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EXISTENCE OF SUPERCRITICAL PASTING ARCS FOR TWO SHEETED SPHERES

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Abstract

Take e.g. two disjoint nondegenerate compact continua A and B in the complex plane $\mathbf{C}$ with connected complements and pick a simple arc γ in the complex sphere $\hat{\mathbf{C}}$ disjoint from $A \cup B$, which we call a pasting arc for A and B. Construct a covering Riemann surface $\hat{\mathbf{C}}_\gamma$ over $\hat{\mathbf{C}}$ by pasting two copies of $\hat{\mathbf{C}} \backslash \gamma$ crosswise along γ. We embed A in one sheet and B in another sheet of two sheets of $\hat{\mathbf{C}}_\gamma$ which are copies of $\hat{\mathbf{C}} \backslash \gamma$ so that $\hat{\mathbf{C}}_\gamma \backslash A \cup B$ is understood as being obtained by pasting $(\hat{\mathbf{C}} \backslash A) \backslash \gamma$ with $(\hat{\mathbf{C}} \backslash B) \backslash \gamma$ crosswise along γ. In the comparison of the variational 2 capacity $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B)$ of the compact set A considered in the open set $\hat{\mathbf{C}}_\gamma \backslash B$ with the corresponding $\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$, we say that the pasting arc γ for A and B is subcritical, critical, or supercritical according as $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B)$ is less than, equal to, or greater than $\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$, respectively. We have shown in our former paper [4] the existence of pasting arc γ of any one of the above three types but that of supercritical and critical type was only shown under the additional requirment on A and B that A and B are symmetric about a common straight line simultaneously. The purpose of the present paper is to show that in the above mentioned result the additional symmetry assumption is redundant: we will show the existence of supercritical and hence of critical arc γ starting from an arbitrarily given point in $\hat{\mathbf{C}} \backslash A \cup B$ for any general admissible pair of A and B without any further requirment whatsoever.

1. Introduction

A nonempty compact subset A of the complex plane $\mathbf{C}$ will be referred to as an *admissible* compact subset if $\hat{\mathbf{C}} \backslash A$ is a regular subregion of the complex sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, i.e. a relatively compact and connected open subset of $\hat{\mathbf{C}}$ whose relative boundary $\partial(\hat{\mathbf{C}} \backslash A)$ consists of a finite number of disjoint analytic Jordan curves. Thus an admissible A may or may not be connected and in general it consists of a finite number of connected components which themselves are also admissible. If B is another admissible compact subset of $\mathbf{C}$ disjoint from A, then $A \cup B$ is again admissible. For a pair of two disjoint admissible compact subsets

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A and B in $\mathbf{C}$, a simple arc γ in $\hat{\mathbf{C}}\backslash A\cup B$ will be referred to as a *pasting arc* for A and B since we will paste $(\hat{\mathbf{C}}\backslash A)\backslash\gamma$ with $(\hat{\mathbf{C}}\backslash B)\backslash\gamma$ crosswise along γ. In general consider two subregions R and S in $\hat{\mathbf{C}}$ and a simple arc γ in $R\cap S$. We will use (cf. [5]) the convenient notation $(R\backslash\gamma)\boxtimes_\gamma(S\backslash\gamma)$ for the Riemann surface obtained from R and S by pasting $R\backslash\gamma$ with $S\backslash\gamma$ crosswise along γ. For a pair of two disjoint admissible compact subsets A and B in $\mathbf{C}$ and a pasting arc γ for A and B we will consider a new Riemann surface

$$\hat{\mathbf{C}}_\gamma := (\hat{\mathbf{C}}\backslash\gamma)\boxtimes_\gamma(\hat{\mathbf{C}}\backslash\gamma)$$

and also its subsurface

$$W_\gamma := \hat{\mathbf{C}}_\gamma\backslash A\cup B, \tag{1.1}$$

where we understand that A (B, resp.) is embedded in the upper (lower, resp.) sheet $\hat{\mathbf{C}}\backslash\gamma$ of $\hat{\mathbf{C}}_\gamma$ although A and B are originally contained in the same $\mathbf{C}$. Hence

$$W_\gamma = ((\hat{\mathbf{C}}\backslash A)\backslash\gamma)\boxtimes_\gamma((\hat{\mathbf{C}}\backslash B)\backslash\gamma). \tag{1.2}$$

Here $\hat{\mathbf{C}}_\gamma$ is a covering Riemann surface $(\hat{\mathbf{C}}_\gamma,\hat{\mathbf{C}},\pi_\gamma)$ of the base surface $\hat{\mathbf{C}}$ with the natural projection π_γ.

