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THE INFINITE DIRECT PRODUCT OF DEHN TWISTS ACTING ON INFINITE DIMENSIONAL TEICHMÜLLER SPACES

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§1. Introduction

We consider the action of the Teichmüller modular group on the Teichmüller space of a topologically infinite Riemann surface. For a Riemann surface S, the Teichmüller space $T(S)$ is the set of all equivalence classes of the pair (f, σ), where $f : S \to S_\sigma$ is a quasiconformal homeomorphism of S onto another Riemann surface S_σ of a complex structure σ. Two pairs (f_1, σ_1) and (f_2, σ_2) are considered to be equivalent if $\sigma_1 = \sigma_2$ and $f_2 \circ f_1^{-1}$ is isotopic to a conformal map. Here the isotopy is considered to be relative to the boundary at infinity. A distance between $p_1 = [f_1, \sigma_1]$ and $p_2 = [f_2, \sigma_2]$ in $T(S)$ is defined by $d(p_1, p_2) = \log K(h)$ for an extremal quasiconformal homeomorphism h whose maximal dilatation $K(h)$ is minimal in the isotopy class of $f_2 \circ f_1^{-1}$. Then d becomes a complete metric on $T(S)$, which is called the Teichmüller distance.

The Teichmüller modular group $\mathrm{Mod}(S)$ of S is a group of the isotopy classes of quasiconformal automorphisms of S. An element g of $\mathrm{Mod}(S)$ acts on $T(S)$ in such a way that $[f, \sigma] \mapsto [f \circ g^{-1}, \sigma]$, where g also denotes a representative of the isotopy class. It is evident from definition that $\mathrm{Mod}(S)$ acts on $T(S)$ isometrically with respect to the Teichmüller distance. In the case that $T(S)$ is finite dimensional (equivalently S is of analytically finite type), $\mathrm{Mod}(S)$ acts on $T(S)$ properly discontinuously and the orbit of any point $p \in T(S)$ is discrete. However, when $T(S)$ is infinite dimensional, these are not always true. See recent works [4], [5] and a monograph [6, Chap. 10].

For finite dimensional Teichmüller spaces, Bers [2] classified the elements of $\mathrm{Mod}(S)$ by certain analytic criteria in comparison with Thurston's topological classification. This can be extended to infinite dimensional Teichmüller spaces in the same way. For example, an element $g \in \mathrm{Mod}(S)$ is elliptic if g has a fixed point in $T(S)$, and parabolic if $\inf d(p, g(p)) = 0$ where the infimum is taken over all points p in $T(S)$. An elliptic element is realized as a conformal automorphism of the Riemann surface corresponding to the fixed point of g. A

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typical parabolic element is the Dehn twist along a simple closed geodesic on a hyperbolic surface S.

In a recent research announcement [10], which will be completed in near future, we investigated the orbit under a cyclic subgroup of $\mathrm{Mod}(S)$ generated by an elliptic element of infinite order. In particular, we proved the existence of a non-closed orbit which accumulates to a point of a different complex structure. In this present paper, we consider similar problems for a parabolic abelian subgroup G of $\mathrm{Mod}(S)$ generated by an infinite number of Dehn twists along mutually disjoint simple closed geodesics on S. In contrast to the elliptic case, convergence of the orbit with respect to the Teichmüller distance implies locally uniform convergence of the elements of the Teichmüller modular group, and in particular the orbit is always closed. From this, we easily see that, if the orbit is not discrete, then it is a perfect set and in particular uncountable. By estimating the maximal dilatation of an element of G in terms of the hyperbolic lengths of the simple closed geodesics, we will refine these results to obtain necessary and sufficient conditions for the orbit to be discrete and to be countable.

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§2. The orbit under the direct product of Dehn twists

Let S be a hyperbolic Riemann surface and $\{c_i\}_{i=1}^{\infty}$ a family of mutually disjoint simple closed geodesics on S. Let δ_i be the Dehn twist along c_i, which is regarded as an element of the Teichmüller modular group $\mathrm{Mod}(S)$. Elements in $\{\delta_i\}_{i=1}^{\infty}$ are mutually commuting to each other. Hence the subgroup of $\mathrm{Mod}(S)$ generated by $\{\delta_i\}_{i=1}^{\infty}$ is nothing but the direct sum $\sum_{i=1}^{\infty}\langle\delta_i\rangle$ of the infinite cyclic groups $\langle\delta_i\rangle$.

