H. MASAOKA AND S. SEGAWA
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QUASICONFORMAL MAPPINGS AND MINIMAL MARTIN BOUNDARY OF p-SHEETED UNLIMITED COVERING SURFACES OF THE COMPLEX PLANE

Hiroaki Masaoka and Shigeo Segawa

To the memory of Professor Nobuyuki Suita

§1. Introduction

Let W be an open Riemann surface. We denote by Δ_1^W the minimal Martin boundary of W. In [L], it was showed that there exist open Riemann surfaces F and F' quasiconformally equivalent to each other such that F' possesses nonconstant positive harmonic functions although F does not possess nonconstant positive harmonic functions. This means that $\#\Delta_1^{F'} \geq 2$ although $\#\Delta_1^F = 1$, where $\#A$ stands for the cardinal number of a set A. Needless to say, the above F and F' are of *positive boundary*, i.e. F and F' admit the Green function (cf. [SN]). However, in case open Riemann surfaces W and W' are of *null boundary* (i.e. not positive boundary), it does not seem to be known whether $\#\Delta_1^W = \#\Delta_1^{W'}$ or not if W and W' are quasiconformally equivalent to each other.

In this paper, we are concerned with p-sheeted unlimited covering surfaces of the complex plane $\mathbf{C}$. Consider p-sheeted unlimited covering surfaces R and R' of the complex plane $\mathbf{C}$ which are quasiconformally equivalent to each other. Then, it seems to be valid that $\#\Delta_1^R = \#\Delta_1^{R'}$ (cf. [Sh], [M]). The purpose of this paper is to give a partial answer to this conjecture. Namely,

Main Theorem. *Let R and R' be p-sheeted unlimited covering surfaces of* $\mathbf{C}$ *which are quasiconformally equivalent to each other. If $p = 2$ or 3, then it holds that $\#\Delta_1^R = \#\Delta_1^{R'}$.*

§2. Preliminaries

Hereafter we consider the punctured sphere $\hat{\mathbf{C}}\backslash\{0\}$ in place of the complex plane $\mathbf{C}$ since $\hat{\mathbf{C}}\backslash\{0\}$ is conformally equivalent to $\mathbf{C}$. Hence we assume that R and R' in Main Theorem are p-sheeted unlimited covering surfaces of $\hat{\mathbf{C}}\backslash\{0\}$. Let Δ^R and Δ_1^R be as in §1, and π the projection map from R onto $\hat{\mathbf{C}}\backslash\{0\}$. Set $\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$, $\mathbf{D}_0 = \mathbf{D}\backslash\{0\}$ and $R_0 = \pi^{-1}(\mathbf{D}_0)$. It is well-known that Δ^{R_0} and $\Delta_1^{R_0}$ are identified with $\Delta^R \cup \pi^{-1}(\partial\mathbf{D})$ and $\Delta_1^R \cup \pi^{-1}(\partial\mathbf{D})$, respectively, where

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$\partial \mathbf{D} = \{|z| = 1\}$. From now on we consider $\mathbf{D}_0$ (resp. R_0) in place of $\hat{\mathbf{C}} \setminus \{0\}$ (resp. R) since $\hat{\mathbf{C}} \setminus \{0\}$ (resp. R) does not admit the Green function. Let g_0 be the Green function on $\mathbf{D}$ with pole at 0.

Definition 2.1 (cf. [B], [BH]). We say that a subset E of $\mathbf{D}_0$ is *thin* at 0 if ${}^{\mathbf{D}}\hat{\mathrm{R}}^E_{g_0} \neq g_0$, where ${}^{\mathbf{D}}\hat{\mathrm{R}}^E_{g_0}$ is the balayage of g_0 relative to E on $\mathbf{D}$.

If E is a closed subset of $\mathbf{D}$, it is well-known that E is thin at 0 if and only if 0 is an irregular boundary point of $\mathbf{D} \setminus E$ in the sense of the Dirichlet problem.

The following lemma gives the quasiconformal invariance for thinness.

Lemma 2.1 (cf. [M], [Sh]). *Let G be a subdomain of $\mathbf{C}$ and φ a quasiconformal mapping from $\mathbf{C}$ onto $\mathbf{C}$. If ζ is an irregular boundary point of G in the sense of Dirichlet problem, $\varphi(\zeta)$ is an irregular boundary point of $\varphi(G)$ in the sense of Dirichlet problem.*

Definition 2.2. A subset U in $\mathbf{D}$ which contains 0 is said to be a *fine neighborhood* of 0 if $\mathbf{D} \setminus U$ is thin at 0.

