dimension filtration $\mathscr{D} : 0 = D_0 \subset D_1 \subset D_2 = M$ where $\dim D_1 = d-1$.*

Proof. (i $\Rightarrow$ ii) is trivial.

(ii $\Rightarrow$ iii). Assume that M/a^2M is Cohen-Macaulay of dimension $d-1$. Then there exists a system of parameters $\underline{x} = (x_1, \ldots, x_{d-1}, x_d)$ of M such that $x_d = a$ and we have

$$\begin{aligned}\ell(M/(x_1^2, \ldots, x_d^2)M) &= e(x_1^2, \ldots, x_{d-1}^2; M/x_d^2 M) \\ &= 2^d e(\underline{x}; M) + 2^{d-1} e(x_1, \ldots, x_{d-1}; 0 :_M x_d).\end{aligned}$$

Since M is not Cohen-Macaulay, $e(x_1, \ldots, x_{d-1}; 0 :_M x_d) > 0$, so $\dim(0 :_M x_d) = d-1$. Therefore the filtration $\mathscr{F} : 0 \subset 0 :_M x_d \subset M$ satisfies the dimension condition. Since $0 :_M x_d = 0 :_M x_d^2$, it is easy to prove that $\underline{x}$ is a good system of parameters with respect to the filtration $\mathscr{F}$. Moreover, we have $I_{\mathscr{F},M}(x_1^2, \ldots, x_d^2) = 0$. Then M is a sequentially Cohen-Macaulay module by Theorem 4.3 and therefore $\mathscr{F}$ is the dimension filtration of M by Lemma 4.1.

(iii $\Rightarrow$ i). Assume that M is a sequentially Cohen-Macaulay module with the dimension filtration $\mathscr{D} : 0 = D_0 \subset D_1 \subset D_2 = M$ where $\dim D_1 = d-1$. Let $\underline{x}$ be a good system of parameters of M. By Proposition 3.4, $\underline{x}$ is a dd-sequence and $I_{\mathscr{D},M}(\underline{x}(\underline{n})) = 0$ for all positive integers $n_1, \ldots, n_d$. Hence $\ell(M/\underline{x}(\underline{n})M) = e(x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}}; M/x_d^{n_d}M)$, since $D_1 = 0 :_M x_d = 0 :_M x_d^{n_d}$. Therefore $M/x_d^{n_d}M$ is a Cohen-Macaulay module of dimension $d-1$ for all $n_d > 0$. $\square$

Remark 4.6. A filtration satisfying the dimension condition

$$\mathscr{F} : M_0 \subset M_1 \subset \cdots \subset M_t = M$$

is called a Cohen-Macaulay filtration if M_i/M_{i-1} is Cohen-Macaulay modules for all $i = 1, \ldots, t$. Then it was showed in [4] that if M admits a Cohen-Macaulay filtration $\mathscr{F}$, M is a sequentially Cohen-Macaulay module and $\mathscr{F}$ is just the dimension filtration of M. Here we want to clarify that there exists filtration satisfying the equivalent conditions of Theorem 4.3 which is not the dimension filtration of M. Let M be a sequentially Cohen-Macaulay module of positive depth and

$$\mathscr{D} : 0 = D_0 \subset D_1 \subset \cdots \subset D_t = M$$

the dimension filtration of M. Let $\underline{x} = (x_1, \ldots, x_d)$ be a good system of parameters of M. Then it is not difficult to check that the following filtration

$$\mathscr{F} : 0 = M_0 \subset M_1 = \underline{x}D_1 \subset \cdots \subset M_{t-1} = \underline{x}D_{t-1} \subset M$$

satisfies the equivalent conditions of Theorem 4.2 but it is not the dimension filtration of M.

It should be mentioned that an R-module M is Cohen-Macaulay if and only if there exists a system of parameters $\underline{x}$ of M such that $\ell(M/\underline{x}M) = e(\underline{x}; M)$. Then the second question in the introduction says that M is sequentially Cohen-Macaulay if there exists a good system of parameters $\underline{x}$ such that $I_{\mathscr{D},M}(\underline{x}) = 0$ where $\mathscr{D}$ is the dimension filtration of M. Unfortunately, the answer is negative. Therefore we can not delete the number 2 in the statement iv) of Theorem 4.3. Below we give two counterexamples for this question, where the first example is due to Goto [5, Remark 2.9].

Example 4.7. (1) Let $R = k[[x, y, z, w]]$ be the ring of formal power series over a field k and $P = (xw - yz, x^3 - z^2, w^2 - xy^2, zw - x^2y)$, $Q = (y^2, z, w)$. Put $M = R/P \cap Q$. Then the dimension filtration of M is $\mathscr{D} : 0 = D_0 \subset D_1 \subset D_2 = M$ where $D_1 = P/P \cap Q$, $\dim D_1 = 1$. Since $D_1 = 0 :_M w = 0 :_M w^2$, it is easy to see that $(x + y + z + w, w)$ is a good system of parameters of M. Moreover, by a simple computation we have

$$I_{\mathscr{D},M}((x + y + z + w)^{n_1}, w^{n_2}) = \begin{cases} 0 & \text{if } n_1 = n_2 = 1 \\ 1 & \text{otherwise.} \end{cases}$$

So M is not a sequentially Cohen-Macaulay module though $I_{\mathscr{D},M}(x + y + z + w, w) = 0$.