On the other hand, the direct product $\prod_{i=1}^{\infty}\langle\delta_i\rangle$ of the infinite cyclic groups $\langle\delta_i\rangle$ lies in the group of the isotopy classes of all orientation preserving homeomorphic automorphisms of S which contains $\mathrm{Mod}(S)$. We consider the intersection

$$G = \mathrm{Mod}(S) \cap \prod_{i=1}^{\infty}\langle\delta_i\rangle,$$

which is an abelian subgroup of $\mathrm{Mod}(S)$ containing the direct sum. We call G the direct product of $\{\delta_i\}_{i=1}^{\infty}$ within $\mathrm{Mod}(S)$.

Let $\phi : \prod_{i=1}^{\infty}\langle\delta_i\rangle \to \boldsymbol{Z}^{\infty}$ be the coordinate map defined by

$$\delta_1^{n_1}\delta_2^{n_2}\cdots \mapsto (n_1, n_2, \ldots).$$

We induce the product topology to $\prod_{i=1}^{\infty}\langle\delta_i\rangle$ by ϕ. This is equivalent to the topology defined by locally uniform convergence of a sequence of representatives.

Theorem 1. *Let $\{c_i\}_{i=1}^{\infty}$ be a family of mutually disjoint simple closed geodesics on a hyperbolic Riemann surface S and ℓ_i the hyperbolic length of c_i. Let δ_i be the Dehn twist along c_i and G the direct product of $\{\delta_i\}_{i=1}^{\infty}$ within* $\mathrm{Mod}(S)$. *If an element $g \in G$ has the coordinate $(n_1, n_2, \ldots)$, then the maximal dilatation $K(g)$ of an extremal quasiconformal automorphism g satisfies*

$$\sup\left\{\left(\frac{(2|n_i|-1)_+\ell_i}{\pi}\right)^2+1\right\}^{1/2} \le K(g) \le \sup\left[\left\{\left(\frac{|n_i|\ell_i}{2\theta_i}\right)^2+1\right\}^{1/2}+\frac{|n_i|\ell_i}{2\theta_i}\right]^2,$$

where $\theta_i = \pi - 2\arctan\{\sinh(\ell_i/2)\}$, $(2|n_i|-1)_+ = \max\{(2|n_i|-1), 0\}$ *and the supremum is taken over all $i \in \boldsymbol{N}$.*

Remark. If we slightly modify the proof given in the next section, we can replace the left hand side of the above inequality with

$$\sup\left\{\left(\frac{(|n_i|-1)_+\ell_i}{\pi}\right)^2+1\right\}.$$

Also see the remark given at the end of this paper where we discuss the better estimate for a special case.

A proof of Theorem 1 is given in the next section. In this section, assuming this estimate, we prove several properties of the orbit $G(p)$.

Proposition 1. *For any point $p \in T(S)$, the correspondence $v_p : G \to G(p)$ defined by $g \mapsto g(p) \in T(S)$ is injective. Moreover, for any points p and q in $T(S)$, the composition $v_q \circ v_p^{-1} : G(p) \to G(q)$ is a biLipschitz homeomorphism with respect to the Teichmüller distance on $T(S)$.*

Proof. The injectivity of v_p is obvious. Let $d = d(p,q)$ be the Teichmüller distance between p and q. Then the ratio of the hyperbolic lengths of simple closed geodesics measured on p and q is bounded by e^d (by Wolpert [12] or implicitly in the proof of Theorem 1). For two elements g and g' in G, compare the Teichmüller distances $d(g(p), g'(p))$ and $d(g(q), g'(q))$. By the estimate in Theorem 1, we see that the ratio of these distances is bounded by a constant depending on d. We omit the details, for it is merely a matter of calculation. □

By this injection $v_p : G \to G(p) \subset T(S)$, we induce a distance on G from $T(S)$, which we call the Teichmüller distance. Of course, this depends on the choice of $p \in T(S)$, however, any such distances are biLipschitz equivalent by Proposition 1.