Consider next the capacity $\mathrm{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B)$, or more precisely the variational 2 capacity (cf. e.g. [2]), of the compact subset A in $\hat{\mathbf{C}}_\gamma$ with respect to the open subset $\hat{\mathbf{C}}_\gamma\backslash B$ of $\hat{\mathbf{C}}_\gamma$ containing A given by

$$\mathrm{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B) := \inf_\varphi D(\varphi; W_\gamma), \tag{1.3}$$

where φ in taking the infimum in (1.3) runs over the family of $\varphi\in C(\hat{\mathbf{C}}_\gamma)\cap C^\infty(W_\gamma)$ with $\varphi|A=1$ and $\varphi|B=0$ and $D(\varphi; W_\gamma)$ indicates the Dirichlet integral of φ over W_γ defined by

$$D(\varphi; W_\gamma) := \int_{W_\gamma} d\varphi\wedge *d\varphi = \int_{W_\gamma}|\nabla\varphi(z)|^2\,dxdy.$$

Here the second term in the above is the coordinate free expression of $D(\varphi, W_\gamma)$ and the third term is the expression of $D(\varphi, W_\gamma)$ in terms of local parameters $z=x+iy$ for W_γ and $\nabla\varphi(z)$ is the gradient vector $(\partial\varphi(z)/\partial x, \partial\varphi(z)/\partial y)$. Clealy we have the following symmetry:

$$\mathrm{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B) = \mathrm{cap}(B,\hat{\mathbf{C}}_\gamma\backslash A).$$

The variation (1.3) has the unique extremal function u_γ:

$$\mathrm{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B) = D(u_\gamma; W_\gamma), \tag{1.4}$$

characterized by the conditions $u_\gamma\in C(\hat{\mathbf{C}}_\gamma)\cap H(W_\gamma)$ with $u_\gamma|A=1$ and $u_\gamma|B=0$ (cf. e.g. [2]), where $H(X)$ denotes the class of harmonic functions defined on an open subset X of a Riemann surface so that the function $u_\gamma|W_\gamma$ is usually referred to as the *harmonic measure* of $A\cap\partial W_\gamma$ (cf. e.g. [8]). The extremal function u_γ for (1.3) is also referred to as the *capacity function* for the compact subset A with respect to $\hat{\mathbf{C}}_\gamma\backslash B$ (cf. [7]).

We also consider the capacity $\mathrm{cap}(A, \hat{\mathbf{C}} \backslash B)$ of the subset A in $\hat{\mathbf{C}}$ contained in the open subset $\hat{\mathbf{C}} \backslash B$. Similarly as in the case of $\mathrm{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B)$ we have the symmetry $\mathrm{cap}(A, \hat{\mathbf{C}} \backslash B) = \mathrm{cap}(B, \hat{\mathbf{C}} \backslash A)$ and that the capacity $\mathrm{cap}(A, \hat{\mathbf{C}} \backslash B)$ is given by the capacity function u for the compact subset A with respect to $\hat{\mathbf{C}} \backslash B$:

$$\mathrm{cap}(A, \hat{\mathbf{C}} \backslash B) = D(u; W) \quad (W := \hat{\mathbf{C}} \backslash A \cup B),$$

where $u \in C(\hat{\mathbf{C}}) \cap H(W)$ with $u|A = 1$ and $u|B = 0$. Motivated by the problem to clarify when the situation $\mathrm{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B) \leq \mathrm{cap}(A, \hat{\mathbf{C}} \backslash B)$ holds, which occured in the study of the classical and modern type problem (cf. e.g. [6], [10], [8], [5], [3], among many others), the following classification problem of pasting arcs started: since the occurence of

$$\mathrm{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B) = \mathrm{cap}(A, \hat{\mathbf{C}} \backslash B) \tag{1.5}$$

