

## SHARP ENDPOINT INEQUALITY FOR MULTILINEAR LITTLEWOOD-PALEY OPERATOR

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### Abstract

We establish a sharp inequality for multilinear Littlewood-Paley operator. As application, we obtain the weighted norm inequalities and  $L \log L$  type endpoint estimate for the multilinear operator.

### 1. Introduction and result

Let  $\psi$  be a function on  $R^n$  which satisfies the following properties:

- (1)  $\int \psi(x) dx = 0$ ,
- (2)  $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$ ,
- (3)  $|\psi(x + y) - \psi(x)| \leq C|y|(1 + |x|)^{-(n+2)}$  when  $2|y| < |x|$ ;

Let  $m$  be a positive integer and  $A$  be a function on  $R^n$ . We denote that  $\Gamma(x) = \{(y, t) \in R_+^{n+1} : |x - y| < t\}$  and the characteristic function of  $\Gamma(x)$  by  $\chi_{\Gamma(x)}$ . The multilinear Littlewood-Paley operator is defined by

$$S_\psi^A(f)(x) = \left[ \iint_{\Gamma(x)} |F_t^A(f)(y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^A(f)(x, y) = \int_{R^n} \frac{f(z)\psi_t(y - z)}{|x - z|^m} R_{m+1}(A; x, z) dz,$$

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha,$$

and  $\psi_t(x) = t^{-n}\psi(x/t)$  for  $t > 0$ . We write  $F_t(f)(x) = f * \psi_t(x)$ . We also define that

$$S_\psi(f)(x) = \left( \iint_{\Gamma(x)} |F_t(f)(x)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

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which is the Littlewood-Paley operator (see [12]).

Let  $H$  be the Hilbert space  $H = \left\{ h : \|h\| = \left( \iint_{R_+^{n+1}} |h(t)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} < \infty \right\}$ .

Then for each fixed  $x \in R^n$ ,  $F_t^A(f)(x)$  and  $F_t(f)(x)$  may be viewed as a mapping from  $(0, +\infty)$  to  $H$ , and it is clear that

$$S_\psi^A(f)(x) = \|\chi_{\Gamma(x)} F_t^A(f)(x, y)\|.$$

Note that when  $m = 0$ ,  $S_\psi^A$  is just the commutator of Littlewood-Paley operator (see [9]), while when  $m > 0$ , it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–5]). In [8], authors establish a variant sharp estimate for the multilinear singular integral operators. The main purpose of this paper is to establish a sharp estimate for the multilinear Littlewood-Paley operator, then the weighted norm inequalities and the  $L \log L$  type endpoint estimate for the multilinear operator are obtained by using the sharp estimate. We point out that some of our ideas come from [8] and [10]. First, let us introduce some notation (see [6] [7] [10]).

For any locally integrable function  $f$ , the sharp function of  $f$  is defined by

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows,  $Q$  will denote a cube with sides parallel to the axes, and  $f_Q = |Q|^{-1} \int_Q f(x) dx$ . It is well-known that

$$f^\#(x) = \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that  $f$  belongs to  $BMO(R^n)$  if  $f^\#$  belongs to  $L^\infty(R^n)$ . For  $0 < r < \infty$ , we denote  $f_r^\#$  by

$$f_r^\#(x) = [(|f|^r)^\#(x)]^{1/r}.$$

Let  $M$  be the Hardy-Littlewood maximal operator, that is

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

we write that  $M_p f = (M(f^p))^{1/p}$ , for  $k \in N$ , we denote by  $M^k$  the operator  $M$  iterated  $k$  times, i.e.,  $M^1 f(x) = Mf(x)$  and

$$M^k f(x) = M(M^{k-1} f)(x) \quad \text{when } k \geq 2.$$

Let  $B$  be a Young function and  $\tilde{B}$  be the complementary associated to  $B$ , we denote that, for a function  $f$