Proposition 2. *A bounded sequence in G with respect to the Teichmüller distance has a convergent subsequence in the product topology. A Cauchy sequence in G is a convergent sequence with respect to the Teichmüller distance. Hence G is complete, that is, the orbit $G(p)$ is a closed subset of $T(S)$.*

Proof. Consider a bounded sequence $\{g_m\}$ in G. We also regard g_m as an extremal quasiconformal automorphism of S representing the isotopy class. Then the maximal dilatations of g_m are bounded. Since they all preserve a free homotopy class of a simple closed curve, say c_1, we can choose a subsequence of $\{g_m\}$ that converges to a quasiconformal automorphism g of S locally uniformly. If $\{g_m\}$ is a Cauchy sequence, then they are bounded and hence any subsequence has a subsequence that converges to some $g \in G$ in the product topology. This also implies the convergence with respect to the Teichmüller distance. Since such limit g must be unique, the entire sequence $\{g_m\}$ converges to $g \in G$. $\square$

If the orbit $G(p)$ is not discrete, then it is a closed perfect set by the group invariance. In particular, $G(p)$ is uncountable. Considering this fact, we obtain the following conditions for discreteness and countability of the orbit in terms of the geodesic lengths $\{\ell_i\}$ of $\{c_i\}$.

Theorem 2. *Let $\{c_i\}_{i=1}^{\infty}$ be a family of mutually disjoint simple closed geodesics on a hyperbolic Riemann surface S and ℓ_i the hyperbolic length of c_i. Let δ_i be the Dehn twist along c_i and G the direct product of $\{\delta_i\}_{i=1}^{\infty}$ within* $\mathrm{Mod}(S)$. *Then the orbit $G(p)$ of any point $p \in T(S)$ is discrete if and only if the lengths $\{\ell_i\}$ are uniformly bounded away from* 0. *On the other hand, $G(p)$ is countable if and only if, for every positive constant L, the number of c_i whose lengths ℓ_i are less than L is finite. In this case, G coincides with $\sum_{i=1}^{\infty} \langle \delta_i \rangle$.*

Proof. Suppose that the lengths of c_i are uniformly bounded away from 0. Then, by the lower estimate in Theorem 1, the distance $d(g(p), g'(p))$ are uniformly bounded away from 0 for any distinct g and g' in G. Hence $G(p)$ is discrete. The converse is also clear by the upper estimate in Theorem 1.

Assume that, for every positive constant L, the number of simple closed geodesics c_i whose lengths are less than L is finite. Then, for every positive constant $K \geq 1$, the number of elements in G the maximal dilatation of whose extremal quasiconformal automorphism is less than K is finite by the lower estimate in Theorem 1. Hence G is countable as well as $G(p)$. Conversely, if the number of simple closed geodesics c_i whose lengths are less than L is infinite for some L, then the choice of doing the Dehn twist or not for each c_i makes an uncountable number of distinct elements whose maximal dilatations are bounded. This can be seen by the upper estimate in Theorem 1. Hence G is uncountable. $\square$

§3. Estimate of the maximal dilatation of Dehn twists

Our estimate in Theorem 1 is carried out by considering the Dehn twist on an annular cover A and on an embedded collar A' of S with respect to each of $\{c_i\}_{i=1}^{\infty}$.

We begin with defining several constants concerning an annulus and giving the relationship between them, though they are well known facts. Let γ be a hyperbolic element acting on the upper half-plane $\mathbf{H}$ whose translation length is $\ell > 0$. We may assume that $\gamma(z) = kz$, where $\log k = \ell$. Set

$$C(\theta) = \left\{ z \in \mathbf{H} \mid \frac{\pi}{2} - \frac{\theta}{2} < \arg z < \frac{\pi}{2} + \frac{\theta}{2} \right\}$$

for $0 < \theta \le \pi$. A fundamental domain of $C(\theta)$ under the action of the cyclic group $\langle \gamma \rangle$ is conformally mapped by log onto a rectangle with the horizontal side $(0, \log k)$ and the vertical side $[\pi/2 - \theta/2, \pi/2 + \theta/2]$. Hence it is conformally equivalent to a rectangle Q with the side lengths 1 and θ/ℓ.