(2) Let $R = k[[x, y, z, w]]$ be the ring of formal power series over a field k and $P = (x, w) \cap (y, z)$, $Q = (x, y^2, z)$. Put $M = R/P \cap Q$. Then the dimension filtration of M is $\mathscr{D} : 0 = D_0 \subset D_1 \subset D_2 = M$ where $D_1 = P/P \cap Q$, $\dim D_1 = 1$. Since $D_1 = 0 :_M (x + y) = 0 :_M (x + y)^2$, it is easy to see that $(z + w, x + y)$ is a good system of parameters of M. Moreover, by a simple computation we have

$$I_{\mathscr{D},M}((z + w)^{n_1}, (x + y)^{n_2}) = \begin{cases} 0 & \text{if } n_2 = 1 \\ 1 & \text{if } n_2 \geqslant 2. \end{cases}$$

So M is not a sequentially Cohen-Macaulay module though $I_{\mathscr{D},M}(z+w, x+y) = 0$.

5. Stanley-Reisner rings

A simplicial complex Δ over n vertices $\{v_1, \dots, v_n\}$ is a collection of subsets of the set $\{v_1, \dots, v_n\}$ such that,

i) $\emptyset \in \Delta$.

ii) For all element $F \in \Delta$ and all subsets $F' \subseteq F$, $F' \in \Delta$.

Each element $F \in \Delta$ is called a face of Δ. Among the faces of Δ, the face F with the property that if $F \subseteq F'$ and $F' \in \Delta$ then $F = F'$ is called a facet of Δ. So a simplicial complex is defined completely if all its facets are given. For a set of n vertices $\{v_1, \dots, v_n\}$ we consider the polynomial ring $R = k[X_1, \dots, X_n]$ over a field k. Then Δ corresponds to an ideal I_Δ of R defined by a set of generators $\{X_{i_1} \cdots X_{i_s} : \{v_{i_1}, \dots, v_{i_s}\} \notin \Delta\}$. The Stanley-Reisner ring of Δ over k is defined by $k[\Delta] = R/I_\Delta$. For each face F of Δ we define $\dim F = \dim R/I_F - 1$ and the corresponding ideal I_F is a prime ideal generated by X_i's. It is well-known that I_Δ is a radical ideal and $I_\Delta = \bigcap_F I_F$, where F runs through over all the facets of Δ. Therefore, the dimension filtration

$$\mathscr{D} : D_0 \subset D_1 \subset \cdots \subset D_{t-1} \subset D_t = k[\Delta]$$

of $k[\Delta]$, $d_i = \dim D_i$, can be determined by $D_i = \bigcap_{\dim F+1 \geqslant d_{i+1}} I_F/I_\Delta$.

For more details on Stanley-Reisner rings, the readers can find in books of Bruns-Herzog [2] and Stanley [11].

Denote by λ_i the number of the facets of dimension $d_i - 1$ of the simplicial complex Δ. We derive from Theorem 4.2 the following criterion, which is very convenient for checking whether $k[\Delta]$ is a sequentially Cohen-Macaulay ring.

Proposition 5.1. *Let Δ be a simplicial complex of dimension $d-1$ on n vertices. Then the Stanley-Reisner ring $k[\Delta]$ is sequentially Cohen-Macaulay if and only if there exists a homogeneous good system of parameters $\underline{x}$ such that*

$$\ell(k[\Delta]/(x_1^2, \dots, x_d^2)k[\Delta]) = \sum_{i=1}^{t} 2^{d_i} \lambda_i \deg(x_1) \cdots \deg(x_{d_i}),$$

where $\deg(x_i)$ is the degree of x_i in the graded ring $k[\Delta]$.

Proof. Let $\underline{x}$ be a homogeneous good system of parameters of M. Then the proposition follows from Theorem 4.3 if we can show that

$$e(x_1, \dots, x_{d_i}; D_i) = \lambda_i \deg(x_1) \cdots \deg(x_{d_i}).$$

In fact, since $\dim(R/(\bigcap_{\dim F+1 \geqslant d_{i+1}} I_F + \bigcap_{\dim F < d_i} I_F)) < d_i$, and

$$D_i = \bigcap_{\dim F+1 \geqslant d_{i+1}} I_F/I_\Delta \simeq \left(\bigcap_{\dim F+1 \geqslant d_{i+1}} I_F + \bigcap_{\dim F < d_i} I_F \right) \Bigg/ \bigcap_{\dim F < d_i} I_F,$$

we obtain

$$e(x_1,\dots,x_{d_i};D_i)=e\left(x_1,\dots,x_{d_i};R\Big/\bigcap_{\dim F<d_i} I_F\right).$$

Moreover, by the association law of multiplicity, we have

$$e\left(x_1,\dots,x_{d_i};R\Big/\bigcap_{\dim F<d_i} I_F\right)=\sum_{\dim F=d_i-1} e(x_1,\dots,x_{d_i};R/I_F).$$

Since R/I_F is a regular ring, $e(x_1,\dots,x_{d_i};R/I_F)=\deg(x_1)\cdots\deg(x_{d_i})$; thus $e(x_1,\dots,x_{d_i};D_i)=\lambda_i\deg(x_1)\cdots\deg(x_{d_i})$. □

We consider following examples, in which the number of vertices is small.