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1 \right\}$$

and the maximal function by

$$M_B f(x) = \sup_{x \in Q} \|f\|_{B, Q};$$

The main Young function to be using in this paper is  $B(t) = t(1 + \log^+ t)$  and its complementary  $\bar{B} = \exp t$ , the corresponding maximal denoted by  $M_{L \log L}$  and  $M_{\exp L}$ . We have the generalized Holder's inequality (see [10])

$$\frac{1}{|Q|} \int_Q |f(y)g(y)| \, dy \leq \|f\|_{B, Q} \|g\|_{B, Q}$$

and the following inequality (in fact they are equivalent), for any  $x \in R^n$

$$M_{L \log L} f(x) \leq CM^2 f(x)$$

and the following inequalities, for all cube  $Q$  any  $b \in BMO(R^n)$

$$\|b - b_Q\|_{\exp L, Q} \leq C \|b\|_{BMO}$$

and

$$|b_{2^{k+1}Q} - b_{2Q}| \leq 2k \|b\|_{BMO}.$$

We denote the Muckenhoupt weights by  $A_p$  for  $1 \leq p < \infty$  (see [6]).

Now we state the results in this paper as following.

**THEOREM 1.** *Let  $D^\alpha A \in BMO(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ . Then for any  $0 < r < 1$ , there exists a constant  $C > 0$  such that for any  $f \in C_0^\infty(R^n)$  and any  $x \in R^n$ ,*

$$(S_\psi^A(f))_r^\#(x) \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M^2 f(x).$$

**THEOREM 2.** *Let  $1 < p < \infty$  and  $D^\alpha A \in BMO(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ ,  $w \in A_p$ . Then  $S_\psi^A$  is bounded on  $L^p(w)$ , that is*

$$\|S_\psi^A(f)\|_{L^p(w)} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^p(w)}.$$

**THEOREM 3.** *Let  $D^\alpha A \in BMO(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ ,  $w \in A_1$ . Then there exists a constant  $C > 0$  such that for each  $\lambda > 0$ ,*

$$\begin{aligned} & w(\{x \in R^n : S_\psi^A(f)(x) > \lambda\}) \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \int_{R^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) w(x) \, dx. \end{aligned}$$

As in [10], Theorem 2 and 3 follow from Theorem 1 and the boundedness of  $S_\psi$  with  $M$ . So we only need to prove Theorem 1.

## 2. Some lemmas

We begin with some preliminary lemmas.

LEMMA 1 (Kolmogorov, [7, p. 485]). *Let  $0 < p < q < \infty$  and for any function  $f \geq 0$ . We define that*

$$\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : f(x) > \lambda\}|^{1/q},$$

$$N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p} / \|\chi_E\|_{L^r}, \quad (1/r = 1/p - 1/q)$$

where the sup is taken for all measurable sets  $E$  with  $0 < |E| < \infty$ . Then

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

LEMMA 2 ([10, p. 165]). *Let  $w \in A_1$ . Then there exists a constant  $C > 0$  such that for any function  $f$  and for all  $\lambda > 0$ ,*

$$w(\{y \in \mathbb{R}^n : M^2 f(y) > \lambda\}) \leq C \lambda^{-1} \int_{\mathbb{R}^n} |f(y)| (1 + \log^+(\lambda^{-1} |f(y)|)) w(y) dy.$$

LEMMA 3 ([3, p. 448]). *Let  $A$  be a function on  $\mathbb{R}^n$  and  $D^\alpha A \in L^q(\mathbb{R}^n)$  for all  $\alpha$  with  $|\alpha| = m$  and some  $q > n$ . Then*

$$|R_m(A; x, y)| \leq C |x - y|^m \sum_{|\alpha|=m} \left( \frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where  $\tilde{Q}(x, y)$  is the cube centered at  $x$  and having side length  $5\sqrt{n}|x - y|$ .

