

# MEROMORPHIC FUNCTION OF INFINITE ORDER WITH MAXIMUM DEFICIENCY SUM\*

WEILING XIONG

## Abstract

In this paper we prove the following theorem: Let  $f(z)$  be a meromorphic function of infinite order. If  $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$ , then for each positive integer  $k$ , we have  $K(f^{(k)}) = 2k(1 - \delta(\infty, f))/(1 + k - k\delta(\infty, f))$ , where  $K(f^{(k)}) = \lim_{r \rightarrow \infty} (N(r, 1/f^{(k)}) + N(r, f^{(k)}))/T(r, f^{(k)})$  exists. This result improves the results by S. K. Singh and V. N. Kulkarni [1] and Mingliang Fang [2].

## 1 Introduction and results

In this paper, we assume that  $f(z)$  is a nonconstant meromorphic function in the complex plane  $C$ . We shall use the standard notations of the Nevanlinna theory of meromorphic functions (see [3]).

$T(r, f), m(r, f), N(r, f), \bar{N}(r, f), \delta(a, f), S(r, f)$  and so on.

We shall also use the following notations (see [4]):

$$\begin{aligned} T_0(r, f) &= \int_1^r \frac{T(t, f)}{t} dt, & N_0(r, f) &= \int_1^r \frac{N(t, f)}{t} dt, \\ m_0(r, f) &= \int_1^r \frac{m(t, f)}{t} dt, & S_0(r, f) &= \int_1^r \frac{S(t, f)}{t} dt. \end{aligned}$$

Similarly, we use the notations  $m_0(r, a, f)$  and  $N_0(r, a, f)$ . Set

$$\delta_0(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_0(r, a, f)}{T_0(r, a, f)},$$

$$K(f^{(k)}) = \limsup_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)}) + N(r, f^{(k)})}{T(r, f^{(k)})}, \quad (k = 0, 1, 2, \dots).$$

In 1973, S. K. Singh and V. N. Kulkarni [1] proved the following result:

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THEOREM 1.1. *Suppose that  $f$  is a transcendental meromorphic function of finite order satisfying*

$$\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2.$$

*Then*

$$\frac{1 - \delta(\infty, f)}{2 - \delta(\infty, f)} \leq K(f') \leq \frac{2(1 - \delta(\infty, f))}{2 - \delta(\infty, f)}.$$

In 2000, Mingliang Fang [2] proved the following result:

THEOREM 1.2. *Suppose that  $f$  is a transcendental meromorphic function of finite order satisfying*

$$\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2.$$

*Then*

$$K(f^{(k)}) = \frac{2k(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}.$$

In this paper, we shall prove the following theorem:

THEOREM 1.3. *Suppose that  $f$  is a meromorphic function of infinite order. If*

$$\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2,$$

*then for each positive integer  $k$ ,*

$$K(f^{(k)}) = \frac{2k(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}.$$

## 2 Lemmas

LEMMA 1 ([4]). *Suppose that  $f$  is a meromorphic function of infinite order. Then for each complex number  $a$ ,*

$$0 \leq \delta(a, f) \leq \delta_0(a, f) \leq 1, \quad \sum_{a \neq \infty} \delta_0(a, f) + \delta_0(\infty, f) \leq 2.$$

LEMMA 2 ([5]). *Suppose that  $f$  is a meromorphic function of infinite order satisfying  $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$ . Then for each  $k \in \mathbb{N}$ ,*

$$T_0(r, f^{(k)}) = ((k+1) - k\delta_0(\infty, f) + o(1))T_0(r, f),$$

as  $r \rightarrow \infty$  through all values.

Using Lemma 2, we can prove

LEMMA 3 ([5]). *Suppose that  $f$  is a meromorphic function of infinite order satisfying  $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$ . Then for each  $k \in \mathbb{N}$ ,*

$$T(r, f^{(k)}) = ((k+1) - k\delta(\infty, f) + o(1))T(r, f),$$

as  $r \rightarrow \infty$  through all values.

