

UNIQUENESS OF ENTIRE FUNCTIONS AND FIXED POINTS

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Abstract

Let f be a nonconstant entire function. If f , f' and f'' have the same fixed points, then $f \equiv f'$.

1. Introduction

Let f be a nonconstant meromorphic function in the whole complex plane. We use the following standard notation of value distribution theory,

$$T(r, f), m(r, f), N(r, f), \bar{N}(r, f), \dots$$

(see Hayman [1], Yang [6]). We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \rightarrow +\infty$, possibly outside a set of r of finite linear measure.

Let g be a meromorphic function, and let a, b be two complex numbers. If $g(z) = b$ whenever $f(z) = a$, then we denote it by $f(z) = a \Rightarrow g(z) = b$. Thus $f(z) = a \Leftrightarrow g(z) = a$ means $f(z) = a$ if and only if $g(z) = a$.

In 1986, Jank-Muse-Volkman [3] proved the following result.

THEOREM A. *Let f be a nonconstant entire function, and a be a nonzero value. If $f(z) = a \Leftrightarrow f'(z) = a$, and $f'(z) = a \Rightarrow f''(z) = a$, then $f \equiv f'$.*

In this paper, we extend Theorem A as follows.

THEOREM 1. *Let f be a nonconstant entire function, and let a, c be two nonzero constants. If $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = c$, then either $f(z) = Ae^{cz/a} + (ac - a^2)/c$ or $f(z) = Ae^{cz/a} + a$, where A is a nonzero constant.*

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COROLLARY 2. *Let f be a nonconstant entire function, and let a be a nonzero constant. If $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = a$, then either $f \equiv f'$ or $f \equiv f' + a$.*

Remark. In Corollary 2, the case $f \equiv f' + a$ occurs.

Let $f(z) = a + Ae^z$. Then $f'(z) = Ae^z$, $f''(z) = Ae^z$. Obviously, f satisfies $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = a$, and $f \equiv f' + a$.

The main result of this paper is the following

THEOREM 3. *Let f be a nonconstant entire function. If $f(z) = z \Leftrightarrow f'(z) = z$, and $f'(z) = z \Rightarrow f''(z) = z$, then $f \equiv f'$.*

If $f(z_0) = z_0$, then z_0 is called a fixed point of f .

COROLLARY 4. *Let f be a nonconstant entire function. If f , f' and f'' have the same fixed points, then $f \equiv f'$.*

2. Proof of Theorem 1

Set

$$(2.1) \quad \psi(z) = \frac{cf'(z) + af''(z)}{f(z) - a} - \frac{2cf''(z)}{f'(z) - a}.$$

Let $f(z_0) = a$. Then by the assumptions we may suppose that, near z_0

$$(2.2) \quad f(z) = a + a(z - z_0) + \frac{c}{2}(z - z_0)^2 + b(z - z_0)^3 + O((z - z_0)^4),$$

where $b = f^{(3)}(z_0)/6$ is a constant. Thus we have

$$(2.3) \quad \begin{aligned} f'(z) &= a + c(z - z_0) + 3b(z - z_0)^2 + O((z - z_0)^3), \\ f''(z) &= c + 6b(z - z_0) + O((z - z_0)^2). \end{aligned}$$

Hence

$$\begin{aligned} \frac{cf'(z) + af''(z)}{f(z) - a} &= \frac{2c}{z - z_0} + 6b + O(z - z_0), \\ \frac{2cf''(z)}{f'(z) - a} &= \frac{2c}{z - z_0} + 6b + O(z - z_0). \end{aligned}$$

Thus we obtain

$$(2.4) \quad \psi(z_0) = 0.$$

Next we consider two cases.

CASE 1. $f''/f' \equiv c/a$, that is $cf' \equiv af''$. Then we get

$$(2.5) \quad f'(z) = \frac{cA}{a} e^{(c/a)z}, \quad f(z) = Ae^{(c/a)z} + B,$$

where $A \neq 0, B$ are two constants.

