

ON A CERTAIN HOLOMORPHIC CURVE EXTREMAL FOR THE DEFECT RELATION

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1. Introduction

Let $f = [f_1, \dots, f_{n+1}]$ be a holomorphic curve from \mathbf{C} into the n -dimensional complex projective space $P^n(\mathbf{C})$ with a reduced representation

$$(f_1, \dots, f_{n+1}) : \mathbf{C} \rightarrow \mathbf{C}^{n+1} - \{\mathbf{0}\},$$

where n is a positive integer.

We use the following notations:

$$\|f(z)\| = (|f_1(z)|^2 + \dots + |f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathbf{C}^{n+1} - \{\mathbf{0}\}$

$$\|\mathbf{a}\| = (|a_1|^2 + \dots + |a_{n+1}|^2)^{1/2},$$

$$(\mathbf{a}, f) = a_1 f_1 + \dots + a_{n+1} f_{n+1},$$

$$(\mathbf{a}, f(z)) = a_1 f_1(z) + \dots + a_{n+1} f_{n+1}(z).$$

The characteristic function of f is defined as follows (see [11]):

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta - \log \|f(0)\|.$$

Due to Cartan ([1]), we have the following relation:

$$(1) \quad T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \max_{1 \leq j \leq n+1} |f_j(re^{i\theta})| d\theta + O(1).$$

We suppose throughout the paper that f is transcendental; that is to say,

$$\lim_{r \rightarrow \infty} \frac{T(r, f)}{\log r} = \infty$$

and that f is linearly non-degenerate over \mathbf{C} ; namely, f_1, \dots, f_{n+1} are linearly independent over \mathbf{C} .

It is well-known that f is linearly non-degenerate over \mathbf{C} if and only if the Wronskian $W = W(f_1, \dots, f_{n+1})$ of f_1, \dots, f_{n+1} is not identically equal to zero.

For meromorphic functions in the complex plane we use the standard notations of Nevanlinna theory of meromorphic functions ([4], [5]).

For $\mathbf{a} \in \mathbf{C}^{n+1} - \{\mathbf{0}\}$, we write

$$m(r, \mathbf{a}, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\mathbf{a}\| \|f(re^{i\theta})\|}{|(\mathbf{a}, f(re^{i\theta}))|} d\theta,$$

$$N(r, \mathbf{a}, f) = N\left(r, \frac{1}{(\mathbf{a}, f)}\right).$$

We then have the first fundamental theorem:

$$(2) \quad T(r, f) = m(r, \mathbf{a}, f) + N(r, \mathbf{a}, f) + O(1)$$

([11], p. 76). We call the quantity

$$\delta(\mathbf{a}, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \mathbf{a}, f)}{T(r, f)} = \liminf_{r \rightarrow \infty} \frac{m(r, \mathbf{a}, f)}{T(r, f)}$$

the deficiency (or defect) of \mathbf{a} with respect to f . We have

$$0 \leq \delta(\mathbf{a}, f) \leq 1$$

by (2) since $N(r, \mathbf{a}, f) \geq 0$ for $r \geq 1$ and $m(r, \mathbf{a}, f) \geq 0$ for $r > 0$.

Let X be a subset of $\mathbf{C}^{n+1} - \{\mathbf{0}\}$ in N -subgeneral position; that is to say, $\#X \geq N + 1$ and any $N + 1$ elements of X generate \mathbf{C}^{n+1} , where N is an integer satisfying $N \geq n$.

Cartan ([1], $N = n$) and Nochka ([6], $N > n$) gave the following

THEOREM A (Defect relation). *For any q elements \mathbf{a}_j ($j = 1, \dots, q$) of X ,*

$$\sum_{j=1}^q \delta(\mathbf{a}_j, f) \leq 2N - n + 1,$$

where $2N - n + 1 \leq q \leq \infty$ (see also [2] or [3]).

We are interested in the holomorphic curve f for which the defect relation is extremal:

$$(3) \quad \sum_{j=1}^q \delta(\mathbf{a}_j, f) = 2N - n + 1.$$

In [9] we proved the following theorem.

THEOREM B. *Suppose that there are vectors \mathbf{a}_j ($j = 1, \dots, q$) in X which satisfy (3), where $2N - n + 1 \leq q \leq \infty$. If $(n + 1, 2N - n + 1) = 1$, then there are at least*

$$\left[\frac{2N - n + 1}{n + 1} \right] + 1$$

vectors $\mathbf{a} \in \{\mathbf{a}_j \ (j = 1, \dots, q)\}$ satisfying $\delta(\mathbf{a}, f) = 1$.

Further, we improved this theorem in [10]. Namely, we weakened the condition “ $(n + 1, 2N - n + 1) = 1$ ” in Theorem B to “ $N > n = 2m \ (m \in \mathbb{N})$ ” and obtained the same conclusion as in Theorem B.

In this paper we consider the holomorphic curve f satisfying (3) from a different point of view.

Let

$$X_k(0) = \{\mathbf{a} = (a_1, \dots, a_{n+1}) \in X \mid a_k = 0\} \quad (1 \leq k \leq n + 1).$$

Then, it is easy to see that

$$0 \leq \#X_k(0) \leq N$$

since X is in N -subgeneral position.

Further we put (see Definition 1 in [7])

$$u_k(z) = \max_{1 \leq j \leq n+1, j \neq k} |f_j(z)|,$$

$$t_k(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \{\log u_k(re^{i\theta}) - \log u_k(e^{i\theta})\} d\theta,$$

and

$$\Omega_k = \limsup_{r \rightarrow \infty} \frac{t_k(r, f)}{T(r, f)}.$$

PROPOSITION A (see [7]).

- (a) $t_k(r, f)$ is independent of the choice of reduced representation of f .
- (b) $t_k(r, f) \leq T(r, f) + O(1)$.
- (c) $N(r, 1/f_j) \leq t_k(r, f) + O(1) \ (j = 1, \dots, n + 1, j \neq k)$.
- (d) $0 \leq \Omega_k \leq 1$.

Our main purpose of this paper is to prove the following theorem:

THEOREM. *Suppose that*

- (i) $N > n \geq 2$;
- (ii) *there are vectors $\mathbf{a}_1, \dots, \mathbf{a}_q \in X \ (2N - n + 1 < q \leq \infty)$ satisfying $\delta(\mathbf{a}_j, f) > 0 \ (j = 1, \dots, q)$ and*

$$\sum_{j=1}^q \delta(\mathbf{a}_j, f) = 2N - n + 1.$$

If $\Omega_k < 1$ for some $k \ (1 \leq k \leq n + 1)$, then

- (a) $\#X_k(0) = N$;
- (b) *there is a subset $P \subset \{1, 2, \dots, q\}$ satisfying*

$$\#P = N - n + 1, \quad d(P) = 1, \quad \delta(\mathbf{a}_j, f) = 1 \quad (j \in P)$$

and

$$X_k(0) \cap \{\mathbf{a}_j \mid j \in P\} = \emptyset,$$

where $d(P)$ is the dimension of the vector space spanned by $\{\mathbf{a}_j \mid j \in P\}$.

- (c) *Any n elements of $X - \{\mathbf{a}_j \mid j \in P\}$ are linearly independent.*

As an application of this theorem, we can prove the following result:

“Let f be any exponential curve. If $N > n \geq 2$, then

$$\sum_{\mathbf{a} \in X} \delta(\mathbf{a}, f) < 2N - n + 1.”$$

This result means that any exponential curve is not extremal for the defect relation when $N > n \geq 2$.