Let k_ζ be the Martin function on R_0 with pole at $\zeta \in \Delta^R$.

Definition 2.3. Let ζ be a point in Δ_1^R and E a subset of R_0. We say that E is *minimally thin* at ζ if ${}^{R_0}\hat{\mathrm{R}}^E_{k_\zeta} \neq k_\zeta$.

Definition 2.4. Let ζ be a point in Δ_1^R and U a subset of R_0. We say that $U \cup \{\zeta\}$ is a *minimal fine neighborhood* of ζ if $R_0 \setminus U$ is minimally thin at ζ.

The following proposition gives the characterization of $\#\Delta_1^R$ in terms of minimal fine topology.

Proposition 2.1 ([MS]). *Let $\mathscr{M}$ be the class of subdomains M of $\mathbf{D}_0$ such that $M \cup \{0\}$ is a fine neighborhood of $z = 0$. Then, it holds that*

$$\#\Delta_1^R = \max_{M \in \mathscr{M}} n_R(M),$$

where $n_R(M)$ is the number of connected components of $\pi^{-1}(M)$ and π is the projection map from R onto $\hat{\mathbf{C}} \setminus \{0\}$.

§3. Proof of Main Theorem in case $p = 2$

Consider the case $p = 2$ in this section. Let R and R' be as in Main Theorem and f be a quasiconformal mapping from R onto R'. It is known that $1 \leq \#\Delta_1^R, \#\Delta_1^{R'} \leq 2$ (cf. [H] and see also [MS]). We have only to prove that $\#\Delta_1^{R'} = 2$ if and only if $\#\Delta_1^R = 2$. Since f^{-1} is a quasiconformal mapping from R' onto R, it is sufficient to prove that if $\#\Delta_1^R = 2$, then $\#\Delta_1^{R'} = 2$. Suppose that $\#\Delta_1^R = 2$. Let π (resp. π') be the projection map from R (resp.

R') onto $\hat{\mathbf{C}}\backslash\{0\}$. By Proposition 2.1 there exists a subdomain U of $\mathbf{D}_0$ such that $\mathbf{D}_0\backslash U$ is thin at 0, $n_R(U) = 2$ and $f(\pi^{-1}(U)) \subset R'_0$ $(:= (\pi')^{-1}(\mathbf{D}_0))$. Let U_j $(j = 1, 2)$ be components of $\pi^{-1}(U)$. Since R is a 2-sheeted unlimited covering surface of $\hat{\mathbf{C}}\backslash\{0\}$, it is easily seen that each U_j is considered as a replica of U. Let $g_z^{f(U_j)}$ $(j = 1, 2)$ be the Green function on $f(U_j)$ with pole at z. Denote by $\mu_{f,j}$ the complex dilatation of f on U_j. Set

$$\mu_j = \begin{cases} \mu_{f,j} \circ \varphi_j & \text{on } U \\ 0 & \text{on } \mathbf{C}\backslash U, \end{cases}$$

where φ_j is the inverse of $\pi \,|\, U_j : U_j \to U$. It is well-known that there exists a quasiconformal mapping f_j from $\mathbf{C}$ onto $\mathbf{C}$ with the complex delatation μ_j (cf. e.g. [LV]). Set $V_j = f_j(U)$. By Lemma 2.1 we find that $f_j(0)$ is an irregular boundary point of V_j in the sense of the usual Dirichlet problem since 0 is an irregular boundary point of U in the sense of the usual Dirichlet problem. On the other hand, the function $z \mapsto g^{f(U_j)}_{f\circ\varphi_j\circ(f_j)^{-1}(z)} \circ f \circ \varphi_j \circ (f_j)^{-1}(\xi)$ $(\xi \in V_j)$ is the Green function on V_j with pole at ξ since $f \circ \varphi_j \circ (f_j)^{-1}$ is conformal. Hence, by [Hl, Theorem 10.16], there exists a fine limit $\mathscr{F} - \lim_{z\to f_j(0)} g^{f(U_j)}_{f\circ\varphi_j\circ(f_j)^{-1}(z)} \circ f \circ \varphi_j \circ (f_j)^{-1}$. Since $f_j(0)$ is an irregular boundary point of V_j in the sense of the usual Dirichlet problem, this limit must be positive by [Hl, Theorem 8.34]. Denote this limit function on V_j by $g_0^{V_j}$ and set $g_0^{f(U_j)} = g_0^{V_j} \circ f_j \circ \pi \circ f^{-1}$. We see that each $g_0^{f(U_j)}$ is a positive harmonic function on $f(U_j)$ since each $g_0^{V_j}$ is a positive harmonic function on V_j and $f_j \circ \pi \circ f^{-1}$ is conformal. For $j = 1, 2$, set