If there exists z_0 satisfying $f(z_0) = a$, then by the assumptions and (2.5) we obtain

$$f(z) = Ae^{(c/a)z} + \frac{ac - a^2}{c}.$$

If there doesn't exist z_0 satisfying $f(z_0) = a$, then by (2.5) we get

$$f(z) = a + Ae^{(c/a)z},$$

where A is a nonzero constant.

CASE 2. $f''/f' \not\equiv c/a$. Then by the assumptions we have

$$(2.6) \quad \begin{aligned} N\left(r, \frac{1}{f' - a}\right) &\leq N\left(r, \frac{1}{f''/f' - c/a}\right) \leq T\left(r, \frac{f''}{f'}\right) + O(1) \\ &= N\left(r, \frac{f''}{f'}\right) + S(r, f) = \bar{N}\left(r, \frac{1}{f'}\right) + S(r, f). \end{aligned}$$

In the following, we consider two subcases.

CASE 2.1. $\psi \neq 0$. Then by (2.1) and (2.4) we get

$$(2.7) \quad \begin{aligned} N\left(r, \frac{1}{f' - a}\right) &\leq N\left(r, \frac{1}{\psi}\right) \leq T(r, \psi) + O(1) \\ &\leq N_0\left(r, \frac{1}{f' - a}\right) + S(r, f), \end{aligned}$$

where $N_0(r, 1/(f' - a))$ is the counting function for those zero points of $f'(z) - a$ which are not zero points of $f(z) - a$.

Thus by the assumption and (2.7) we obtain

$$(2.8) \quad 2N\left(r, \frac{1}{f' - a}\right) \leq N\left(r, \frac{1}{f' - a}\right) + S(r, f).$$

On the other hand, by Nevanlinna first fundamental theorem we have

$$\begin{aligned} m\left(r, \frac{1}{f' - a}\right) &\leq m\left(r, \frac{1}{f'}\right) + S(r, f) \\ &\leq T(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f) \\ &\leq T(r, f) - N\left(r, \frac{1}{f'}\right) + S(r, f) \end{aligned}$$

$$\begin{aligned}
&= T\left(r, \frac{1}{f-a}\right) - N\left(r, \frac{1}{f'}\right) + S(r, f) \\
&= m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) - N\left(r, \frac{1}{f'}\right) + S(r, f).
\end{aligned}$$

Thus

$$(2.9) \quad N\left(r, \frac{1}{f'}\right) \leq N\left(r, \frac{1}{f-a}\right) + S(r, f).$$

Hence by (2.6), (2.8) and (2.9) we get

$$(2.10) \quad N\left(r, \frac{1}{f-a}\right) = N\left(r, \frac{1}{f'-a}\right) + S(r, f) = S(r, f).$$

By Milloux's inequality (see [1, 6])

$$(2.11) \quad T(r, f) \leq N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f'-a}\right) + S(r, f).$$

Thus by (2.10) and (2.11) we get $T(r, f) = S(r, f)$, a contradiction.

CASE 2.2. $\psi \equiv 0$. That is

$$(2.12) \quad \frac{cf'(z) + af''(z)}{f(z) - a} \equiv \frac{2cf''(z)}{f'(z) - a}.$$

Thus by $f(z) = a \Rightarrow f'(z) = a$ and (2.12) we deduce that $f(z) = a \Leftrightarrow f'(z) = a$. Hence, $f''(z) = 0 \Rightarrow f'(z) = 0$.

Set

$$(2.13) \quad \phi(z) = \frac{af''(z) - cf'(z)}{f(z) - a}.$$

Since $f''/f' \not\equiv c/a$, we get $\phi \not\equiv 0$.

Let $f'(z_0) = 0$ and $f''(z_0) \neq 0$. Then by (2.12) we get

$$(2.14) \quad f(z_0) = \frac{2ac - a^2}{2c}.$$

Differentiating the two sides of (2.12) we get

$$\begin{aligned}
(2.15) \quad & \frac{[cf''(z) + af'''(z)][f(z) - a] - f'(z)[cf'(z) + af''(z)]}{[f(z) - a]^2} \\
& \equiv \frac{2cf'''(z)[f'(z) - a] - 2c[f''(z)]^2}{[f'(z) - a]^2}.
\end{aligned}$$

Thus by (2.14), (2.15), $f'(z_0) = 0$ and $f''(z_0) \neq 0$, we obtain

$$(2.16) \quad f''(z_0) = c.$$

Hence we have

$$(2.17) \quad \phi(z_0) = -\frac{2c^2}{a}.$$

Next we divide two subcases.