is very delicate in the sense that the relation is easily destroyed even if we change γ slightly, the pasting arc γ for $\hat{\mathbf{C}}_\gamma$ for which we have (1.5) is referred to as being *critical*. In contrast the situation

$$\mathrm{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B) < \mathrm{cap}(A, \hat{\mathbf{C}} \backslash B) \tag{1.6}$$

and also the situation

$$\mathrm{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B) > \mathrm{cap}(A, \hat{\mathbf{C}} \backslash B) \tag{1.7}$$

are quite stable with respect to the small perturbation of γ and the pasting arc γ for which the relation (1.6) ((1.7), resp.) holds is referred to as being *subcritical* (*supercritical*, resp.). The occurence for a pasting arc γ to be subcritical is just very common. For example, if the daiameter of γ is sufficiently small, then γ is subcritical (cf. [4]). In view of this it was expected in one time that every pasting arc γ satisfies (1.5) or (1.6) and there is no γ for which (1.7) is valid. We found, however, (1.7) can really occur in our former paper [4] when the pair of disjoint admissible compact subsets A and B are symmetric with respect to a common straight line. The *purpose* of the present paper is to show that without any additional condition the above is correct: for any pair of disjoint admissible compact subsets A and B in $\mathbf{C}$ there always exists a supercritical pasting arc γ for A and B. Hence we have

THEOREM. *For any pair of disjoint admissible compact subsets A and B of $\mathbf{C}$, there always exist pasting arcs γ_1, γ_2, and γ_3 in $\mathbf{C}$ for A and B starting from an arbitrarily given nonsingular point a in $\mathbf{C} \backslash A \cup B$ of the gradient of the capacity function on $\hat{\mathbf{C}}$ for A and B such that γ_1 is critical, γ_2 is subcritical, and γ_3 is supercritical.*

Not only the mere existence but also the criteria for a given pasting arc γ to be subcrical are discussed in detail in [4]. For a pasting arc γ starting from a point $a \in \mathbf{C}$, we denote by γ_z for any $z \in \gamma \backslash \{a\}$ the subarc of γ starting from a and terminating at z. We have also shown in [4] that if γ is supercritical, then there are points s and c in $\gamma \backslash \{a\}$ such that γ_s (γ_c, resp.) is subcritical (critical, resp.). Hence to complete the proof of the above theorem, we only have to show the existence of a supercritical arc, which is the actual work in this paper.

2. Proof of the theorem

As mentioned in the introduction we only have to prove the existence of a supercritical pasting arc γ for an arbitrarily given general pair of disjoint admissible compact subsets A and B in the complex plane $\mathbf{C}$. We set

$$W := \hat{\mathbf{C}} \backslash (A \cup B). \tag{2.1}$$

We denote by u the capacity function on $\hat{\mathbf{C}}$ for the capacity of A with respect to $\hat{\mathbf{C}} \backslash B$:

$$\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B) = D(u; \hat{\mathbf{C}}) \tag{2.2}$$

so that $u \in C(\hat{\mathbf{C}}) \cap H(\hat{\mathbf{C}} \backslash A \cup B)$, $u|A = 1$, and $u|B = 0$. Hence $u|W$ is the harmonic measure of $A \cap \partial W$ on W. Choose an arbitrary but then fixed nonsingular point $a \in \hat{\mathbf{C}} \backslash A \cup B$ of the gradient vector field of $u : du(a) \neq 0$. There is an arc l containing a as its interior point, *on* which $du \neq 0$, such that

$$*du = 0 \tag{2.3}$$

along l, i.e. l is a u conjugate level arc. We pick an arbitrary interior point b in l other than a such that $u(z)$ decreases as z traces l from a to b. We take an arbitrary but then fixed smooth Jordan curve σ encircling B and intersecting with l only once at b. We give a negative direction to σ. We denote by (σ) the region bounded by σ. Then $\bar{B} \subset (\sigma)$ and a is an interior point in the arc $l \backslash (\sigma)$. Then consider the subarc τ of l whose initial point is a and the terminal point is b. Thus τ is a u conjugate level arc with the positive direction starting from a and ending at b. In general we denote by $|\gamma|$ the length of an arc γ measured by the plane metric. For each $t \in (0, |\sigma|/100)$ we pick the point $c(t) \in \sigma$ such that the subarc σ_t' of σ starting from $c(t)$ and ending at b in the direction of σ satisfies $|\sigma_t'| = t$. We then put $\sigma_t := \overline{\sigma \backslash \sigma_t'}$, the subarc of σ starting from b and ending at $c(t)$ in the direction of σ. Finally we consider the arc

$$\gamma_t := \tau + \sigma_t \quad (0 < t < |\sigma|/100). \tag{2.4}$$

Here 100 in (2.4) has no particular meaning other than suggesting the point $c(t)$ is situated enough close to the point b since we are making $t \downarrow 0$ later.