2. Preliminaries and lemma

We shall give some lemmas for later use. Let $f = [f_1, \dots, f_{n+1}]$, X and $X_k(0)$ etc. be as in Section 1, q any integer satisfying $2N - n + 1 < q < \infty$ and we put $Q = \{1, 2, \dots, q\}$.

Let $\{\mathbf{a}_j \mid j \in Q\}$ be a family of vectors in X . For a non-empty subset P of Q , we denote

$$V(P) = \text{the vector space spanned by } \{\mathbf{a}_j \mid j \in P\} \quad \text{and} \quad d(P) = \dim V(P)$$

and we put

$$\mathcal{O} = \{P \subset Q \mid 0 < \#P \leq N + 1\}.$$

LEMMA 1 ((2.4.3) in [3], p. 68). For $P \in \mathcal{O}$, $\#P - d(P) \leq N - n$.

For $\{\mathbf{a}_j \mid j \in Q\}$, let

$$\omega : Q \rightarrow (0, 1]$$

be the Nochka weight function given in [3, p. 72] and θ the reciprocal number of the Nochka constant given in [3, p. 72]. Then they have the following properties:

LEMMA 2 (see [3], Theorem 2.11.4).

- (a) $0 < \omega(j)\theta \leq 1$ for all $j \in Q$;
- (b) $q - 2N + n - 1 = \theta(\sum_{j=1}^q \omega(j) - n - 1)$;
- (c) $(N + 1)/(n + 1) \leq \theta \leq (2N - n + 1)/(n + 1)$;
- (d) If $P \in \mathcal{O}$, then $\sum_{j \in P} \omega(j) \leq d(P)$.

Note 1. (c) of Lemma 2 can be refined as follows: $\frac{N}{n} \leq \theta \leq \frac{2N - n + 1}{n + 1}$.

Proof. When $\theta = (2N - n + 1)/(n + 1)$, there is nothing to prove as $N/n \leq (2N - n + 1)/(n + 1)$.

When $\theta < (2N - n + 1)/(n + 1)$, there is an element $P \in \mathcal{O}$ satisfying

$$\theta = \frac{2N - n + 1 - \#P}{n + 1 - d(P)} \quad (1 \leq d(P) \leq n)$$

by the definition of θ . By Lemma 1 we have

$$\theta = \frac{2N - n + 1 - \#P}{n + 1 - d(P)} \geq \frac{N + 1 - d(P)}{n + 1 - d(P)} \geq \frac{N}{n}$$

since $d(P) \geq 1$. □

Put

$$P_k(0) = \{j \in \mathcal{Q} \mid \mathbf{a}_j \in X_k(0)\} \quad \text{and} \quad d_k = \sum_{j \in P_k(0)} \omega(j).$$

Then, we have the inequality

$$(4) \quad d_k \leq n$$

since $d_k \leq d(P_k(0))$ by Lemma 2(d) and $d(P_k(0)) \leq n$ by the definition of $X_k(0)$.

LEMMA 3 (Defect relation) (see Theorem 3 in [8]). *For any $\mathbf{a}_1, \dots, \mathbf{a}_q \in X$, we have the following inequalities:*

$$(I) \quad \sum_{j=1}^q \omega(j) \delta(\mathbf{a}_j, f) \leq d_k + 1 + (n - d_k) \Omega_k;$$

$$(II) \quad \sum_{j=1}^q \delta(\mathbf{a}_j, f) \leq 2N - n + 1 - \frac{N}{n} (n - d_k) (1 - \Omega_k).$$

By applying Lemma 2 and Note 1 to (I) we obtain (II) as usual. (II) is an amelioration of Theorem 3 (II) in [8].

Remark 1. This is an amelioration of Theorem A. Since $\Omega_k \leq 1$ and $d_k \leq n$ we have the inequalities:

$$d_k + 1 + (n - d_k) \Omega_k \leq n + 1 \quad \text{and}$$

$$2N - n + 1 - N(n - d_k)(1 - \Omega_k)/n \leq 2N - n + 1.$$

LEMMA 4. *For any $\mathbf{a} \in X_k(0)$, $\delta(\mathbf{a}, f) \geq 1 - \Omega_k$.*

Proof. For $\mathbf{a} \in X_k(0)$ we have the inequality

$$|(\mathbf{a}, f(z))| \leq Ku_k(z)$$

for a positive constant K by the definitions of $X_k(0)$ and $u_k(z)$. From this inequality we have the inequality

$$\begin{aligned} N(r, \mathbf{a}, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log |(\mathbf{a}, f(re^{i\theta}))| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \log u_k(re^{i\theta}) d\theta + \log K \\ &= t_k(r, f) + O(1) \quad (r > 0), \end{aligned}$$

from which we obtain the inequality $\delta(\mathbf{a}, f) \geq 1 - \Omega_k$. \square

LEMMA 5 ([9], Lemma 3). *Suppose that $N > n$. For $\mathbf{a}_1, \dots, \mathbf{a}_q \in X$ the maximal deficiency sum*

$$\sum_{j=1}^q \delta(\mathbf{a}_j, f) = 2N - n + 1$$

holds if and only if the following two relations hold:

- 1) $(1 - \theta\omega(j))(1 - \delta(\mathbf{a}_j, f)) = 0 \quad (j = 1, \dots, q);$
- 2) $\sum_{j=1}^q \omega(j)\delta(\mathbf{a}_j, f) = n + 1.$

COROLLARY 1 ([9], Corollary 1(I)). *Suppose that $N > n$ and that for $\mathbf{a}_1, \dots, \mathbf{a}_q \in X$, the equality*

$$\sum_{j=1}^q \delta(\mathbf{a}_j, f) = 2N - n + 1$$

holds. If $\theta\omega(j) < 1$ for some j , then $\delta(\mathbf{a}_j, f) = 1$.

DEFINITION 1 ([9], Definition 1). We put

$$\lambda = \min_{P \in \mathcal{O}} \frac{d(P)}{\#P}.$$

Then, λ has the following property.

LEMMA 6 ([9], Proposition 2). $1/(N - n + 1) \leq \lambda \leq (n + 1)/(N + 1)$.

Remark 2. (a) If $\lambda < (n + 1)/(2N - n + 1)$, then $\lambda = \min_{1 \leq j \leq q} \omega(j)$ and $\omega(j) = \lambda$, $\theta\omega(j) < 1$ ($j \in P_0$) for an element $P_0 \in \mathcal{O}$ satisfying $\lambda = d(P_0)/\#P_0$.

(b) If $\lambda \geq (n + 1)/(2N - n + 1)$, then $\omega(j) = 1/\theta = (n + 1)/(2N - n + 1)$ ($j = 1, \dots, q$).

In fact, the first assertion of (a) is given in the proof of Proposition 2.4.4 ([3], p. 68) with the definition of $\omega(j)$ ([3], p. 72). For the second assertion of (a), as $\omega(j) = \lambda$ ($j \in P_0$) by the definition of ω and $(n + 1)/(2N - n + 1) \leq 1/\theta$, we have the conclusion.

(b) See the definitions of $\omega(j)$ and θ ([3], p. 72).