LEMMA 4. *Let  $1 < p < \infty$ ,  $1 < r \leq \infty$ ,  $1/q = 1/p + 1/r$  and  $D^\alpha A \in BMO(\mathbb{R}^n)$  for all  $\alpha$  with  $|\alpha| = m$ . Then  $S_\psi^A$  is bound from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ , that is*

$$\|S_\psi^A(f)\|_{L^q} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^p}.$$

*Proof.* By Minkowski inequality and the condition of  $\psi$ , we have

$$\begin{aligned} S_\psi^A(f)(x) &\leq \int_{\mathbb{R}^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^m} \left( \int_{\Gamma(x)} |\psi_t(y - z)|^2 \frac{dy dt}{t^{1+n}} \right)^{1/2} dz \\ &\leq C \int_{\mathbb{R}^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^m} \left( \int_0^\infty \int_{|x-y| \leq t} \frac{t^{-2n}}{(1 + |y - z|/t)^{2n+4}} \frac{dy dt}{t^{1+n}} \right)^{1/2} dz \\ &\leq C \int_{\mathbb{R}^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^m} \left( \int_0^\infty \int_{|x-y| \leq t} \frac{t^{-2n}}{(1 + |y - z|/t)^{2n+4}} \frac{dy dt}{t^{n+1}} \right)^{1/2} dz, \\ &\leq C \int_{\mathbb{R}^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^m} \left( \int_0^\infty \int_{|x-y| \leq t} \frac{2^{2n+4} \cdot t^{1-n}}{(2t + |y - z|)^{2n+2}} dy dt \right)^{1/2} dz, \end{aligned}$$

noting that  $2t + |y - z| \geq 2t + |x - z| - |x - y| \geq t + |x - z|$  when  $|x - y| \leq t$  and

$$\int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+2}} = C|x - z|^{-2n},$$

we obtain

$$\begin{aligned} S_\psi^A(f)(x) &\leq C \int_{R^n} \frac{|f(z)|}{|x - z|^m} |R_{m+1}(A; x, z)| \left( \int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+2}} \right)^{1/2} dz \\ &= C \int_{R^n} \frac{|f(z)|}{|x - z|^{m+n}} |R_{m+1}(A; x, z)| dz, \end{aligned}$$

thus, the lemma follows from [4] [5].

### 3. Proof of Theorems

We first prove Theorem 1.

*Proof of Theorem 1.* For  $\tilde{x} \in R^n$ , let  $Q = Q(x_0, l)$  be a cube centered at  $x_0$  and having side length  $l$  such that  $\tilde{x} \in Q$ . It suffice to prove for  $f \in C_0^\infty(R^n)$  and some constant  $C_0$ , the following inequality holds:

$$\left( \frac{1}{|Q|} \int_Q |S_\psi^A(f)(x) - C_0|^r dx \right)^{1/r} \leq CM^2 f(\tilde{x}).$$

Set  $\tilde{Q} = 5\sqrt{n}Q$  and  $\tilde{A}(x) = A(x) - \sum_{|z|=m} \frac{1}{z!} (D^z A)_{\tilde{Q}} x^z$ , then  $R_m(A; x, y) = R_m(\tilde{A}; x, y)$  and  $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$  for  $|\alpha| = m$ . We write, for  $f_1 = f\chi_{\tilde{Q}}$  and  $f_2 = f\chi_{R^n \setminus \tilde{Q}}$ ,

$$\begin{aligned} F_t^A(f)(x, y) &= \int \frac{R_{m+1}(A; x, z)}{|x - z|^m} \psi_t(y - z) f(z) dz \\ &= \int \frac{R_{m+1}(\tilde{A}; x, z)}{|x - z|^m} \psi_t(y - z) f_2(z) dz + \int \frac{R_m(\tilde{A}; x, z)}{|x - z|^m} \psi_t(y - z) f_1(z) dz \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \frac{(x - z)^\alpha D^\alpha \tilde{A}(z)}{|x - z|^m} \psi_t(y - z) f_1(z) dz \end{aligned}$$