We will show that γ_t in (2.4) is a supercritical arc if we choose $t \in (0, |\sigma|/100)$ sufficiently small:

$$\operatorname{cap}(A, \hat{\mathbf{C}}_{\gamma_t} \backslash B) > \operatorname{cap}(A, \hat{\mathbf{C}} \backslash B) \tag{2.5}$$

for sufficiently small $t \in (0, |\sigma|/100)$. For simplicity we set

$$W_t := W_{\gamma_t} = \hat{\mathbf{C}}_{\gamma_t} \backslash A \cup B = ((\hat{\mathbf{C}} \backslash A) \backslash \gamma_t) \bigotimes_{\gamma_t} ((\hat{\mathbf{C}} \backslash B) \backslash \gamma_t) \tag{2.6}$$

and also we denote by $u_t := u_{\gamma_t}$, the capacity function on $\hat{\mathbf{C}}_{\gamma_t}$ for $\operatorname{cap}(A, \hat{\mathbf{C}}_{\gamma_t} \backslash B)$ so that $u_t \in C(\hat{\mathbf{C}}_{\gamma_t}) \cap H(W_t)$ with $u_t|A = 1$ and $u_t|B = 0$. We also consider an auxiliary surface

$$W_0 := ((\hat{\mathbf{C}} \backslash A \cup B) \backslash \tau) \bigotimes_{\tau} (\hat{\mathbf{C}} \backslash \tau). \tag{2.7}$$

We denote by δ_t (δ_{0t}, resp.) the part of W_t (W_0, resp.) lying over $\sigma_t' = \overline{\sigma \backslash \sigma_t}$, which consists of two copies of σ_t' situated in each of two sheets of $\hat{\mathbf{C}}_{\gamma_t}$ ($\hat{\mathbf{C}}_\tau$, resp.). Finally we put $W_t' = W_t \backslash \delta_t$ and observe the following two and, especially the second, crucial relations in our proof:

$$W_t' \subset W_s' \quad (0 < s \leq t < |\sigma|/100) \tag{2.8}$$

and

$$W_t' = W_0 \backslash \delta_{0t} \quad (0 < t < |\sigma|/100). \tag{2.9}$$

The function u_t, originally defined on W_t so that on W_t', may also be considered as being defined on $W_0 \backslash \delta_{0t}$ by (2.9) but its boundary values at δ_{0t} must be considered in the sense of Carathéodory, i.e. a single point in δ_{0t} is considered as two boundary elements in the Carathéodory compactification of $W_0 \backslash \delta_{0t}$ (cf. [10]). Let w_t be the function on $\hat{\mathbf{C}}_\tau$ such that $w_t \in C(\hat{\mathbf{C}}_\tau) \cap H(W_0 \backslash \delta_{0t})$ with $w_t | A = w_t | B = 0$ and $w_t | \delta_{0t} = 1$. By comparing boundary values we see that $0 \leq w_s \leq w_t \leq 1$ $(0 < s \leq t)$ on $W_0 \backslash \delta_{0t}$. Hence $(w_t)_{t \downarrow 0}$ converges to a function $w \in C(\hat{\mathbf{C}}_\tau \backslash \{\tilde{b}\}) \cap H(W_0 \backslash \{\tilde{b}\})$ with $0 \leq w \leq 1$ on $\hat{\mathbf{C}}_\tau \backslash \{\tilde{b}\}$ and $w | A = w | B = 0$ almost uniformly on $\hat{\mathbf{C}}_\tau \backslash \{\tilde{b}\}$, where $\tilde{b}$ is the branch point of $\hat{\mathbf{C}}_\tau$ lying over b. By the Riemann removability theorem we see that $w \in H(W_0)$ with boundary values 0 so that $w = 0$ and a fortiori