LEMMA 7 ([9], Corollary 2). For $\mathbf{a}_1, \dots, \mathbf{a}_q \in X$, we have the inequality

$$\sum_{j=1}^q \delta(\mathbf{a}_j, f) \leq \min\left(2N - n + 1, \frac{n+1}{\lambda}\right).$$

3. Proof of Theorem when $q < \infty$

Let f , X , $X_k(0)$ and ω etc. be as in the previous sections and q an integer satisfying

$$2N - n + 1 < q < \infty.$$

Throughout this section we suppose that

- (i) $N > n \geq 2$;
- (ii) there are vectors $\mathbf{a}_1, \dots, \mathbf{a}_q \in X$ satisfying $\delta(\mathbf{a}_j, f) > 0$ ($j = 1, \dots, q$) and

$$\sum_{j=1}^q \delta(\mathbf{a}_j, f) = 2N - n + 1;$$

- (iii) $\Omega_k < 1$ for some k ($1 \leq k \leq n+1$).

PROPOSITION 1. $X_k(0) \subset \{\mathbf{a}_1, \dots, \mathbf{a}_q\}$.

Proof. If there exists a vector $\mathbf{a} \in X_k(0)$ satisfying $\mathbf{a} \notin \{\mathbf{a}_1, \dots, \mathbf{a}_q\}$, then $\delta(\mathbf{a}, f) > 0$ by Lemma 4 and (iii), and so by Theorem A we have the inequality

$$\sum_{j=1}^q \delta(\mathbf{a}_j, f) \leq 2N - n + 1 - \delta(\mathbf{a}, f) < 2N - n + 1,$$

which is a contradiction to our assumption (ii). □

PROPOSITION 2. $d_k = n$.

Proof. From Lemma 3(II) and the assumption (ii) we have the inequality

$$(1 - \Omega_k)(n - d_k) \leq 0.$$

Then, by the assumption (iii) and (4), we obtain the equality $d_k = n$. □

PROPOSITION 3. (a) $\theta = N/n$, (b) $\#P_k(0) = N$ and (c) $\theta\omega(j) = 1$ ($j \in P_k(0)$).

Proof. As X is in N -subgeneral position, we have $\#X_k(0) \leq N$. From Proposition 2 and Lemma 2(a)

$$(A) \quad \theta n = \sum_{j \in P_k(0)} \theta\omega(j) \leq \sum_{j \in P_k(0)} 1 = \#P_k(0) = \#X_k(0) \leq N,$$

so that we have $\theta \leq N/n$. By Note 1 we obtain $\theta = N/n$.

Combining this result with the inequality (A), we have

$$\#P_k(0) = N \quad \text{and} \quad \theta\omega(j) = 1 \quad (j \in P_k(0)). \quad \square$$

COROLLARY 2. $\lambda < (n+1)/(2N-n+1)$.

Proof. By Lemma 7 and the assumption (ii), we have

$$\lambda \leq \frac{n+1}{2N-n+1}.$$

If $\lambda = (n+1)/(2N-n+1)$, then by Remark 2(b) and Proposition 3(a)

$$\theta = \frac{2N-n+1}{n+1} = \frac{N}{n},$$

which is a contradiction, since $N/n < (2N-n+1)/(n+1)$ when $N > n \geq 2$. This implies that our corollary holds. \square

Put

$$P_1 = \{j \mid \theta\omega(j) < 1, 1 \leq j \leq q\}.$$

Then,

$$(5) \quad P_1 \cap P_k(0) = \emptyset$$

by Proposition 3(c). We have the following

PROPOSITION 4. $N - n + 1 \leq \#P_1 < 2N - n + 1$.

Proof. (a) From Lemma 2(b), we have

$$q - (2N - n + 1) = \theta \left(\sum_{j=1}^q \omega(j) - n - 1 \right) = \sum_{j \notin P_1} \theta\omega(j) + \sum_{j \in P_1} \theta\omega(j) - \theta n - \theta.$$

Here, as $\theta\omega(j) = 1$ for $j \notin P_1$ we have that

$$\sum_{j \notin P_1} \theta\omega(j) = q - \#P_1$$

and by Proposition 3(a) we have

$$\sum_{j \in P_1} \theta\omega(j) - \theta n - \theta = -N + \frac{N}{n} \left(\sum_{j \in P_1} \omega(j) - 1 \right).$$

Combining these three equalities we obtain

$$(B_1) \quad q - (2N - n + 1) = q - \#P_1 - N + \frac{N}{n} \left(\sum_{j \in P_1} \omega(j) - 1 \right).$$

Here, as $1/(N - n + 1) \leq \lambda \leq \omega(j)$ ($j \in P_1$) due to Lemma 6, Corollary 2 and Remark 2(a) we obtain the inequality

$$(B_2) \quad q - \#P_1 - N + \frac{N}{n} \left(\sum_{j \in P_1} \omega(j) - 1 \right) \geq q - \#P_1 - N + \frac{N}{n} \left(\frac{\#P_1}{N - n + 1} - 1 \right).$$

From (B₁) and (B₂) we have the inequality

$$q - (2N - n + 1) \geq q - \#P_1 - N + \frac{N}{n} \left(\frac{\#P_1}{N - n + 1} - 1 \right),$$

which reduces to the inequality

$$(N - n + 1 - N/n)(\#P_1 - N + n - 1) \geq 0.$$

As

$$N - n + 1 - \frac{N}{n} = \frac{(N - n)(n - 1)}{n} > 0$$

by the assumption (i), we have

$$\#P_1 \geq N - n + 1.$$

(b) As $\delta(\mathbf{a}_j, f) = 1$ ($j \in P_1$) by Corollary 1, from Propositions 1, 3(b) and the assumption (ii) we have

$$\#P_1 < 2N - n + 1$$

as $P_1 \cap P_k(0) = \emptyset$ by (5). □

Let P_0 be an element of \mathcal{O} satisfying

$$\frac{d(P_0)}{\#P_0} = \lambda,$$

where $\lambda = \min_{P \in \mathcal{O}} d(P)/\#P$. Then, $\omega(j) = \lambda$ ($j \in P_0$) and

$$(6) \quad \emptyset \neq P_0 \subset P_1$$

since $\theta\lambda < 1$ by Corollary 2 and Remark 2(a).

PROPOSITION 5. (a) $\#P_0 = N - n + 1$, (b) $d(P_0) = 1$ and
(c) $\omega(j) = \lambda = 1/(N - n + 1)$ ($j \in P_0$).

Proof. By Proposition 3(a), θ is equal to N/n , which is smaller than $(2N - n + 1)/(n + 1)$. By the definition of θ , there exists a set $P \in \mathcal{O}$ satisfying

$$P_0 \subset P, \quad 1 \leq d(P) \leq n$$

and

$$\theta = \frac{2N - n + 1 - \#P}{n + 1 - d(P)} = \frac{N}{n}.$$

By Proposition 3(a) and Lemma 1 we have the inequality

$$\begin{aligned} 0 = \theta - \frac{N}{n} &= \frac{2N - n + 1 - \#P}{n + 1 - d(P)} - \frac{N}{n} = \frac{(N - n)(n - 1) + Nd(P) - n\#P}{n(n + 1 - d(P))} \\ &\geq \frac{(N - n)(d(P) - 1)}{n(n + 1 - d(P))} \geq 0, \end{aligned}$$

which implies that

$$d(P) = 1 \quad \text{and} \quad \#P = N - n + 1.$$

By Lemma 2(d), Remark 2(a) with Corollary 2 and Lemma 6 we obtain the inequality

$$1 = d(P) \geq \sum_{j \in P} \omega(j) \geq (N - n + 1)\lambda \geq 1$$

and we have

$$\lambda = \frac{1}{N - n + 1} = \omega(j) \quad (j \in P).$$

By the choice of P_0 , $1 \leq d(P_0) \leq d(P) = 1$ and so we have

$$d(P_0) = 1 \quad \text{and} \quad \#P_0 = N - n + 1. \quad \square$$

PROPOSITION 6. $P_1 = P_0$.