*Proof.* Because the paper [5] is written in Chinese, we give here a sketch of the proof of Lemma 3. Set  $b = (k+1) - k\delta(\infty, f)$ ,  $F_1(x) = T_0(e^x, f^{(k)})$  and  $G_1 = b \cdot T_0(e^x, f)$ . Then by Lemma 2 and by a similar method to the part (II) of the proof of Theorem 1.3 below, we see that  $F_1(x)$  and  $G_1(x)$  satisfy the conditions of Lemma 9. Thus we have

$$\lim_{x \rightarrow \infty} \frac{F_1'(x)}{G_1'(x)} = 1.$$

Hence we obtain the conclusion of Lemma 3.

LEMMA 4 ([6]). *Suppose that  $f$  is a transcendental meromorphic function, and  $a_i$  ( $i = 1, 2, \dots, p$ ) be  $p$  distinct complex numbers. Then for each  $k \in \mathbb{N}$ ,*

$$\sum_{i=1}^p m(r, a_i, f) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

LEMMA 5 ([6]). *Suppose that  $f$  is a transcendental meromorphic function of infinite order,  $S(r, f)$  be any quantity satisfying*

$$S(r, f) \leq A \log^+ T(R, f) + B \log^+ \frac{1}{R-r} + C \log^+ R + D,$$

where  $0 < r < R$ , and  $A, B, C, D$  are positive constants. Then  $\lim_{r \rightarrow \infty} S_0(r, f)/T_0(r, f) = 0$ .

LEMMA 6 ([7]). *Suppose that  $f$  is a transcendental meromorphic function. Then for each positive number  $\varepsilon_0$  and for each  $k \in \mathbb{N}$ ,*

$$\bar{N}(r, f) < \frac{1}{k}N\left(r, \frac{1}{f^{(k)}}\right) + \frac{1}{k}N(r, f) + \varepsilon_0 T(r, f) + S(r, f).$$

LEMMA 7. *Suppose that  $f$  is a meromorphic function of infinite order. If  $\sum_{a \neq \infty} \delta_0(a, f) + \delta_0(\infty, f) = 2$ , then for each positive integer  $k$ ,*

$$\lim_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} = 2 - \delta_0(\infty, f) \quad \text{and} \quad \lim_{r \rightarrow \infty} \frac{N_0(r, f)}{T_0(r, f)} = \lim_{r \rightarrow \infty} \frac{\bar{N}_0(r, f)}{T_0(r, f)}.$$

*Proof.* Since  $\sum_{a \neq \infty} \delta_0(a, f) + \delta_0(\infty, f) = 2$ , then for any positive number  $\varepsilon_0$ , there exist distinct complex numbers  $a_i$  ( $i = 1, 2, \dots, p$ ) such that

$$\sum_{i=1}^p \delta_0(a_i, f) + \delta_0(\infty, f) \geq 2 - \varepsilon_0. \quad (1)$$

By Lemma 4, we have

$$\sum_{i=1}^p m(r, a_i, f) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \quad (2)$$

From (1), (2) and Lemma 5, we have

$$\liminf_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} \geq \sum_{i=1}^p \delta_0(a_i, f) \geq 2 - \delta_0(\infty, f) - \varepsilon_0.$$

Taking  $\varepsilon_0 \rightarrow 0$ , we deduce

$$\liminf_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} \geq 2 - \delta_0(\infty, f). \quad (3)$$

On the other hand, by Lemma 6, we have

$$\begin{aligned} m\left(r, \frac{1}{f^{(k)}}\right) &\leq T(r, f^{(k)}) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ &\leq T(r, f) + k\bar{N}(r, f) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ &\leq T(r, f) + N(r, f) + k\varepsilon_0 T(r, f) + S(r, f). \end{aligned} \quad (4)$$