then

$$\begin{aligned} &|S_\psi^A(f)(x) - S_\psi^{\tilde{A}}(f_2)(x_0)| \\ &= \|\chi_{\Gamma(x)} F_t^A(f)(x, y)\| - \|\chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f)(x_0, y)\| \\ &\leq \|\chi_{\Gamma(x)} F_t^A(f)(x, y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f)(x_0, y)\| \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \chi_{\Gamma(x)} F_t \left( \frac{R_m(\tilde{A}; x, \cdot)}{|x - \cdot|^m} f_1 \right) (y) \right\| + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| \chi_{\Gamma(x)} F_t \left( \frac{(x - \cdot)^\alpha}{|x - \cdot|^m} D^\alpha \tilde{A} f_1 \right) (y) \right\| \\
&\quad + \left\| \chi_{\Gamma(x)} F_t^{\tilde{A}}(f_2)(x, y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0, y) \right\| \\
&\equiv I(x) + II(x) + III(x),
\end{aligned}$$

thus,

$$\begin{aligned}
&\left( \frac{1}{|Q|} \int_Q |S_\psi^{\tilde{A}}(f)(x) - S_\psi^{\tilde{A}}(f_2)(x_0)|^r dx \right)^{1/r} \\
&\leq \left( \frac{C}{|Q|} \int_Q I(x)^r dx \right)^{1/r} + \left( \frac{C}{|Q|} \int_Q II(x)^r dx \right)^{1/r} + \left( \frac{C}{|Q|} \int_Q III(x)^r dx \right)^{1/r} \\
&\equiv I + II + III.
\end{aligned}$$

Now, let us estimate  $I, II$  and  $III$ , respectively. First, for  $x \in Q$  and  $y \in \tilde{Q}$ , using Lemma 2, we get

$$R_m(\tilde{A}; x, y) \leq C|x - y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO},$$

thus, by Lemma 1 and the weak type  $(1, 1)$  of  $S_\psi$  (see [9] [12]), we obtain

$$\begin{aligned}
I &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |Q|^{-1} \frac{\|S_\psi(f_1)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^{r/(1-r)}}} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |Q|^{-1} \|S_\psi(f_1)(f_1)\|_{WL^1} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |\tilde{Q}|^{-1} \int_{\tilde{Q}} |f(y)| dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M(f)(\tilde{x});
\end{aligned}$$

For  $II$ , similar to the proof of  $I$ , we get

$$\begin{aligned}
II &\leq C \sum_{|\alpha|=m} |Q|^{-1} \frac{\|S_\psi(D^\alpha \tilde{A} f_1)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^{r/(1-r)}}} \leq C \sum_{|\alpha|=m} |Q|^{-1} \|S_\psi(D^\alpha \tilde{A} f_1)\|_{WL^1} \\
&\leq C \sum_{|\alpha|=m} |Q|^{-1} \int_{\tilde{Q}} |D^\alpha \tilde{A}(y)| |f(y)| dy \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\exp L, \tilde{Q}} \|f\|_{L \log L, \tilde{Q}} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M_{L \log L} f(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M^2 f(\tilde{x});
\end{aligned}$$

Now let us estimate *III*. We write

$$\begin{aligned}
& \chi_{\Gamma(x)} F_t^{\tilde{A}}(f_2)(x, y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0, y) \\
&= \int_{R^n} \left[ \frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m} \right] \chi_{\Gamma(x)} \psi_t(y-z) R_m(\tilde{A}; x, z) f_2(z) dz \\
&+ \int \frac{\chi_{\Gamma(x)} \psi_t(y-z) f_2(z)}{|x_0-z|^m} [R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)] dz \\
&+ \int (\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}) \frac{\psi_t(y-z) R_m(\tilde{A}; x_0, z) f_2(z)}{|x_0-z|^m} dz \\
&- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \left[ \frac{\chi_{\Gamma(x)} (x-z)^\alpha}{|x-z|^m} - \frac{\chi_{\Gamma(x_0)} (x_0-z)^\alpha}{|x_0-z|^m} \right] \psi_t(y-z) D^\alpha \tilde{A}(z) f_2(z) dz \\
&= III_1 + III_2 + III_3 + III_4.
\end{aligned}$$