Proof. By Lemma 2(b) we have the equality

$$\begin{aligned} q - (2N - n + 1) &= \theta \left(\sum_{j=1}^q \omega(j) - n - 1 \right) \\ &= \theta \left(\sum_{j \notin P_k(0) \cup P_0} \omega(j) + \sum_{j \in P_k(0) \cup P_0} \omega(j) - n - 1 \right). \end{aligned}$$

Here, as $P_k(0) \cap P_0 = \emptyset$ by (5) and (6), $\sum_{j \in P_k(0)} \omega(j) = d_k = n$ (Proposition 2) and $\sum_{j \in P_0} \omega(j) = 1$ (Proposition 5(a), (c)), we have

$$\sum_{j \in P_k(0) \cup P_0} \omega(j) = \sum_{j \in P_k(0)} \omega(j) + \sum_{j \in P_0} \omega(j) = n + 1,$$

so that we have the equality

$$(C_1) \quad q - (2N - n + 1) = \theta \sum_{j \notin P_k(0) \cup P_0} \omega(j).$$

As $P_0 \subset P_1$, $\theta\omega(j) = 1$ for $j \notin P_1$ and $\theta\omega(j) < 1$ for $j \in P_1$ by the definition

of P_1 , $\#P_k(0) = N$ (Proposition 3(b)), $\#P_0 = N - n + 1$ (Proposition 5(a)) and $P_k(0) \cap P_1 = \phi$ by (5), we have

$$\begin{aligned} (C_2) \quad \theta \sum_{j \notin P_k(0) \cup P_0} \omega(j) &= q - (2N - n + 1) - \#(P_1 - P_0) + \theta \sum_{j \in P_1 - P_0} \omega(j) \\ &= q - (2N - n + 1) - \sum_{j \in P_1 - P_0} (1 - \theta\omega(j)), \end{aligned}$$

From (C₁) and (C₂) we have the equality

$$\sum_{j \in P_1 - P_0} (1 - \theta\omega(j)) = 0.$$

If $P_0 \subsetneq P_1$, we have a contradiction since $1 - \theta\omega(j) > 0$ for $j \in P_1$. This means that $P_1 = P_0$ must hold. \square

PROPOSITION 7. *Any n elements of $X - \{\mathbf{a}_j \mid j \in P_0\}$ are linearly independent. In particular, any n elements of $X_k(0)$ are linearly independent.*

Proof. Let $\mathbf{b}_1, \dots, \mathbf{b}_n$ be any n elements of $X - \{\mathbf{a}_j \mid j \in P_0\}$. Then, the set

$$\{\mathbf{b}_1, \dots, \mathbf{b}_n\} \cup \{\mathbf{a}_j \mid j \in P_0\}$$

contains $n + 1$ linearly independent elements since X is in N -subgeneral position. As $d(P_0) = 1$, $\mathbf{b}_1, \dots, \mathbf{b}_n$ must be linearly independent. As $X_k(0) \subset X - \{\mathbf{a}_j \mid j \in P_0\}$, we have the last assertion. \square

Summarizing Propositions from 1 through 7, we obtain the following

THEOREM 1. *Suppose that*

- (i) $N > n \geq 2$;
- (ii) *there are vectors $\mathbf{a}_1, \dots, \mathbf{a}_q \in X$ ($2N - n + 1 < q < \infty$) satisfying*

$$\sum_{j=1}^q \delta(\mathbf{a}_j, f) = 2N - n + 1.$$

If $\Omega_k < 1$ for some k ($1 \leq k \leq n + 1$), then

- (a) $X_k(0) \subset \{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ and $\#X_k(0) = N$;
- (b) *there is a subset $P \subset Q$ satisfying*

$$\#P = N - n + 1, \quad d(P) = 1, \quad \delta(\mathbf{a}_j, f) = 1 \quad (j \in P)$$

and

$$X_k(0) \cap \{\mathbf{a}_j \mid j \in P\} = \phi;$$

- (c) *any n elements of $X - \{\mathbf{a}_j \mid j \in P\}$ are linearly independent.*

4. Proof of Theorem when $q = \infty$

Let $[f_1, \dots, f_{n+1}]$, X , $X_k(0)$, θ and ω etc. be as in Section 1, 2 or 3. From Theorem A, it is easy to see that the set

$$\{\mathbf{a} \in X \mid \delta(\mathbf{a}, f) > 0\}$$

is at most countable and

$$\sum_{\mathbf{a} \in X} \delta(\mathbf{a}, f) \leq 2N - n + 1.$$

In this section we consider a holomorphic curve f with an infinite number of vectors $\mathbf{a}_j \in X$ such that

$$\delta(\mathbf{a}_j, f) > 0 \quad (j = 1, 2, 3, \dots).$$

We put

$$N = \{1, 2, 3, \dots\} \quad (\text{the set of positive integers});$$

$$Y = \{\mathbf{a}_j \mid j \in N\};$$

$$\mathcal{O}_\infty = \{P \subset N \mid 0 < \#P \leq N + 1\}$$

and for any non-empty, finite subset P of N , we use

$$V(P) \quad \text{and} \quad d(P)$$

as in Section 2.

DEFINITION 2 (see [9], p. 144). We put

$$\lambda_\infty = \min_{P \in \mathcal{O}_\infty} \frac{d(P)}{\#P}.$$

Note that the set $\{d(P)/\#P \mid P \in \mathcal{O}_\infty\}$ is a finite set. We have the followings ([9], p. 144):

$$(a_\infty) \quad 1/(N - n + 1) \leq \lambda_\infty \leq (n + 1)/(N + 1);$$

(b_∞) (the inequality (12) in [9])

$$\sum_{j=1}^{\infty} \delta(\mathbf{a}_j, f) \leq (n + 1)/\lambda_\infty.$$

From now on throughout this section we suppose that

(i) $N > n \geq 2$;

(ii) there exists a subset $Y = \{\mathbf{a}_j \mid j \in N\}$ of X satisfying $\delta(\mathbf{a}_j, f) > 0$ and

$$(7) \quad \sum_{j=1}^{\infty} \delta(\mathbf{a}_j, f) = 2N - n + 1;$$

(iii) $\Omega_k < 1$ for some k ($1 \leq k \leq n + 1$).

Note that we obtain the inequality

$$(8) \quad \lambda_\infty \leq (n+1)/(2N-n+1)$$

from (7) and (b_∞) . Let P_0 be an element of \mathcal{O}_∞ satisfying

$$\frac{d(P_0)}{\#P_0} = \lambda_\infty$$

and let ε be any positive number satisfying

$$(9) \quad 0 < \varepsilon < (N-n)(1-\Omega_k)/(N-n+1)(n+1).$$

We restrict the number ε as in (9) for the forthcoming Propositions from 8 through 13 to hold.

Now, for the number ε in (9), there exists $p \in N$ satisfying $\{1, 2, \dots, p\} \supset P_0$, $p > 2N-n+1$ and

$$(10) \quad 2N-n+1-\varepsilon < \sum_{j=1}^p \delta(\mathbf{a}_j, f).$$

For an integer q not less than p , we put

$$Q = \{1, 2, \dots, q\}.$$

Note that $2N-n+1 < q < \infty$. For this Q , we use θ_q , ω_q and λ_q instead of θ , ω and λ in Section 2 respectively. Note that

$$(11) \quad \lambda_q = \lambda_\infty$$

since $Q \supset P_0$. Further we obtain the following inequalities from the equality (2) in [9]:

$$(12) \quad n+1 - \frac{\varepsilon}{\theta_q} < \sum_{j=1}^q \omega_q(j) \delta(\mathbf{a}_j, f)$$

$$(13) \quad \sum_{j=1}^q (1 - \theta_q \omega_q(j)) (1 - \delta(\mathbf{a}_j, f)) < \varepsilon.$$

From now on we put $\varepsilon_1 = \varepsilon/(1-\Omega_k)$ for simplicity. Then,

$$(14) \quad 0 < \varepsilon_1 < (N-n)/(N-n+1)(n+1).$$

PROPOSITION 8. $X_k(0) \subset \{\mathbf{a}_1, \dots, \mathbf{a}_q\}$.