Thus we have

$$m_0\left(r, \frac{1}{f^{(k)}}\right) \leq T_0(r, f) + N_0(r, f) + k\varepsilon_0 T_0(r, f) + S_0(r, f).$$

Hence we obtain

$$\limsup_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} \leq 2 - \delta_0(\infty, f) + k\varepsilon_0.$$

Taking  $\varepsilon_0 \rightarrow 0$ , we deduce

$$\limsup_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} \leq 2 - \delta_0(\infty, f). \quad (5)$$

From (3) and (5), we have

$$\lim_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} = 2 - \delta_0(\infty, f).$$

Choosing  $k = 1$ , from (2) and (4) we have

$$\sum_{i=1}^p \delta_0(a_i, f) \leq 1 + \liminf_{r \rightarrow \infty} \frac{\bar{N}_0(r, f)}{T_0(r, f)} \leq 1 + \limsup_{r \rightarrow \infty} \frac{N_0(r, f)}{T_0(r, f)} = 2 - \delta_0(\infty, f).$$

From (1), we have

$$2 - \delta_0(\infty, f) - \varepsilon_0 \leq \sum_{i=1}^p \delta_0(a_i, f).$$

Taking  $\varepsilon_0 \rightarrow 0$ , we obtain

$$\liminf_{r \rightarrow \infty} \frac{\bar{N}_0(r, f)}{T_0(r, f)} = \limsup_{r \rightarrow \infty} \frac{N_0(r, f)}{T_0(r, f)},$$

hence

$$\lim_{r \rightarrow \infty} \frac{\bar{N}_0(r, f)}{T_0(r, f)} = \lim_{r \rightarrow \infty} \frac{N_0(r, f)}{T_0(r, f)}.$$

This completes the proof of Lemma 7.

**LEMMA 8** ([8]). *Suppose that  $f(r)$  is a nonnegative and increasing function with  $\lim_{r \rightarrow \infty} f(r)/r^\alpha = \infty$  for each positive number  $\alpha$ . We set  $F(x) = \int_0^x f(r) dr$ . Then*

$$\lim_{x \rightarrow \infty} \frac{F(x)}{f^2(x)/f'(x)} = 1.$$

**LEMMA 9** ([9]). *Let  $f(x)$  and  $g(x)$  satisfy the following four conditions for  $x \geq 0$ :*

- (i)  $f'(x)$  and  $g'(x)$  are two continuous functions
- (ii)  $f(x)$  is an increasing convex function
- (iii)  $1/g(x)$  is a convex function
- (iv)  $\lim_{x \rightarrow \infty} f'(x)/g'(x) = 1$ .

*Then  $\lim_{x \rightarrow \infty} f'(x)/g'(x) = 1$ .*

### 3 Proof of Theorem 1.3

Since  $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$ , by Lemma 1 and Lemma 3, we have  $\delta(\infty, f) = \delta_0(\infty, f)$  and

$$T(r, f^{(k)}) = ((k+1) - k\delta(\infty, f) + o(1))T(r, f), \quad (6)$$

as  $r \rightarrow \infty$  through all values.

From (6), Lemma 3 and Lemma 7, we have

$$\lim_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} = 2 - \delta_0(\infty, f), \quad (7)$$

$$\lim_{r \rightarrow \infty} \frac{N_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} = 1 - \lim_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} \lim_{r \rightarrow \infty} \frac{T_0(r, f)}{T_0(r, f^{(k)})} = A_0, \quad (8)$$

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{N_0(r, f^{(k)})}{T_0(r, f^{(k)})} &= \lim_{r \rightarrow \infty} \frac{N_0(r, f) + k\bar{N}_0(r, f)}{T_0(r, f^{(k)})} \\ &= \lim_{r \rightarrow \infty} \frac{(k+1)N_0(r, f)}{T_0(r, f)} \lim_{r \rightarrow \infty} \frac{T_0(r, f)}{T_0(r, f^{(k)})} = B_0, \end{aligned} \quad (9)$$

where  $A_0 = (k-1)(1-\delta(\infty, f))/(1+k-k\delta(\infty, f))$  and  $B_0 = (k+1) \cdot (1-\delta(\infty, f))/(1+k-k\delta(\infty, f))$ .