Consider an annulus $A(\theta) = C(\theta)/\langle \gamma \rangle$ obtained by the identification of the sides of the fundamental domain. This is realized in the complex plane $\boldsymbol{C}$ as the image of the rectangle Q by the exponential map (combined with a euclidean similarity): $A(\theta) = \{z \in \boldsymbol{C} \mid 1 < |z| < r(\theta)\}$. Then the conformal modulus $\log r(\theta)$ of $A(\theta)$ is $2\pi\theta/\ell$. In the case of $\theta = \pi$, the conformal modulus $\log R := \log r(\pi)$ is $2\pi^2/\ell$.

Let Γ be a Fuchsian group acting on $\mathbf{H}$ and assume that Γ contains $\gamma(z) = kz$ as an element corresponding to a simple closed geodesic c on a hyperbolic Riemann surface $S = \mathbf{H}/\Gamma$. The collar lemma (cf. [3, Chap. 4], [7] and [9]) asserts that the annulus $A(\theta) = C(\theta)/\langle \gamma \rangle$ can be conformally embedded into $\mathbf{H}/\Gamma$ for a constant $\theta = \pi - 2\arctan\{\sinh(\ell/2)\}$. We call this annulus the collar for c. Moreover, if two simple closed geodesics c_1 and c_2 are disjoint, then the corresponding collars A_1 and A_2 are also disjoint.

Proof of Theorem 1. We represent $S = \mathbf{H}/\Gamma$ by a Fuchsian group Γ acting on $\mathbf{H}$. Choose one of the simple closed geodesics $\{c_i\}_{i=1}^{\infty}$ and denote it by c. We may assume that a lift $\tilde{c}$ of c is the imaginary axis and the corresponding hyperbolic element of Γ is $\gamma(z) = kz$, where $\log k > 0$ is the hyperbolic length $\ell = \ell_i$ of c. Let g denote an element (an isotopy class) of G having the coordinate $\phi(g) = (n_1, n_2, \ldots)$ as well as a quasiconformal automorphism representing this isotopy class. Set $n = n_i$.

Consider all the lifts of the simple closed geodesics $\{c_i\}_{i=1}^{\infty}$ and denote the union of them by L. They do not intersect each other. Then the complement of L in $\mathbf{H}$ consists of simply connected components, each of which is bounded by complete geodesic lines and the non-empty ideal boundary on $\partial\mathbf{H} = \boldsymbol{R} \cup \{\infty\}$. Let E_1 and E_2 be the adjacent components of $\mathbf{H} - L$ facing to each other along $\tilde{c}$. Both of them are invariant under the cyclic group $\langle \gamma \rangle$.

Take a lift $\tilde{g}$ of g so that $\tilde{g}$ is the identity on $\partial E_1 \cap \partial\mathbf{H}$. The whole lift $\tilde{g}$ is obtained by shifting the other components of $\mathbf{H} - L$ along L by hyperbolic translation and then smoothing. However the boundary value of $\tilde{g}$ on $\partial\mathbf{H}$ is exactly obtained only by this shifting operation and it does not depend on the choice of a representative of $g \in G$. The component E_2 is invariant under $\langle \gamma \rangle$ and it moves along the imaginary axis $\tilde{c}$ by γ^n. Although E_2 has other geodesic boundaries than $\tilde{c}$ and the shifting operations are also carried out along them, they do not affect $\tilde{g}$ on $\partial E_2 \cap \partial \mathbf{H}$. This is because, once E_2 moves along $\tilde{c}$, it is not moved by the succeeding shifts any longer. Therefore the boundary value of $\tilde{g}$ restricted to $\partial E_2 \cap \partial \mathbf{H}$ coincides with γ^n.

Set $A = \mathbf{H}/\langle \gamma \rangle$, which is an annular cover of S with respect to c. The ideal boundary of A consists of two components, $\partial_1 A = \boldsymbol{R}_{<0}/\langle \gamma \rangle$ and $\partial_2 A = \boldsymbol{R}_{>0}/\langle \gamma \rangle$, where $\boldsymbol{R}_{<0}$ and $\boldsymbol{R}_{>0}$ are the negative and the positive real axes respectively. Let g_A be the lift of g to A that is the projection of $\tilde{g}$ onto A. Its maximal dilatation $K(g_A)$ is the same as $K(g)$.