Note that  $|x-z| \sim |x_0-z|$  for  $x \in Q$  and  $z \in R^n \setminus \tilde{Q}$ . By Lemma 3 and the following inequality (see [7])

$$|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO}, \quad \text{for } Q_1 \subset Q_2,$$

we know that, for  $x \in Q$  and  $z \in 2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}$ ,

$$\begin{aligned}
|R_m(\tilde{A}; x, z)| &\leq C |x-z|^m \sum_{|\alpha|=m} (\|D^\alpha A\|_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,z)} - (D^\alpha A)_{\tilde{Q}}|) \\
&\leq Ck |x-z|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO};
\end{aligned}$$

For  $III_1$ , by the condition on  $\psi$  and similar to the proof of Lemma 4, we get

$$\begin{aligned}
\|III_1\| &\leq \int_{R^n \setminus \tilde{Q}} \frac{|x-x_0| |f(z)|}{|x-z|^{n+m+1}} |R_m(\tilde{A}; x, z)| dz \\
&\leq \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} \frac{|x-x_0| |f(z)|}{|x-z|^{n+m+1}} |R_m(\tilde{A}; x, z)| dz \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=1}^{\infty} k 2^{-k} M(f)(\tilde{x}) \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M(f)(\tilde{x});
\end{aligned}$$

For  $III_2$ , by the formula (see [3]):

$$R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z) = \sum_{|\beta| < m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; x, x_0) (x-z)^\beta$$

and Lemma 3, we have

$$\begin{aligned} |R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)| &\leq C \sum_{|\beta| < m} \sum_{|z|=m} |x - x_0|^{m-|\beta|} |x - z|^{|\beta|} \|D^\alpha A\|_{BMO} \\ &\leq C \sum_{|z|=m} \|D^\alpha A\|_{BMO} |x - x_0| |x - z|^{m-1}, \end{aligned}$$

thus, similar to the proof of Lemma 4

$$\begin{aligned} \|III_2\| &\leq \int_{R^n \setminus \tilde{Q}} \frac{|R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^{m+n}} |f(z)| \, dz \\ &\leq C \sum_{|z|=m} \|D^\alpha A\|_{BMO} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|Q|^{1/n}}{|x_0 - z|^{n+1}} |f(z)| \, dz \\ &\leq C \sum_{|z|=m} \|D^\alpha A\|_{BMO} \sum_{k=1}^{\infty} 2^{-k} \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(z)| \, dz \\ &\leq C \sum_{|z|=m} \|D^\alpha A\|_{BMO} M(f)(\tilde{x}); \end{aligned}$$

For  $III_3$ , similar to the proof of Lemma 4, we obtain

$$\begin{aligned} \|III_3\| &\leq C \int_{R^n} \left( \iint_{R_+^{n+1}} \left[ \frac{|\psi_t(y-z)| |f_2(z)| |R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^m} \right. \right. \\ &\quad \left. \left. \times |\chi_{\Gamma(x)}(y, t) - \chi_{\Gamma(x_0)}(y, t)| \frac{dydt}{t^{n+1}} \right]^2 \frac{dydt}{t^{n+1}} \right)^{1/2} dz \\ &\leq C \int_{R^n} \frac{|f_2(z)| |R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^m} \\ &\quad \times \left| \iint_{\Gamma(x)} \frac{t^{1-n} dydt}{(t + |y - z|)^{2n+2}} - \iint_{\Gamma(x_0)} \frac{t^{1-n} dydt}{(t + |y - z|)^{2n+2}} \right|^{1/2} dz \\ &\leq C \int_{R^n} \frac{|f_2(z)| |R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^m} \\ &\quad \times \left( \iint_{|y| \leq t} \left| \frac{1}{(t + |x + y - z|)^{2n+2}} - \frac{1}{(t + |x_0 + y - z|)^{2n+2}} \right| \frac{dydt}{t^{n-1}} \right)^{1/2} dz \\ &\leq C \int_{R^n} \frac{|f_2(z)| |R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^m} \left( \iint_{|y| \leq t} \frac{|x - x_0| t^{1-n} dydt}{(t + |x + y - z|)^{2n+3}} \right)^{1/2} dz \end{aligned}$$