Proof. If there exists a vector $\mathbf{a} \in X_k(0)$ satisfying $\mathbf{a} \notin \{\mathbf{a}_1, \dots, \mathbf{a}_q\}$, then by Lemma 4, Theorem A and (10) we have the inequality

$$\begin{aligned}
2N - n + 1 - \varepsilon &< \sum_{j=1}^q \delta(\mathbf{a}_j, f) \leq 2N - n + 1 - \delta(\mathbf{a}, f) \leq 2N - n + 1 - (1 - \Omega_k) \\
&< 2N - n + 1 - \varepsilon
\end{aligned}$$

as $p \leq q$ and $\varepsilon < 1 - \Omega_k$ from (9). This is a contradiction. \square

We put

$$P_k(0) = \{j \in Q \mid \mathbf{a}_j \in X_k(0)\} \quad \text{and} \quad d_k(q) = \sum_{j \in P_k(0)} \omega_q(j).$$

Note that

$$(15) \quad \#P_k(0) \leq N \quad \text{and} \quad d_k(q) \leq d(P_k(0)) \leq n.$$

In fact, we have $\#P_k(0) = \#X_k(0) \leq N$ as X is in N -subgeneral position. We have $d_k(q) \leq d(P_k(0))$ by Lemma 2(d) and $d(P_k(0)) \leq n$ by the definition of $X_k(0)$.

PROPOSITION 9. $n - \varepsilon_1/\theta_q < d_k(q)$.

Proof. From (12) and Lemma 3(I) we have the inequality

$$n + 1 - \varepsilon/\theta_q < \sum_{j=1}^q \omega_q(j) \delta(\mathbf{a}_j, f) \leq d_k(q) + 1 + (n - d_k(q))\Omega_k$$

from which we obtain

$$(n - d_k(q))(1 - \Omega_k) < \varepsilon/\theta_q$$

and so $n - \varepsilon_1/\theta_q < d_k(q)$ as $\Omega_k < 1$ and $\varepsilon_1 = \varepsilon/(1 - \Omega_k)$. \square

PROPOSITION 10. (a) $\theta_q < (N + \varepsilon_1)/n$ and (b) $\#P_k(0) = N$.

Proof. From Proposition 9, Lemma 2(a) and (15), we have the inequality

$$(D) \quad \theta_q(n - \varepsilon_1/\theta_q) < \theta_q d_k(q) = \theta_q \sum_{j \in P_k(0)} \omega_q(j) \leq \#P_k(0) \leq N,$$

from which we obtain $\theta_q < (N + \varepsilon_1)/n$ easily. Next, from (D) and Note 1, we obtain that $N - \varepsilon_1 < \#P_k(0) \leq N$, so that $\#P_k(0) = N$ as $\varepsilon_1 < 1$ from (14). \square

COROLLARY 3. (a) $\theta_q \lambda_q < 1$ and (b) $\lambda_q < (n + 1)/(2N - n + 1)$.

Proof. (a) From (8), (11) and Proposition 10(a), we have

$$\theta_q \lambda_q < \frac{N + \varepsilon_1}{n} \frac{n + 1}{2N - n + 1}$$

and by (14) and the assumption (i) it is easy to see that

$$\frac{N + \varepsilon_1}{n} \frac{n + 1}{2N - n + 1} < 1.$$

We have (a) of this corollary.

(b) By (8) and (11), we have $\lambda_q \leq (n + 1)/(2N - n + 1)$. If λ_q is equal to $(n + 1)/(2N - n + 1)$, then by Remark 2(b) we have $\theta_q \lambda_q = 1$, which contradicts (a) of this corollary. We have (b) of this corollary. \square

Put

$$P_1 = \{j \in Q \mid \theta_q \omega_q(j) < 1, j \notin P_k(0)\}.$$

Note that

$$(16) \quad P_1 \cap P_k(0) = \emptyset.$$

PROPOSITION 11. $N - n + 1 \leq \#P_1$.

Proof. From Lemma 2(b) and (16) we have the equality

$$\begin{aligned} (E_1) \quad q - (2N - n + 1) &= \theta_q \left\{ \sum_{j=1}^q \omega_q(j) - n - 1 \right\} \\ &= \theta_q \left\{ \sum_{j \in P_k(0)} \omega_q(j) + \sum_{j \in P_1} \omega_q(j) + \sum_{j \notin P_k(0) \cup P_1} \omega_q(j) - n - 1 \right\} \end{aligned}$$

and by Proposition 9

$$> \theta_q \left\{ \sum_{j \in P_1} \omega_q(j) + \sum_{j \notin P_k(0) \cup P_1} \omega_q(j) - (1 + \varepsilon_1/\theta_q) \right\}.$$

Here, by (a_∞) , (11) and Remark 2(a) with Corollary 3(b) we have

$$\sum_{j \in P_1} \omega_q(j) \geq \frac{\#P_1}{N - n + 1}$$

and as $\theta_q \omega_q(j) = 1$ for $j \notin P_k(0) \cup P_1$ by Lemma 2(a) and the definition of P_1 , we have

$$\theta_q \sum_{j \notin P_k(0) \cup P_1} \omega_q(j) = q - \#P_k(0) - \#P_1,$$

so that we have the inequality

$$(E_2) \quad \text{the last term of } (E_1) \geq \frac{\theta_q \#P_1}{(N-n+1)} + q - \#P_k(0) - \#P_1 - \theta_q - \varepsilon_1.$$

From (E_1) and (E_2) we obtain the following inequality by Proposition 10(b)

$$\#P_1 \left(1 - \frac{\theta_q}{N-n+1} \right) > N-n+1 - \theta_q - \varepsilon_1,$$

which reduces to the inequality

$$\#P_1(N-n+1 - \theta_q) > (N-n+1)(N-n+1 - \theta_q - \varepsilon_1).$$

Here, by Proposition 10(a) and by the fact that $0 < \varepsilon_1 < 1$ from (14) we have the inequality

$$N-n+1 - \theta_q > N-n+1 - \frac{N+\varepsilon_1}{n} = \frac{(N-n)(n-1) - \varepsilon_1}{n} > 0$$

as $N > n \geq 2$ (the assumption (ii)), so that we have

$$\begin{aligned} \#P_1 &> (N-n+1) \left(1 - \frac{\varepsilon_1}{N-n+1 - \theta_q} \right) \\ &> (N-n+1) \left(1 - \frac{\varepsilon_1}{(N-n+1 - (2N-n+1)/(n+1))} \right) \\ &= (N-n+1) \left(1 - \frac{(n+1)\varepsilon_1}{(N-n)(n-1)} \right) \\ &> N-n \end{aligned}$$

by Lemma 2(c) and (14). This means that $\#P_1 \geq N-n+1$. \square

PROPOSITION 12. (a) $\#P_0 = N-n+1$, (b) $d(P_0) = 1$ and (c) $\theta_q = N/n$.