(I) We shall first prove that either for any positive number  $\beta$

$$\lim_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)})}{r^\beta} = \infty \quad \text{and} \quad \lim_{r \rightarrow \infty} \frac{N(r, f^{(k)})}{r^\beta} = \infty,$$

or the conclusion of Theorem 1.3 holds. Since  $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$ ,  $f(z)$  is of regular growth [6]. Note that  $f(z)$  is a meromorphic function of infinite order and regular growth. Then for any positive number  $\beta$ , we have  $\lim_{r \rightarrow \infty} T(r, f^{(k)})/r^\beta = \infty$ . Thus we have

$$\lim_{r \rightarrow \infty} \frac{T(r, 1/f^{(k)})}{r^\beta} = \infty. \quad (10)$$

(i) If

$$\liminf_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = B \neq 0,$$

then

$$\liminf_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)})}{r^\beta} \geq \liminf_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} \liminf_{r \rightarrow \infty} \frac{T(r, f^{(k)})}{r^\beta} = \infty.$$

Thus we have  $\lim_{r \rightarrow \infty} N(r, 1/f^{(k)})/r^\beta = \infty$ .

(ii) If

$$\limsup_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = 0,$$

then

$$\lim_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} \frac{m(r, 1/f^{(k)})}{T(r, f^{(k)})} = 1.$$

Thus we have

$$\lim_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} = 1.$$

From (6) and (7), we obtain  $\delta(\infty, f) = 1$ . Therefore the conclusion of Theorem 1.3 holds.

(iii) Suppose that

$$\limsup_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = B \neq 0 \quad \text{and} \quad \liminf_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = 0.$$

Then we have

$$\limsup_{r \rightarrow \infty} \frac{m(r, 1/f^{(k)})}{T(r, f^{(k)})} = 1.$$

Hence there exists an increasing sequence  $r_n \rightarrow \infty$  such that  $\lim_{n \rightarrow \infty} m(r_n, 1/f^{(k)})/T(r_n, f^{(k)}) = 1$ . By (6), we have

$$\lim_{n \rightarrow \infty} \frac{m(r_n, 1/f^{(k)})}{T(r_n, f)} = 1 + k - k\delta(\infty, f). \quad (11)$$

Hence by (7) and (11), we have

$$\begin{aligned} 2 - \delta(\infty, f) &= \lim_{r \rightarrow \infty} \frac{m_0(r, 1/f^k)}{T_0(r, f)} = \lim_{n \rightarrow \infty} \frac{m_0(r_n, 1/f^k)}{T_0(r_n, f)} \\ &= \lim_{n \rightarrow \infty} \frac{m(r_n, 1/f^k)}{T(r_n, f)} = 1 + k - k\delta(\infty, f). \end{aligned}$$

This yields  $\delta(\infty, f) = 1$ . Hence  $\delta_0(\infty, f) = 1$  by Lemma 1. Thus we have

$$\begin{aligned} \delta_0(0, f^{(k)}) &= \liminf_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} \\ &= \liminf_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} \lim_{r \rightarrow \infty} \frac{T_0(r, f)}{T_0(r, f^{(k)})} = \frac{2 - \delta_0(\infty, f)}{(k+1) - k\delta(\infty, f)} = 1 \end{aligned}$$

by  $\delta(\infty, f) = \delta_0(\infty, f) = 1$ . Then

$$1 = \delta_0(0, f^{(k)}) = \liminf_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} \leq \limsup_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} \leq 1.$$