$$S_j(g_0^{f(U_j)})(x) := \inf_s s(x),$$

where s runs over the space of positive superharmonic functions s on R'_0 satisfying $s \geq g_0^{f(U_j)}$ on $f(U_j)$. By Perron-Wiener-Brelot method we find that each $S_j(g_0^{f(U_j)})$ is a positive harmonic function on R'_0. Then, the following inequality

$$(*) \qquad S_j(g_0^{f(U_j)}) - {}^{R'_0}\hat{\mathbf{R}}^{R'_0\backslash f(U_j)}_{S_j(g_0^{f(U_j)})} \geq g_0^{f(U_j)}$$

holds on $f(U_j)$. In fact, to prove the inequality $(*)$ note that

$${}^{R'_0}\hat{\mathbf{R}}^{R'_0\backslash f(U_j)}_{S_j(g_0^{f(U_j)})} = H^{f(U_j)}_{S_j(g_0^{f(U_j)})}$$

on $f(U_j)$, where $H^{f(U_j)}_{S_j(g_0^{f(U_j)})}$ is the Dirichlet solution for $S_j(g_0^{f(U_j)})$ on $f(U_j)$ (cf. e.g. [Hl], [CC]). By definition $S_j(g_0^{f(U_j)}) \geq g_0^{f(U_j)}$ on $f(U_j)$. Hence, by the definition of the Dirichlet solution in the sense of Perron-Wiener-Brelot,

$$S_j(g_0^{f(U_j)}) - g_0^{f(U_j)} \geq H^{f(U_j)}_{S_j(g_0^{f(U_j)})}$$

on $f(U_j)$. Thus $(*)$ is proved.

We shall proceed the proof of Main Theorem in case $p = 2$. By the Martin representation theorem, there exist at most two minimal functions $h_{j,k}$ $(k = 1, 2)$ on R'_0 with $S_j(g_0^{f(U_j)}) = h_{j,1} + h_{j,2}$ on R'_0. Hence, by the above inequality $(*)$, we have

$$h_{j,1}+h_{j,2}=S_j(g_0^{f(U_j)})\geq {}^{R_0'}\hat{\mathrm{R}}_{h_{j,1}+h_{j,2}}^{R_0'\backslash f(U_j)}+g_0^{f(U_j)}>{}^{R_0'}\hat{\mathrm{R}}_{h_{j,1}}^{R_0'\backslash f(U_j)}+{}^{R_0'}\hat{\mathrm{R}}_{h_{j,2}}^{R_0'\backslash f(U_j)}$$

on $f(U_j)$. Therefore, we find that there exists a minimal function h_j on R_0' such that $h_j\neq {}^{R_0'}\hat{\mathrm{R}}_{h_j}^{R_0'\backslash f(U_j)}$. Hence, by the definition of minimal thinness, $R_0'\backslash f(U_j)$ is minimally thin at the minimal boundary point corresponding to h_j. Since $f(U_1)\cap f(U_2)=\emptyset$, we find that $\#\Delta_1^{R'}=2$.

§4. Proof of Main Theorem in case $p=3$

Consider the case $p=3$ in this section. As in §3, it is known that $1\leq \#\Delta_1^R$, $\#\Delta_1^{R'}\leq 3$ (cf. [H] and see also [MS]). By the same argument as in the proof of Main Theorem in case $p=2$ we find that $\#\Delta_1^R=3$ if and only if $\#\Delta_1^{R'}=3$. Hence, to prove the statement of Main Theorem in case $p=3$, we have only to prove that $\#\Delta_1^{R'}=2$ if and only if $\#\Delta_1^R=2$. Since f^{-1} is a quasiconformal mapping from R' onto R, it is sufficient to prove that if $\#\Delta_1^R=2$, then $\#\Delta_1^{R'}=2$. Contrary to this, we suppose that $\#\Delta_1^R=2$ and that $\#\Delta_1^{R'}\neq 2$. Then, by the above observation, we see that $\#\Delta_1^{R'}=1$.