Proof. By the definition of θ_q and the choice of P_0 , there exists a set P satisfying

$$P_0 \subset P, \quad 1 \leq d(P) \leq n$$

and

$$(17) \quad \theta_q = \frac{2N-n+1 - \#P}{n+1 - d(P)}.$$

By Proposition 10(a), (17) and Lemma 1 we have the inequality

$$\begin{aligned}
(\text{F}) \quad 0 > \theta_q - (N + \varepsilon_1)/n &= \theta_q - N/n - \varepsilon_1/n \\
&= \frac{(N - n)(n - 1) + Nd(P) - n\#P}{n(n + 1 - d(P))} - \frac{\varepsilon_1}{n} \\
&\geq \frac{(N - n)(d(P) - 1)}{n(n + 1 - d(P))} - \frac{\varepsilon_1}{n}.
\end{aligned}$$

First we prove that $d(P) = 1$. Suppose that $d(P) \geq 2$. Then, from (F) we have the inequality

$$\frac{\varepsilon_1}{n} > \frac{N - n}{n(n - 1)},$$

which reduces to the inequality

$$\varepsilon_1 > (N - n)/(n - 1),$$

which contradicts (14). This means that $d(P)$ must be equal to 1.

As $d(P) = 1$, we have from (17) and Note 1 that

$$\theta_q = \frac{2N - n + 1 - \#P}{n} \geq \frac{N}{n},$$

from which we have that $\#P \leq N - n + 1$. On the other hand, as

$$\theta_q = \frac{2N - n + 1 - \#P}{n} < \frac{N + \varepsilon_1}{n}$$

by Proposition 10(a), we have the following inequality by (14)

$$\#P > N - n + 1 - \varepsilon_1 > N - n + 1 - \frac{N - n}{(N - n + 1)(n + 1)} > N - n.$$

We have that $\#P = N - n + 1$. Substituting $\#P = N - n + 1$ and $d(P) = 1$ in (17) we obtain that $\theta_q = N/n$.

Next, by Lemma 2(d), (a_∞) , (11) and Remark 2(a) with Corollary 3(b) we have the inequality

$$1 = d(P) \geq \sum_{j \in P} \omega_q(j) \geq (N - n + 1)\lambda_q \geq 1$$

since $d(P) = 1$ as is proved above, so that we have

$$\lambda_q = \frac{1}{N - n + 1} = \omega_q(j) \quad (j \in P).$$

As $1 \leq d(P_0) \leq d(P) = 1$, we have $d(P_0) = 1$. By the choice of P_0 , we have the equality

$$\frac{1}{\#P_0} = \frac{d(P_0)}{\#P_0} = \lambda_q = \frac{1}{N-n+1},$$

from which we have that $\#P_0 = N - n + 1$. □

COROLLARY 4. $\lambda_q = \lambda_\infty = 1/(N - n + 1) = \omega_q(j)$ ($j \in P_0$).

PROPOSITION 13. (a) $P_1 = P_0$ and (b) $d_k(q) = n$.

Proof. First we note that

$$(18) \quad \theta_q \omega_q(j) = \frac{N}{n(N-n+1)} < 1 \quad (j \in P_0)$$

as $\theta_q = N/n$ (Proposition 12(c)) and $\omega_q(j) = 1/(N - n + 1)$ for $j \in P_0$ (Corollary 4). Next, we prove that $P_0 \cap P_k(0) = \phi$. Suppose to the contrary that $P_0 \cap P_k(0) \neq \phi$. As $d(P_0) = 1$, we have $P_0 \subset P_k(0)$. Then, by Propositions 9, 12(a), Corollary 4 and Lemma 2(a) we have

$$\begin{aligned} n - \varepsilon_1 / \theta_q < d_k(q) &= \sum_{j \in P_k(0)} \omega_q(j) = \sum_{j \in P_0} \omega_q(j) + \sum_{j \in P_k(0) - P_0} \omega_q(j) \\ &\leq 1 + \frac{\#(P_k(0) - P_0)}{\theta_q}. \end{aligned}$$

By Propositions 10(b), 12(a) and 12(c) the last term of this inequality is equal to

$$1 + \frac{(n-1)n}{N},$$

so that we have the inequality

$$\frac{(n-1)(N-n)}{n} < \varepsilon_1.$$

This contradicts (14). This implies that

$$(19) \quad P_0 \cap P_k(0) = \phi.$$

(18) and (19) mean that $P_0 \subset P_1$. By Lemma 2(b) we have the equality

$$\begin{aligned} q - (2N - n + 1) &= \theta_q \left(\sum_{j=1}^q \omega_q(j) - n - 1 \right) \\ &= \theta_q \left(\sum_{j \notin P_k(0) \cup P_0} \omega_q(j) + \sum_{j \in P_k(0) \cup P_0} \omega_q(j) - n - 1 \right). \end{aligned}$$

Here, as $P_k(0) \cap P_0 = \phi$, $\sum_{j \in P_k(0)} \omega_q(j) = d_k(q)$ (the definition of $d_k(q)$) and $\sum_{j \in P_0} \omega_q(j) = 1$ (Proposition 12(a), Corollary 4), we have

$$\sum_{j \in P_k(0) \cup P_0} \omega_q(j) = \sum_{j \in P_k(0)} \omega_q(j) + \sum_{j \in P_0} \omega_q(j) = d_k(q) + 1,$$

so that we have the equality

$$(G_1) \quad q - (2N - n + 1) = \theta_q \sum_{j \notin P_k(0) \cup P_0} \omega_q(j) - \theta_q(n - d_k(q)).$$

As $P_0 \subset P_1$, $\theta_q \omega_q(j) = 1$ for $j \notin P_k(0) \cup P_1$ and $\theta_q \omega_q(j) < 1$ for $j \in P_1$ by Lemma 2(a) and the definition of P_1 , $\#P_k(0) = N$ (Proposition 10(b)), $\#P_0 = N - n + 1$ (Proposition 12(a)) and $P_k(0) \cap P_1 = \emptyset$ by the definition of P_1 , we have

$$\begin{aligned} (G_2) \quad \theta_q \sum_{j \notin P_k(0) \cup P_0} \omega_q(j) &= q - (2N - n + 1) - \#(P_1 - P_0) + \theta_q \sum_{j \in P_1 - P_0} \omega_q(j) \\ &= q - (2N - n + 1) - \sum_{j \in P_1 - P_0} (1 - \theta_q \omega_q(j)). \end{aligned}$$

From (G₁) and (G₂) we have the equality

$$q - (2N - n + 1) = q - (2N - n + 1) - \sum_{j \in P_1 - P_0} (1 - \theta_q \omega_q(j)) - \theta_q(n - d_k(q)),$$

so that we have the equality

$$\sum_{j \in P_1 - P_0} (1 - \theta_q \omega_q(j)) + \theta_q(n - d_k(q)) = 0.$$

If either $P_0 \subsetneq P_1$ or $d_k(q) < n$ holds, we have a contradiction since $\theta_q \omega_q(j) < 1$ for $j \in P_1$ and $d_k(q) \leq n$ by (15). This means that it must hold both $P_1 = P_0$ and $d_k(q) = n$. \square

PROPOSITION 14. For any $j \in P_0$, $\delta(\mathbf{a}_j, f) = 1$.