To obtain the lower bound of $K(g)$, we estimate $K(g_A)$. Take a simple arc $\tilde{\alpha}$ in $\overline{E_1 \cup E_2}$ connecting boundary points $\tilde{a}_1 \in \partial E_1 \cap \partial \mathbf{H}$ and $\tilde{a}_2 \in \partial E_2 \cap \partial \mathbf{H}$. It projects injectively onto an arc α on A connecting the boundary points $a_1 \in \partial_1 A$ and $a_2 \in \partial_2 A$. Then g_A fixes both of a_1 and a_2 because $\tilde{g}$ fixes $\tilde{a}_1$ and moves $\tilde{a}_2$ by γ^n. However, the image $g_A(\alpha)$ is not homotopic to α in A relatively to $\{a_1, a_2\}$. Applying the following Lemma 1, we have the lower bound as stated in Theorem 1.

Lemma 1. *Let A be an annulus of the modulus* $\log R$ $(R > 1)$ *and α an arc in A with the end points a_1 and a_2 on the distinct boundary components $\partial_1 A$ and $\partial_2 A$ of A respectively. Let g be a quasiconformal automorphism of A such that g fixes both of a_1 and a_2, but the image $g(\alpha)$ wraps n times around A. Then the maximal dilatation $K(g)$ of g satisfies*

$$K(g) \geq \left[\left\{ (2|n| - 1) \frac{2\pi}{\log R} \right\}^2 + 1 \right]^{1/2}.$$

Proof. We consider the composition $g^2 = g \circ g$ of the quasiconformal automorphism g of A and estimate the maximal dilatation of g^2 instead of g itself. We remark here that, in case $|n| \geq 2$, a method shown below works for g itself and we can obtain another estimate, which was given in the remark after Theorem 1, however in case $|n| = 1$, we have to work with g^2.

We may assume that $A = \{z \in \boldsymbol{C} \mid 1 < |z| < R\}$, $\partial_1 A = \{|z| = 1\}$ and $\partial_2 A = \{|z| = R\}$. Let $F = \{\beta\}$ be a curve family on A consists of all the radial segments β connecting $e^{i\tau} \in \partial_1 A$ and $Re^{i\tau} \in \partial_2 A$. We consider the extremal length

$$\lambda(F) = \sup_{\rho} \frac{\{\inf_{\beta \in F} \int_{\beta} \rho(z)|dz|\}^2}{\iint_A \rho(z)^2 \, dxdy}$$

of the curve family F, where the supremum is taken over all Borel measurable non-negative functions $\rho(z)$ on A. See [1, Chap. 4] and [11]. Then it is known that

$$\rho_0(z)|dz| := \frac{|dz|}{|z| \log R}$$

is the extremal metric, which is just the translation of the euclidean metric on the rectangle Q by exp. Namely $\lambda(F) = \log R/(2\pi)$. By this metric, the length of any radial segment β is 1, the length of any concentric circle is $2\pi/\log R$ and the area of A is $2\pi/\log R$.

Consider the length $\int_{g^2(\beta)} \rho_0(z)|dz|$ for any $\beta \in F$. Then it is greater than or equal to

$$\left[\left\{(2|n| - 1)\frac{2\pi}{\log R}\right\}^2 + 1\right]^{1/2},$$

which is the euclidean length of the diagonal of a rectangle consisting of $2|n| - 1$ many Q straight in line. The point of our estimate is here: the $2|n|$-time Dehn twist moves α and it forces every segment β to wrap around at least $2|n| - 1$ times. Therefore the extremal length $\lambda(g^2(F))$ of the curve family $g^2(F)$ can be estimated by

$$\lambda(g^2(F)) \geq \frac{\{(2|n| - 1)2\pi/\log R\}^2 + 1}{2\pi/\log R}.$$

The extremal length of the curve families and the maximal dilatation of g satisfy

$$\lambda(g^2(F)) \leq K(g^2)\lambda(F) \leq K(g)^2\lambda(F).$$

Hence

$$\frac{\{(2|n| - 1)2\pi/\log R\}^2 + 1}{2\pi/\log R} \leq K(g)^2 \frac{\log R}{2\pi},$$

from which we have $K(g) \geq [\{(2|n| - 1)2\pi/\log R\}^2 + 1]^{1/2}$. □

Proof of Theorem 1 *continued.* Next we estimate the upper bound. To this end, we take collars for the simple closed geodesics $\{c_i\}_{i=1}^{\infty}$ so that they are mutually disjoint. Each collar is conformally equivalent to an annulus A_i. We consider a quasiconformal automorphism of S that is the n_i-time Dehn twist along c_i on each A_i fixing every boundary point of A_i and is the identity outside of $\bigcup A_i$. Then the estimate on each A_i gives the upper bound of $K(g)$.