By Proposition 2.1 there exists a subdomain U of $\mathbf{D}_0$ such that $\mathbf{D}_0\backslash U$ is thin at 0, $n_R(U)=2$ and $f(\pi^{-1}(U))\subset R_0'$. Hence $\pi^{-1}(U)$ consists of two connected components U_1 and U_2. Since R is a 3-sheeted unlimited covering surface of $\hat{\mathbf{C}}\backslash\{0\}$, we assume that U_1 is a 1-sheeted unlimited covering surface of U, that is, U_1 is a replica of U and U_2 is a 2-sheeted unlimited covering surface of U. Let $g_z^{f(U_1)}$ be the Green function on $f(U_1)$ with pole at z. Denote by μ_f the complex dilatation of f on U_1. Set

$$\mu=\begin{cases}\mu_f\circ\varphi & \text{on } U\\ 0 & \text{on } \mathbf{C}\backslash U,\end{cases}$$

where φ is the inverse of $\pi\,|\,U_1:U_1\to U$. It is well-known that there exists a quasiconformal mapping ψ from $\mathbf{C}$ onto $\mathbf{C}$ with the complex delatation μ (cf. [LV]). By the same method as in §3, there exists a positive fine limit $\mathscr{F}-\lim_{z\to\psi(0)} g_z^{f(U_1)}\circ f\circ\varphi\circ\psi^{-1}$. Denote by $g_0^{f(U_1)}$ the pull-back of this limit function on $\psi(U_1)$ by $\psi\circ\pi\circ f^{-1}$. We see that $g_0^{f(U_1)}$ is a positive harmonic function on $f(U_1)$. Set

$$S(g_0^{f(U_1)})(x):=\inf_s s(x),$$

where s runs over the space of positive superharmonic functions s on R_0' satisfying $s\geq g_0^{f(U_1)}$ on $f(U_1)$. By Perron-Wiener-Brelot method we find that $S(g_0^{f(U_1)})$ is a positive harmonic function on R_0'. By the same consideration as in the proof of Main Theorem in case $p=2$, we have

$$S(g_0^{f(U_1)})-{}^{R_0'}\hat{\mathrm{R}}_{S(g_0^{f(U_1)})}^{R_0'\backslash f(U_1)}\geq g_0^{f(U_1)} \tag{$**$}$$

on $f(U_1)$. By the assumption, $S(g_0^{f(U_1)})$ is only one minimal harmonic function on R_0'. Hence, by $(**)$, we find that $R_0'\backslash f(U_1)$ is minimally thin at the minimal boundary point corresponding to $S(g_0^{f(U_1)})$. Take a curve γ in U such that γ reaches 0 and that $\pi^{-1}(\gamma)$ does not possess any branch points of R. Hence there exists a curve $\tilde{\gamma}$ in $U_2(\subset R_0 \backslash U_1)$ with $f(\tilde{\gamma}) \subset R_0'$ and $\pi(\tilde{\gamma}) = \gamma$ which reaches the ideal boundary since R is unlimited. Hence this implies that

i) $f(\tilde{\gamma})$ is a subset of $R_0' \backslash f(U_1)$;

ii) $\pi'(f(\tilde{\gamma}))$ is not thin at 0 in the usual sense, where π' is the projection map from R' onto $\hat{\mathbf{C}} \backslash \{0\}$.

By the above fact i), $f(\tilde{\gamma})$ is minimally thin at the minimal boundary point corresponding to $S(g_0^{f(U_1)})$. On the other hand, by the above fact ii) and [MS, Proposition 3.1], $f(\tilde{\gamma})$ is not minimally thin at the minimal boundary point corresponding to $S(g_0^{f(U_1)})$. This is a contradiction. We have the desired result.

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Hiroaki Masaoka
Department of Mathematics
Faculty of Science
Kyoto Sangyo University
Kamigamo–Motoyama, Kitaku
Kyoto 603-8555
Japan
e-mail: masaoka@cc.kyoto-su.ac.jp

Shigeo Segawa
Department of Mathematics
Daido Institute of Technology
Nagoya 457-8530
Japan
e-mail: segawa@daido-it.ac.jp