Example 5.2. For $n\leqslant 3$, $k[\Delta]$ are always sequentially Cohen-Macaulay rings.

Proof. There are five cases (up to an isomorphism of $k[\Delta]$) as follows.

i) $n\leqslant 2$ is trivial.

ii) $k[\Delta]=R/(X_1X_2X_3)$: It is easy to see that $\underline{x}=(x_1=X_1+X_2,x_2=X_1+X_3)$ is a good system of parameters of $k[\Delta]$. By a simple computation we have $\ell(k[\Delta]/(x_1^2,x_2^2)k[\Delta])=12=2^2.3$; so $k[\Delta]$ is Cohen-Macaulay, since Δ has three facets of dimension 1.

iii) $k[\Delta]=R/(X_2X_3)$: Similarly to (ii) we can find a good system of parameters $\underline{x}=(x_1=X_1,x_2=X_2+X_3)$. Then $\ell(k[\Delta]/(x_1^2,x_2^2)k[\Delta])=8=2^2.2$. Since Δ has two facets of dimension 1, $k[\Delta]$ is Cohen-Macaulay.

iv) $k[\Delta]=R/(X_1X_2,X_2X_3,X_3X_1)$ is Cohen-Macaulay, since $\dim k[\Delta]=\operatorname{depth} k[\Delta]=1$.

v) $k[\Delta]=R/(X_1X_3,X_2X_3)$: The simplicial complex Δ has a facet of dimension 1 and a facet of dimension 0. There is a good system of parameters $\underline{x}=(x_1=X_2+X_3,x_2=X_1)$ of $k[\Delta]$ and we have $\ell(k[\Delta]/(x_1^2,x_2^2)k[\Delta])=6=2.2+2$. Therefore $k[\Delta]$ is sequentially Cohen-Macaulay but not Cohen-Macaulay. □

Example 5.3. For $n=4$, there is only one simplicial complex Δ for which the ring $k[\Delta]$ is not sequentially Cohen-Macaulay: Consider the simplicial complex Δ given by two facets $\{v_1,v_4\}$, $\{v_2,v_3\}$ of dimension 1. Then $k[\Delta]=R/(X_1X_2,X_1X_3,X_2X_4,X_3X_4)$. There is a good system of parameters $\underline{x}=(x_1=X_1+X_2,x_2=X_3+X_4)$ of $k[\Delta]$. We obtain by a simple computation that $\ell(k[\Delta]/(x_1^2,x_2^2)k[\Delta])=9=2^2.2+1$. So $k[\Delta]$ is not sequentially Cohen-Macaulay. Moreover, it holds $\ell(k[\Delta]/(x_1^{n_1},x_2^{n_2})k[\Delta])=2n_1n_2+1$, and therefore $k[\Delta]$ is a Buchsbaum ring. For all other simplicial complexes Δ, the ring $k[\Delta]$ are sequentially Cohen-Macaulay. Par example, Δ is given by the facets $\{X_1,X_2,X_3\}$, $\{X_1,X_4\}$, $\{X_2,X_4\}$, $\{X_3,X_4\}$. Then $k[\Delta]=R/(X_1X_2X_4,X_1X_3X_4,X_2X_3X_4)$ has a good system of parameters $\underline{x}=(x_1=X_3+X_4,x_2=X_1+X_2+X_3,x_3=X_1X_2)$. We have $\ell(k[\Delta]/(x_1^2,x_2^2,x_3^2)k[\Delta])=28=2^3.1.2+2^2.3$, since Δ has one facet of dimension 2 and three facets of dimension 1. Therefore $k[\Delta]$ is sequentially Cohen-Macaulay.

Example 5.4. For higher n, we can find many "bad" rings. Consider a simplicial complex Δ over five vertices given by the facets $\{v_1, v_2, v_3\}$, $\{v_1, v_4, v_5\}$. Then

$$k[\Delta] = R/(X_2X_4, X_2X_5, X_3X_4, X_3X_5) = R/(X_2, X_3) \cap (X_4, X_5).$$

The ring $k[\Delta]$ has a good system of parameters $\underline{x} = (x_1 = X_1; x_2 = X_2 + X_4; x_3 = X_3 + X_5)$. Moreover,

$$\ell(k[\Delta]/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3})k[\Delta]) = 2n_1n_2n_3 + n_1.$$

Since Δ has two facets of dimension 2, $k[\Delta]$ is not sequentially Cohen-Macaulay. Note here that, although $k[\Delta]/x_3k[\Delta]$ is a sequentially Cohen-Macaulay ring of dimension 2 and x_3 is a regular element of $k[\Delta]$, it does not imply the sequential Cohen-Macaulayness of $k[\Delta]$. This is one of different features between sequential Cohen-Macaulayness and Cohen-Macaulayness.

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