CASE 2.2.1. $\phi(z) \not\equiv -2c^2/a$. Then by (2.17),

$$(2.18) \quad \bar{N}\left(r, \frac{1}{f'}\right) - \bar{N}\left(r, \frac{1}{f''}\right) \leq \bar{N}\left(r, \frac{1}{\phi + 2c^2/a}\right) \leq T(r, \phi) + S(r, f).$$

Obviously, by (2.13), Logarithmic Derivative Lemma (see [1, 6]) and the assumptions we get

$$(2.19) \quad T(r, \phi) = S(r, f).$$

Thus we get

$$(2.20) \quad \bar{N}\left(r, \frac{1}{f'}\right) - \bar{N}\left(r, \frac{1}{f''}\right) = S(r, f).$$

By $f''(z) = 0 \Rightarrow f'(z) = 0$, (2.13) and (2.19) we have

$$(2.21) \quad \bar{N}\left(r, \frac{1}{f''}\right) \leq \bar{N}\left(r, \frac{1}{\phi}\right) = T(r, \phi) + O(1) = S(r, f).$$

Hence

$$(2.22) \quad \bar{N}\left(r, \frac{1}{f'}\right) = S(r, f).$$

Thus by Milloux's inequality, (2.22) and (2.6), we get $T(r, f) = S(r, f)$, a contradiction.

CASE 2.2.2. $\phi \equiv -2c^2/a$. That is

$$(2.23) \quad af''(z) - cf'(z) + \frac{2c^2}{a}[f(z) - a] = 0,$$

for $z \in \mathbb{C}$.

If there exists z_0 such that $f''(z_0) = 0$, then by $f''(z) = 0 \Rightarrow f'(z) = 0$ and (2.23) we get $f'(z_0) = 0$ and $f(z_0) = a$, which contradicts $f(z) = a \Rightarrow f'(z) = a$. Hence, $f''(z) \neq 0$.

Solving the equation (2.23) we obtain

$$(2.24) \quad f(z) = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + a,$$

where λ_1 and λ_2 are solutions of the equation $az^2 - cz + 2c^2/a = 0$, and c_1, c_2 are two constants.

Thus

$$(2.25) \quad f''(z) = c_1 \lambda_1^2 e^{\lambda_1 z} + c_2 \lambda_2^2 e^{\lambda_2 z}.$$

Since $f''(z) \neq 0$, we deduce from (2.25) that either $c_1 = 0$ or $c_2 = 0$. Without loss of generality, we assume that $c_2 = 0$, then

$$(2.26) \quad f(z) = c_1 e^{\lambda_1 z} + a.$$

Thus by $f'(z) = a \Rightarrow f''(z) = c$ and (2.26) we get

$$f(z) = A e^{(c/a)z} + a.$$

The proof of Theorem 1 is complete.

3. Proof of Theorem 3

Firstly, we consider the case that f is a transcendental entire function. Obviously, we have

$$\begin{aligned} & m\left(r, \frac{1}{f-z}\right) + m\left(r, \frac{1}{f'-z}\right) \\ & \leq m\left(r, \frac{1}{f''}\right) + m\left(r, \frac{1}{f''-1}\right) + S(r, f) \\ & \leq m\left(r, \frac{1}{f''} + \frac{1}{f''-1}\right) + S(r, f) \\ & \leq m\left(r, \frac{1}{f'''}\right) \leq T(r, f''') + S(r, f) \\ & \leq T(r, f') + S(r, f). \end{aligned}$$

Hence by Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} (3.1) \quad & T(r, f-z) + T(r, f'-z) \\ & \leq N\left(r, \frac{1}{f-z}\right) + N\left(r, \frac{1}{f'-z}\right) + T(r, f') + S(r, f). \end{aligned}$$