$$\lim_{t \downarrow 0} w_t = 0 \tag{2.10}$$

almost uniformly on $\hat{\mathbf{C}}_\tau \backslash \{\tilde{b}\}$. Clearly, by comparing the boundary values, we see that

$$|u_t - u_s| \leq w_t \quad (0 < s \leq t)$$

on $\overline{W}_0 \backslash \delta_{0t}$ and hence on $\hat{\mathbf{C}}_\tau \backslash \delta_{0t}$. Therefore, by (2.10), we see that $(u_t)_{t \downarrow 0}$ converges to a function $v \in C(\hat{\mathbf{C}}_\tau \backslash \{\tilde{b}\}) \cap H(W_0 \backslash \{\tilde{b}\})$ almost uniformly on $\hat{\mathbf{C}}_\tau \backslash \{\tilde{b}\}$ such that $v | A = 1$ and $v | B = 0$ and $0 \leq v \leq 1$ on $\hat{\mathbf{C}}_\tau \backslash \{\tilde{b}\}$. Again by the Riemann removability theorem, $v \in H(W_0)$ and thus of course $v \in (\hat{\mathbf{C}}_\tau) \cap H(W_0)$.

Set $\alpha := A \cap \partial W_0$ and $\beta := B \cap \partial W_0$. Then we also see that $\alpha = A \cap \partial W_t$ and $\beta = B \cap \partial W_t$ for any $0 < t < |\sigma|/100$. We give the positive orientation to α and β with respect to the region W_0 and hence to W_t for every $0 < t < |\sigma|/100$. Since α and β are analytic with $u_t | \alpha = v | \alpha = 1$ and $u_t | \beta = v | \beta = 0$, u_t and v are extendable as uniformly bounded harmonic functions to a fixed vicinity of $\alpha \cup \beta$ and $(u_t)_{t \downarrow 0}$ converges uniformly to v there. Hence

$$\lim_{t \downarrow 0} *du_t = *dv \tag{2.11}$$

uniformly on $\alpha \cup \beta$ in the sense that coefficients of $*du_t$ converge uniformly to the corresponding coefficients of $*dv$ in any small parametric disc centered at any point of $\alpha \cup \beta$. Hence in particular we see that

$$\lim_{t \downarrow 0} \int_\alpha *du_t = \int_\alpha *dv. \tag{2.12}$$

Observe that, by the Stokes formula, we see

$$\operatorname{cap}(A, \hat{\mathbf{C}}_{\gamma_t} \backslash B) = D(u_{\gamma_t}; \hat{\mathbf{C}}_{\gamma_t}) = D(u_{\gamma_t}; \hat{\mathbf{C}}_{\gamma_t} \backslash A \cup B) = \int_\alpha *du_{\gamma_t} = \int_\alpha *du_t.$$

Understanding this time that A and B are contained in the same one sheet $\hat{\mathbf{C}} \backslash \tau$ of $\hat{\mathbf{C}}_\tau = (\hat{\mathbf{C}} \backslash \tau) \boxtimes_\tau (\hat{\mathbf{C}} \backslash \tau)$ as in the case of W_0, we compute the capacity $\operatorname{cap}(A, W_0 \cup A)$ of the compact subset A of $\hat{\mathbf{C}}_\tau$, considered as $\hat{\mathbf{C}}_\tau = [((\hat{\mathbf{C}} \backslash A) \backslash \tau) \boxtimes_\tau (\hat{\mathbf{C}} \backslash \tau)] \cup A = W_0 \cup (A \cup B) \supset A$, with respect to the open subset $W_0 \cup A = ((\hat{\mathbf{C}} \backslash B) \backslash \tau) \boxtimes_\tau (\hat{\mathbf{C}} \backslash \tau)$ of $\hat{\mathbf{C}}_\tau$ containing A. Then, since v is the capacity function for $\operatorname{cap}(A, W_0 \cup A)$, by exactly the same argument as above we see that

$$\operatorname{cap}(A, W_0 \cup A) = D(v; \hat{\mathbf{C}}_\tau) = D(v; \hat{\mathbf{C}}_\tau \backslash A \cup B) = \int_\alpha *dv.$$