Fix an index i and set $c = c_i$, $\ell = \ell_i$ and $n = n_i$ as before. By the collar lemma, we can take the disjoint collar of the conformal modulus $\log r = 2\pi\theta/\ell$, where $\theta = \pi - 2\arctan\{\sinh(\ell/2)\}$. By the following well known Lemma 2, we can calculate the maximal dilatation of a particular quasiconformal automorphism in the isotopy class of the Dehn twist of the collar $A' = A_i$, which is actually the extremal one (cf. [8]). Then, substituting $2\pi\theta/\ell$ for $\log r$, we have the upper bound as stated in Theorem 1. □

Lemma 2. *Let A' be an annulus of the conformal modulus* $\log r$ $(r > 1)$. *Then there exists a quasiconformal automorphism δ of A' in the isotopy class of the Dehn twist fixing every point on the boundary of A' such that the maximal dilatation of δ^n is*

$$\frac{\{1+(|n|\pi/\log r)^2\}^{1/2}+|n|\pi/\log r}{\{1+(|n|\pi/\log r)^2\}^{1/2}-|n|\pi/\log r}.$$

Proof. We may assume that $A' = \{z \in \boldsymbol{C} \mid 1 < |z| < r\}$. Set

$$\delta(z) = z\exp\left(2\pi i \frac{\log|z|}{\log r}\right),$$

which represents the Dehn twist of A' fixing every point on the boundary. Then

$$\delta^n(z) = z\exp\left(2n\pi i \frac{\log|z|}{\log r}\right).$$

If we map A' conformally onto the rectangle Q' by log, the δ is conjugate to a linear map

$$\hat{\delta}(\zeta) = \zeta + \frac{2\pi i \operatorname{Re}\zeta}{\log r}$$

of Q' onto a parallel quadrangle. Then

$$\hat{\delta}^n(\zeta) = \zeta + \frac{2n\pi i \operatorname{Re}\zeta}{\log r}.$$

We can calculate the complex dilatation of $\hat{\delta}^n$ easily:

$$|\partial(\hat{\delta}^n)| = \left\{1+\left(\frac{|n|\pi}{\log r}\right)^2\right\}^{1/2}, \quad |\bar{\partial}(\hat{\delta}^n)| = \frac{|n|\pi}{\log r}.$$

Since the maximal dilatation of δ^n is the same as that of $\hat{\delta}^n$, we obtain the formula as in the statement of this lemma. □

Finally we remark a difference between the estimates for a simple Dehn twist and for the direct product of Dehn twists. Assuming that S is analytically finite, we take a simple closed geodesic c and consider conformally embedded annuli in S whose core curves are freely homotopic to c. Among them, there exists a unique annulus A^* that has the largest conformal modulus, which is foliated by the trajectories of the simple Jenkins-Strebel quadratic differential. Then the extremal quasiconformal automorphism g in the isotopy class of the Dehn twist along c is represented by the canonical transformation of the annulus as in Lemma 2. See Marden and Masur [8]. The maximal dilatation $K(g)$ of g can be written in terms of the modulus of A^*, which is between the modulus of the collar A' and that of the annular cover A. Hence $K(g)$ has an estimate from above and from below in terms of the hyperbolic length of c.

Even if S is not analytically finite, the annulus with the maximal modulus exists (cf. [6, Chap. 11]) and the canonical transformation still gives an extremal quasiconformal automorphism. Hence the maximal dilatation of simple Dehn twists (n times) has the estimate in terms of the hyperbolic length ℓ of c in any case, which is better than the one we obtain in Theorem 1. Namely, we have

$$\left(\frac{|n|\ell}{\pi}\right)^2 \le \left[\left\{\left(\frac{|n|\ell}{2\pi}\right)^2 + 1\right\}^{1/2} + \frac{|n|\ell}{2\pi}\right]^2 \le K(g).$$

However, if g is composed by multiple Dehn twists and positive and negative twists are mixed, this method does not work even in the analytically finite and finitely many case, as is remarked in [8].

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