By $f(z) = z \Leftrightarrow f'(z) = z$, and $f'(z) = z \Rightarrow f''(z) = z$, it is easy to see that

$$(3.2) \quad N\left(r, \frac{1}{f'-z}\right) = N\left(r, \frac{1}{f-z}\right) + S(r, f).$$

Thus by (3.1) and (3.2) we have

$$(3.3) \quad T(r, f) \leq 2N\left(r, \frac{1}{f-z}\right) + S(r, f).$$

Set

$$(3.4) \quad H(z) = \frac{f''(z)}{f'(z)-1} - \frac{z}{z-1}.$$

If $H \equiv 0$, then by (3.4) we get

$$\frac{f''(z)}{f'(z) - 1} \equiv \frac{z}{z - 1}.$$

Thus we have

$$(3.5) \quad f'(z) = 1 + C(z - 1)e^z,$$

$$(3.6) \quad f(z) = z + C(z - 2)e^z + A.$$

where A, C ($\neq 0$) are two constants.

Hence by (3.5), (3.6) and $f(z) = z \Leftrightarrow f'(z) = z$, we know that $f'(z) = z$ have the unique solution $z_0 = 2 - A$ with $z_0 \neq 0, 1$. But it is clear that $f'(z) = z$ have infinitely many solutions, a contradiction. Hence $H \neq 0$, that is

$$\frac{f''(z)}{f'(z) - 1} \not\equiv \frac{z}{z - 1}.$$

Thus by the assumption of the theorem, we have

$$(3.7) \quad \begin{aligned} N\left(r, \frac{1}{f - z}\right) &\leq N\left(r, \frac{1}{H}\right) + O(\log r) \\ &\leq T(r, H) + S(r, f) \\ &\leq N\left(r, \frac{f''}{f' - 1}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f' - 1}\right) + S(r, f). \end{aligned}$$

Hence by (3.7) and (3.3) we get

$$(3.8) \quad T(r, f) \leq 2\bar{N}\left(r, \frac{1}{f' - 1}\right) + S(r, f).$$

Set

$$(3.9) \quad \phi(z) = \frac{f'(z) - 1}{f(z) - z} - \frac{f''(z) - 1}{f'(z) - z},$$

$$(3.10) \quad \psi(z) = \frac{(z - 1)f''(z) - z[f'(z) - 1]}{f(z) - z}.$$

Obviously, by Logarithmic Derivative Lemma (see [1, 6])

$$(3.11) \quad m(r, \phi) = S(r, f), \quad m(r, \psi) = S(r, f).$$

Let z_0 satisfy $f(z_0) = z_0$ and $z_0 \neq 0, 1$. Then by assumption we may assume that, near z_0

$$(3.12) \quad f(z) = z_0 + z_0(z - z_0) + \frac{z_0}{2}(z - z_0)^2 + \frac{f'''(z_0)}{6}(z - z_0)^3 + \cdots,$$

Thus we have

$$(3.13) \quad f'(z) = z_0 + z_0(z - z_0) + \frac{f'''(z_0)}{2}(z - z_0)^2 + \cdots,$$

$$(3.14) \quad f''(z) = z_0 + f'''(z_0)(z - z_0) + \cdots.$$

By (3.12)–(3.14) we get

$$(3.15) \quad N(r, \phi) = S(r, f), \quad N(r, \psi) = S(r, f).$$

Thus we have

$$(3.16) \quad T(r, \phi) = S(r, f), \quad T(r, \psi) = S(r, f).$$

By (3.12)–(3.14) and (3.9)–(3.10) we get

$$\phi(z_0) = \frac{f'''(z_0) - z_0}{2(1 - z_0)},$$

and

$$\psi(z_0) = f'''(z_0) - z_0 - 1.$$

Thus we obtain

$$2(z_0 - 1)\phi(z_0) + \psi(z_0) + 1 = 0.$$

If $2(z - 1)\phi(z) + \psi(z) + 1 \not\equiv 0$, then by (3.16)

$$(3.17) \quad \begin{aligned} N\left(r, \frac{1}{f - z}\right) &\leq N\left(r, \frac{1}{2(z - 1)\phi + \psi + 1}\right) + O(\log r) \\ &\leq T(r, \phi) + T(r, \psi) + S(r, f) \leq S(r, f). \end{aligned}$$