Hence by (2.12) we deduce that

$$\lim_{t \downarrow 0} \operatorname{cap}(A, \hat{\mathbf{C}}_{\gamma_t} \backslash B) = \operatorname{cap}(A, W_0 \cup A). \tag{2.13}$$

Finally we compare $\operatorname{cap}(A, W_0 \cup A)$ with $\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$. Recall that $W = \hat{\mathbf{C}} \backslash A \cup B$ and u is the capacity function for $\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$ so that $u \in C(\overline{W}) \cap H(W)$ with $u|\alpha = 1$ and $u|\beta = 0$, where α and β can also be viewed as being $\alpha = A \cap \partial W$ and $\beta = B \cap \partial W$ with positive directions with respect to W. Thus

$$\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B) = D(u; W) = D(u; W \backslash \tau).$$

Viewing $W \backslash \tau$ is a subregion of

$$W_0 = ((\hat{\mathbf{C}} \backslash A \cup B) \backslash \tau) \boxtimes_\tau (\hat{\mathbf{C}} \backslash \tau) = (W \backslash \tau) \boxtimes_\tau (\hat{\mathbf{C}} \backslash \tau) \subset \hat{\mathbf{C}}_\tau$$

$W \backslash \tau$ is a subregion R of $\hat{\mathbf{C}}_\tau$ whose relative boundary ∂R consists of α, β, and γ: $\partial R = \alpha + \beta + \gamma$, where γ arises from τ as a smooth Jordan curve positively oriented with respect to R by considering it in the Carathéodory compactification of R. Therefore we have

$$\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B) = D(u; W \backslash \tau) = D(u; R). \tag{2.14}$$

By (2.3) $*du = 0$ along l and of course along τ so that finally along γ. Restricting v defined on $(W \backslash \tau) \boxtimes_\tau (\hat{\mathbf{C}} \backslash \tau) = R \cup \gamma \cup S$ to R, where $S := \hat{\mathbf{C}} \backslash \tau$, we compute the mutual Dirichlet integral $D(u - v, u; R)$ of two functions $u - v$ and u over R by using the Stokes formula as follows:

$$D(u - v, u; R) := \int_R d(u - v) \wedge *du = \int_{\alpha+\beta+\gamma} (u - v) * du = \int_\gamma (u - v) * du = 0$$

since $*du = 0$ along γ. Hence $D(u; R) = D(v, u; R)$ and the Schwarz inequality yields

$$D(u; R)^2 = D(v, u; R)^2 \leq D(v; R) \cdot D(u; R).$$

Hence we conclude that $D(u; R) \le D(v; R)$ since $D(u; R) > 0$. On the other hand we see that

$$\begin{aligned} D(v; R) &= D(v; W\backslash\tau) < D(v; W\backslash\tau) + D(v; \hat{\mathbf{C}}\backslash\tau) \\ &= D(v; W_\tau) = D(v; ((\hat{\mathbf{C}}\backslash A \cup B)\backslash\tau) \uplus_\tau (\hat{\mathbf{C}}\backslash\tau)) \end{aligned}$$

since $D(v; \hat{\mathbf{C}}\backslash\tau) > 0$. The last term of the above is $\mathrm{cap}(A, W_0 \cup A)$ so that, by (2.14), we obtain

$$\mathrm{cap}(A, \hat{\mathbf{C}}\backslash B) < \mathrm{cap}(A, W_0 \cup A).$$

This with (2.13) we finally conclude that

$$\mathrm{cap}(A, \hat{\mathbf{C}}\backslash B) < \lim_{t\downarrow 0} \mathrm{cap}(A, \hat{\mathbf{C}}_{\gamma_t}\backslash B)$$

and therefore we see that

$$\mathrm{cap}(A, \hat{\mathbf{C}}\backslash B) < \mathrm{cap}(A, \hat{\mathbf{C}}_{\gamma_t}\backslash B)$$

for every sufficiently small $t \in (0, |\sigma|/100)$, i.e. γ_t for sufficiently small $0 < t < |\sigma|/100$ is a supercritical pasting arc for A and B in $\hat{\mathbf{C}}$. $\square$

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