Proof. Suppose to the contrary that

$$(20) \quad \min_{j \in P_0} \delta(\mathbf{a}_j, f) = \delta < 1.$$

Now, for any positive number ε_2 satisfying

$$(21) \quad 0 < \varepsilon_2 < \min \left\{ \left(1 - \frac{N}{n(N - n + 1)} \right) (1 - \delta), \frac{(N - n)(1 - \Omega_k)}{(N - n + 1)(n + 1)} \right\},$$

we choose $s \in N$ satisfying $S = \{1, \dots, s\} \supset P_0$, $s \geq p$ and

$$(22) \quad 2N - n + 1 - \varepsilon_2 < \sum_{j=1}^s \delta(\mathbf{a}_j, f).$$

Note that $2N - n + 1 < s < \infty$. For this S we use θ_s , ω_s and λ_s instead of θ , ω and λ in Section 2 respectively. Then, by the choice of s the following relations hold from the results obtained in this section:

- (a) $\lambda_s = \lambda_\infty = 1/(N - n + 1) = \omega_s(j)$ for $j \in P_0$ (Corollary 4);
- (b) $\theta_s = N/n$ (Proposition 12(c)).

By the equality (2) in the proof of Lemma 3 in [9], Lemma 3, Remark 1 and (22) we obtain

$$\sum_{j=1}^s (1 - \theta_s \omega_s(j))(1 - \delta(\mathbf{a}_j, f)) < \varepsilon_2$$

so that for any $j \in S$

$$(1 - \theta_s \omega_s(j))(1 - \delta(\mathbf{a}_j, f)) < \varepsilon_2.$$

By the definition of δ , (a) and (b) given above we have the inequality

$$\left(1 - \frac{N}{n(N - n + 1)}\right)(1 - \delta) < \varepsilon_2,$$

which is a contradiction to (21). This means that $\delta = 1$ and we completes the proof of this proposition. \square

As in Proposition 7, we have the following

PROPOSITION 15. *Any n elements of $X - \{\mathbf{a}_j \mid j \in P_0\}$ are linearly independent.*

Summarizing Propositions from 8 through 15 given above we obtain the following

THEOREM 2. *Suppose that*

- (i) $N > n \geq 2$;
- (ii) *there are an infinite number of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots \in X$ satisfying $\delta(\mathbf{a}_j, f) > 0$ ($j \in \mathbf{N}$) and*

$$\sum_{j=1}^{\infty} \delta(\mathbf{a}_j, f) = 2N - n + 1.$$

If $\Omega_k < 1$ for some k ($1 \leq k \leq n + 1$), then

- (a) $X_k(0) \subset \{\mathbf{a}_1, \mathbf{a}_2, \dots\}$ and $\#X_k(0) = N$;
- (b) *there is a subset P of \mathbf{N} satisfying*

$$\#P = N - n + 1, \quad d(P) = 1, \quad \delta(\mathbf{a}_j, f) = 1 \quad (j \in P)$$

and

$$X_k(0) \cap \{\mathbf{a}_j \mid j \in P\} = \emptyset;$$

- (c) *any n elements of $X - \{\mathbf{a}_j \mid j \in P\}$ are linearly independent.*

5. Application

In this section we shall apply the result obtained in Section 3 to exponential curves. For any $n+1$ distinct complex numbers $\mu_1, \mu_2, \dots, \mu_{n+1}$ we define a holomorphic curve f_e by

$$f_e = [e^{\mu_1 z}, e^{\mu_2 z}, \dots, e^{\mu_{n+1} z}].$$

We call it an exponential curve ([11], p. 94). It is easy to see that f_e is transcendental and non-degenerate. We use the notations $X_k(0)$, Ω_k etc. given in Section 1 in this section. We denote by e_1, e_2, \dots, e_{n+1} the standard basis of \mathbb{C}^{n+1} .

Let D be the convex polygon spanned around the $n+1$ points $\mu_1, \mu_2, \dots, \mu_{n+1}$ and ℓ the length of the polygon, where $\ell = 2|\mu_j - \mu_k|$ if the polygon reduces to a segment with the endpoints μ_j and μ_k .

LEMMA 8 ([11], pp. 95–98). $T(r, f_e) = (\ell/2\pi)r + O(1)$.

LEMMA 9. $\#\{k \mid \Omega_k < 1; 1 \leq k \leq n+1\} \geq 2$.

Proof. (a) The case when D is an $n+1$ -gon.

In this case, the points $\mu_1, \mu_2, \dots, \mu_{n+1}$ are the vertices of D . We number without loss of generality the vertices μ_j ($j = 1, \dots, n+1$) in ascending sequence as one goes around D in the positive direction. For any k ($1 \leq k \leq n+1$), the n -gon D_k with the vertices $\mu_1, \dots, \mu_{k-1}, \mu_{k+1}, \dots, \mu_{n+1}$ is convex. Let ℓ_k be the length of the circumference of D_k . By the representation (1) of $T(r, f)$ due to Cartan given in Introduction, by the definition of $t_k(r, f)$ and by Lemma 8 we have

$$t_k(r, f_e) = \frac{\ell_k}{2\pi}r + O(1),$$

and so we have

$$\Omega_k = \limsup_{r \rightarrow \infty} \frac{t_k(r, f)}{T(r, f)} = \frac{\ell_k}{\ell} < 1$$

since $\ell_k < \ell$ as is easily seen.

(b) The case when D is an $m+1$ -gon ($2 \leq m \leq n-1$).

We may suppose without loss of generality that the vertices of D are $\mu_1, \mu_2, \dots, \mu_{m+1}$. The other points $\mu_{m+2}, \dots, \mu_{n+1}$ are on the circumference of D or inside D .

For any k ($1 \leq k \leq m+1$), let D_k be the convex polygon surrounding the points $\mu_1, \dots, \mu_{k-1}, \mu_{k+1}, \dots, \mu_{n+1}$ and let ℓ_k be the length of the circumference of D_k . Then as in (a), we have

$$t_k(r, f_e) = \frac{\ell_k}{2\pi}r + O(1),$$

and so we have

$$\Omega_k = \limsup_{r \rightarrow \infty} \frac{t_k(r, f)}{T(r, f)} = \frac{\ell_k}{\ell} < 1$$

since $\ell_k < \ell$ as is easily seen by an application of the triangle inequality.

(c) The case when D reduces to a segment L .

We may suppose without loss of generality that

(i) μ_1 and μ_{n+1} are the endpoints of L ;

(ii) The points $\mu_1, \mu_2, \dots, \mu_{n+1}$ are in ascending sequence as one goes from μ_1 to μ_{n+1} on L .

Then, as in (a) we have

$$t_1(r, f_e) = \frac{1}{\pi} |\mu_2 - \mu_{n+1}| r + O(1), \quad t_{n+1}(r, f_e) = \frac{1}{\pi} |\mu_1 - \mu_n| r + O(1)$$

and

$$T(r, f_e) = \frac{1}{\pi} |\mu_1 - \mu_{n+1}| r + O(1),$$

from which we obtain

$$\square \quad \Omega_1 = \frac{|\mu_2 - \mu_{n+1}|}{|\mu_1 - \mu_{n+1}|} < 1 \quad \text{and} \quad \Omega_{n+1} = \frac{|\mu_1 - \mu_n|}{|\mu_1 - \mu_{n+1}|} < 1.$$

LEMMA 10. 1) $\#\{\mathbf{a} \in X \mid \delta(\mathbf{a}, f_e) > 0\} \leq N(n+1)$.

2) $\delta(\mathbf{a}, f_e) = 1$ if and only if $\mathbf{a} = a\mathbf{e}_k$ ($a \neq 0$) for some k ($1 \leq k \leq n+1$) and for some nonzero constant a .

Proof. 1) Let $\mathbf{a} = (a_1, a_2, \dots, a_{n+1})$ be an element of X satisfying $\delta(\mathbf{a}, f_e) > 0$. Then, at least one of a_1, a_2, \dots, a_{n+1} is equal to zero.