Thus  $\lim_{r \rightarrow \infty} m_0(r, 1/f^{(k)})/T_0(r, f^{(k)})$  exists. Therefore by using l'Hospital's rule, we obtain

$$\delta_0(0, f^{(k)}) = \lim_{r \rightarrow \infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} = \lim_{r \rightarrow \infty} \frac{m'_0(r, 1/f^{(k)})}{T'_0(r, f^{(k)})} = \lim_{r \rightarrow \infty} \frac{m(r, 1/f^{(k)})}{T(r, f^{(k)})} = \delta(0, f^{(k)}),$$

that is,  $\delta(0, f^{(k)}) = 1$ . Hence  $\lim_{r \rightarrow \infty} N(r, 1/f^{(k)})/T(r, f^{(k)}) = 0$ . This is a contradiction. Therefore we deduce that  $\lim_{r \rightarrow \infty} N(r, 1/f^{(k)})/r^\beta = \infty$  or the conclusion of Theorem 1.3 holds in this case.

Similarly, we have  $\lim_{r \rightarrow \infty} N(r, f^{(k)})/r^\beta = \infty$  or the conclusion of Theorem 1.3 holds.

(II) Let  $F(x) = N_0(e^x, 1/f^{(k)})$  and  $G(x) = A_0 T_0(e^x, f^{(k)})$ . Then we have

$$F(x) = \int_1^{e^x} \frac{N(t, 1/f^{(k)})}{t} dt = \int_0^x N\left(e^r, \frac{1}{f^{(k)}}\right) dr, \quad (12)$$

$$G(x) = A_0 \int_1^{e^x} \frac{T(t, f^{(k)})}{t} dt = A_0 \int_0^x T(e^r, f^{(k)}) dr. \quad (13)$$

Since  $N(e^x, 1/f^{(k)})$  is increasing, from the result of (I) and Lemma 8, we get

$$\lim_{x \rightarrow \infty} F(x) / \left( \frac{N(e^x, 1/f^{(k)})^2}{N'(e^x, 1/f^{(k)})e^x} \right) = 1. \quad (14)$$

Now we shall show that  $F(x)$  and  $G(x)$  satisfy the conditions of Lemma 9.

(i) From (12) and (13), we get  $F'(x) = N(e^x, 1/f^{(k)})$ ,  $G'(x) = A_0 T(e^x, f^{(k)})$ . Obviously  $F'(x)$  and  $G'(x)$  are continuous functions.

(ii) Since  $T(r, f^{(k)})$  is a convex function of  $\log r$ ,  $G'(x) > 0$ . Thus  $G(x)$  is an increasing convex function.

(iii) Since  $F(x) > 0$  and

$$\frac{d^2}{dx^2} \left( \frac{1}{F(x)} \right) = \frac{1}{F^3(x)} \left\{ 2N^2\left(e^x, \frac{1}{f^{(k)}}\right) - F(x)N'\left(e^x, \frac{1}{f^{(k)}}\right)e^x \right\}.$$

From (14), if  $x$  is sufficiently large, we have

$$\frac{d^2}{dx^2} \left( \frac{1}{F(x)} \right) > 0.$$

Thus  $F(x)$  is a convex function. From the result of (II) and Lemma 9, we obtain  $\lim_{r \rightarrow \infty} N(r, 1/f^{(k)})/T(r, f^{(k)}) = A_0$ . Similarly, we have  $\lim_{r \rightarrow \infty} N(r, f^{(k)})/T(r, f^{(k)}) = B_0$ . Thus we obtain

$$K(f^{(k)}) = \lim_{r \rightarrow \infty} \frac{N(r, 1/f^{(k)}) + N(r, f^{(k)})}{T(r, f^{(k)})} = A_0 + B_0 = \frac{2k(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}.$$

The proof of Theorem 1.3 is now complete.

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DEPARTMENT OF INFORMATION  
AND COMPUTING SCIENCE  
GUANGXI INSTITUTE OF TECHNOLOGY  
545006, LIUZHOU GUANGXI  
CHINA