$$\begin{aligned}
&\leq C \int_{R^n} \frac{|f_2(z)| |x - x_0|^{1/2} |R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^{m+n+1/2}} dz \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=1}^{\infty} 2^{-k/2} \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |f(z)| dz \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M(f)(\tilde{x});
\end{aligned}$$

For  $III_4$ , similar to the proof of  $III_1$  and  $III_3$ , we get

$$\begin{aligned}
\|III_4\| &\leq C \sum_{|\alpha|=m} \int_{R^n} \left( \frac{|x - x_0|}{|x_0 - z|^{n+1}} + \frac{|x - x_0|^{1/2}}{|x_0 - z|^{n+1/2}} \right) |D^\alpha \tilde{A}(y)| |f_2(z)| dz \\
&\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} (2^{-k} + 2^{-k/2}) \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |f(z)| |D^\alpha A(z) - (D^\alpha A)_{\tilde{Q}}| dz \\
&\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-k/2}) (\|D^\alpha A\|_{\exp L, 2^k \tilde{Q}} \|f\|_{L \log L, 2^k \tilde{Q}} \\
&\quad + \|D^\alpha A\|_{BMO} M(f)(\tilde{x})) \\
&\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-k/2}) \|D^\alpha A\|_{BMO} M_{L \log L}(f)(\tilde{x}) \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M^2(f)(\tilde{x}).
\end{aligned}$$

Thus,

$$III \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M^2(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

From Theorem 1 and the weighted boundedness of  $S_\psi$  and  $M$ , we may obtain the conclusion of Theorem 2.

From Theorem 1 and Lemma 2, we may obtain the conclusion of Theorem 3.

## REFERENCES

- [1] J. COHEN, A sharp estimate for a multilinear singular integral on  $R^n$ , Indiana Univ. Math. J. **30** (1981), 693–702.
- [2] J. COHEN AND J. GOSSELIN, On multilinear singular integral operators on  $R^n$ , Studia Math. **72** (1982), 199–223.
- [3] J. COHEN AND J. GOSSELIN, A BMO estimate for multilinear singular integral operators, Illinois J. Math. **30** (1986), 445–465.

- [4] Y. DING, A note on multilinear fractional integrals with rough kernel, *Adv. in Math. (China)* **30** (2001), 238–246.
- [5] Y. DING AND S. Z. LU, Weighted Boundedness for a class of rough multilinear operators, *Acat Math. Sinica, (China)* **17** (2001), 517–526.
- [6] G. B. FOLLAND AND E. M. STEIN, *Hardy spaces on homogenous groups*, Princeton Univ. Press, Princeton, NJ, 1982.
- [7] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Math. Stud. **16**, Amsterdam, 1985.
- [8] G. HU AND D. C. YANG, A variant sharp estimate for multilinear singular integral operators, *Studia Math.* **141** (2000), 25–42.
- [9] L. Z. LIU, Weighted weak type estimates for commutators of Littlewood-Paley operator, *Japanese J. of Math.* **29(1)** (2003), 1–16.
- [10] C. PEREZ, Endpoint estimate for commutators of singular integral operators, *J. Func. Anal.* **128** (1995), 163–185.
- [11] C. PEREZ, Weighted norm inequalities for singular integral operators, *J. London Math. Soc.* **49** (1994), 296–308.
- [12] A. TORCHINSKY, *The real variable methods in harmonic analysis*, Pure and Applied Math. **123**, Academic Press, New York, 1986.

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