In fact, suppose to the contrary that $a_j \neq 0$ ($j = 1, \dots, n+1$). Then $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+1}$ and \mathbf{a} are in general position and by Theorem A for $N = n$, we have

$$\sum_{j=1}^{n+1} \delta(\mathbf{e}_j, f_e) + \delta(\mathbf{a}, f_e) \leq n+1,$$

from which we have $\delta(\mathbf{a}, f_e) = 0$ since $\delta(\mathbf{e}_j, f_e) = 1$ ($j = 1, \dots, n+1$).

This means that

$$\{\mathbf{a} \in X \mid \delta(\mathbf{a}, f_e) > 0\} \subset \bigcup_{k=1}^{n+1} X_k(0)$$

and as X is in N -subgeneral position, $\#X_k(0) \leq N$ ($k = 1, \dots, n+1$). Due to these facts we reach to the fact that

$$\#\{\mathbf{a} \in X \mid \delta(\mathbf{a}, f_e) > 0\} \leq N(n+1).$$

2) If $\mathbf{a} = a\mathbf{e}_k$ ($a \neq 0$), then it is trivial that $\delta(\mathbf{a}, f_e) = 1$.
Conversely, suppose that

$$\mathbf{a} = a_{j_1}\mathbf{e}_{j_1} + \cdots + a_{j_m}\mathbf{e}_{j_m} \quad (a_{j_1} \neq 0, \dots, a_{j_m} \neq 0; 2 \leq m \leq n).$$

Let

$$g_e = [e^{\alpha_1 z}, \dots, e^{\alpha_m z}] \quad (\alpha_p = \mu_{j_p} \quad (p = 1, \dots, m)).$$

Then, g_e is a transcendental and non-degenerate exponential curve and by Lemma 8

$$T(r, g_e) = \frac{\ell'}{2\pi}r + O(1),$$

where $(0 < \ell' (\leq \ell))$ is the length of the convex polygon spanned around the points $\alpha_1, \dots, \alpha_m$.

As $N(r, \mathbf{a}, f_e) = N(r, \mathbf{a}, g_e)$ and

$$\delta(\mathbf{a}, g_e) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \mathbf{a}, g_e)}{T(r, g_e)} = 0$$

by 1) of this lemma, we have

$$\begin{aligned} \delta(\mathbf{a}, f_e) &= 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \mathbf{a}, f_e)}{T(r, f_e)} \\ &= 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \mathbf{a}, g_e)}{T(r, g_e)} \cdot \frac{T(r, g_e)}{T(r, f_e)} \\ &= 1 - \frac{\ell'}{\ell} < 1. \end{aligned} \quad \square$$

Using these lemmas we obtain the following

THEOREM 3. *When $N > n \geq 2$, for any exponential curve f_e*

$$\sum_{\mathbf{a} \in X} \delta(\mathbf{a}, f_e) < 2N - n + 1.$$

Proof. Suppose to the contrary that there exists an exponential curve f_e satisfying

$$\sum_{\mathbf{a} \in X} \delta(\mathbf{a}, f_e) = 2N - n + 1.$$

Then, as the number of $\mathbf{a} \in X$ satisfying $\delta(\mathbf{a}, f_e) > 0$ is finite by Lemma 10-1), let $\mathbf{a}_1, \dots, \mathbf{a}_q$ be the elements of X satisfying

$$\delta(\mathbf{a}_j, f_e) > 0 \quad (j = 1, \dots, q)$$

and

$$(23) \quad \sum_{j=1}^q \delta(\mathbf{a}_j, f_e) = 2N - n + 1,$$

where $2N - n + 1 \leq q < \infty$.

(I) The case when $q = 2N - n + 1$.

In this case, as $q = 2N - n + 1$ and $\delta(\mathbf{a}_j, f_e) \leq 1$ we obtain from (23) that

$$\delta(\mathbf{a}_j, f_e) = 1 \quad (j = 1, \dots, 2N - n + 1).$$

By Lemma 10-2), for each $j = 1, \dots, 2N - n + 1$ there exists some k ($1 \leq k \leq n + 1$) satisfying $\mathbf{a}_j = \alpha_j \mathbf{e}_k$.

Put for each $k = 1, \dots, n + 1$

$$x_k = \#\{\mathbf{a}_j \mid \mathbf{a}_j = \alpha_j \mathbf{e}_k; \alpha_j \neq 0, 1 \leq j \leq 2N - n + 1\}.$$

Then, by (23) and $q = 2N - n + 1$ we have

$$(24) \quad \sum_{k=1}^{n+1} x_k = 2N - n + 1.$$

As $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{2N-n+1}$ are in N -subgeneral position and $2N - n + 1 > N + 1$, it must hold that $1 \leq x_k$ for each k and

$$(25) \quad \sum_{k=1}^{n+1} x_k - x_p \leq N, \quad (p = 1, 2, \dots, n + 1).$$

Summing up $n + 1$ inequalities of (25), we obtain

$$(26) \quad n \sum_{k=1}^{n+1} x_k \leq N(n + 1).$$

From (24) and (26) we obtain the inequality

$$n(2N - n + 1) \leq N(n + 1),$$

from which we have the inequality

$$(N - n)(n - 1) \leq 0,$$

which is impossible since $N > n \geq 2$.

(II) The case when $2N - n + 1 < q < \infty$.

By Lemma 9 we may suppose that

$$\Omega_\mu < 1 \quad \text{and} \quad \Omega_\nu < 1 \quad (1 \leq \mu \neq \nu \leq n + 1).$$

By Theorem 1 for $k = \mu$

(a) $X_\mu(0) \subset \{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ and $\#X_\mu(0) = N$;

(b) There exists a subset P of $Q = \{1, 2, \dots, q\}$ satisfying

$$\#P = N - n + 1, \quad d(P) = 1, \quad \delta(\mathbf{a}_j, f_e) = 1 \quad (j \in P)$$

and

$$X_\mu(0) \cap \{\mathbf{a}_j \mid j \in P\} = \emptyset.$$

Note that $\#P = N - n + 1 \geq 2$. By Lemma 10-2) and (b) given above we obtain that

$$\mathbf{a}_j = \beta_j \mathbf{e}_\mu \quad (j \in P).$$

This means that $\mathbf{a}_j \in X_\nu(0)$ ($j \in P$), and so if we choose n vectors containing at least two vectors of $\{\mathbf{a}_j \mid j \in P\}$ from $X_\nu(0)$, they are linearly dependent. On the other hand, by Theorem 1(c) for $k = \nu$, any n elements of $X_\nu(0)$ must be linearly independent. This is a contradiction.

From (I) and (II) we have that there is no exponential curve f_e satisfying

$$\sum_{\mathbf{a} \in X} \delta(\mathbf{a}, f_e) = 2N - n + 1.$$

We complete the proof of this theorem. □

Remark 3. When $n = 1$, there is an example of exponential curve f_e which satisfies (23) for any $N \geq 2$. Put $f_e = [e^z, e^{2z}]$ and

$$X = \{\mathbf{a}_j = j\mathbf{e}_1 \quad (j = 1, 2, \dots, N), \mathbf{a}_j = j\mathbf{e}_2 \quad (j = N + 1, N + 2, \dots, 2N)\}.$$

Then, X is in N -subgeneral position and

$$\sum_{j=1}^{2N} \delta(\mathbf{a}_j, f_e) = 2N.$$

Acknowledgments. The author thanks the referee for his/her valuable comments to improve the paper.

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