Thus by (3.3) and (3.17) we get a contradiction: $T(r, f) = S(r, f)$. Hence

$$(3.18) \quad 2(z - 1)\phi(z) + \psi(z) + 1 \equiv 0.$$

Now let z_1 satisfy $f'(z_1) = 1$ and $z_1 \neq 1$. Then by $f(z) = z \Leftrightarrow f'(z) = z$, we know that $f(z_1) \neq z_1$. Thus by (3.12) and (3.13) we have

$$(3.19) \quad \phi(z_1) = \frac{f''(z_1) - 1}{z_1 - 1},$$

$$(3.20) \quad \psi(z_1) = \frac{(z_1 - 1)f''(z_1)}{f(z_1) - z_1}.$$

Hence by (3.18)–(3.20) we obtain

$$[2f(z_1) - z_1 - 1]f''(z_1) = f(z_1) - z_1.$$

If $2f(z_1) - z_1 - 1 = 0$, then $f(z_1) = z_1$, a contradiction. Hence $2f(z_1) - z_1 - 1 \neq 0$. Thus

$$(3.21) \quad f''(z_1) = \frac{f(z_1) - z_1}{2f(z_1) - z_1 - 1}.$$

Therefore by (3.19)–(3.21) we get

$$(3.22) \quad \phi(z_1) = \frac{1 - f(z_1)}{(z_1 - 1)[2f(z_1) - z_1 - 1]},$$

$$(3.23) \quad \psi(z_1) = \frac{z_1 - 1}{2f(z_1) - z_1 - 1}.$$

By (3.9), (3.10) and (3.21) we get

$$(3.24) \quad \phi'(z_1) = \frac{f'''(z_1)}{z_1 - 1} + \frac{1}{2f(z_1) - z_1 - 1} + \frac{[f(z_1) - 1]^2}{(z_1 - 1)^2[2f(z_1) - z_1 - 1]^2},$$

and

$$(3.25) \quad \psi'(z_1) = \frac{(z_1 - 1)f'''(z_1)}{f(z_1) - z_1} - \frac{z_1 - 1}{2f(z_1) - z_1 - 1}.$$

By (3.18) we get

$$(3.26) \quad 2\phi(z) + 2(z - 1)\phi'(z) + \psi'(z) \equiv 0.$$

Thus we have

$$(3.27) \quad 2\phi(z_1) + 2(z_1 - 1)\phi'(z_1) + \psi'(z_1) = 0.$$

By (3.22)–(3.24) and (3.27) we get

$$(3.28) \quad \frac{2[1 - f(z_1)]}{(z_1 - 1)^2}\psi(z_1) + 2f'''(z_1) + 2\psi(z_1) + \frac{2[f(z_1) - 1]^2}{(z_1 - 1)^3}\psi^2(z_1) + \psi'(z_1) = 0.$$

By (3.25) we get

$$(3.29) \quad f'''(z_1) = \frac{[\psi'(z_1) + \psi(z_1)][f(z_1) - z_1]}{z_1 - 1}.$$

Thus by (3.28) and (3.29) we have

$$(3.30) \quad \begin{aligned} & \frac{2[1 - f(z_1)]}{(z_1 - 1)^2}\psi(z_1) + \frac{2[\psi'(z_1) + \psi(z_1)][f(z_1) - z_1]}{z_1 - 1} \\ & + 2\psi(z_1) + \frac{2[f(z_1) - 1]^2}{(z_1 - 1)^3}\psi^2(z_1) + \psi'(z_1) = 0. \end{aligned}$$

By (3.23) we get

$$(3.31) \quad f(z_1) = \frac{z_1 + 1}{2} + \frac{z_1 - 1}{2\psi(z_1)}.$$

Hence by (3.30) and (3.31), we have

$$(3.32) \quad 2(z_1 - 1)\psi'(z_1) + \psi^3(z_1) + 2(z_1 - 1)\psi^2(z_1) + (2z_1 - 3)\psi(z_1) = 0.$$

Let

$$\Delta = 2(z - 1)\psi'(z) + \psi^3(z) + 2(z - 1)\psi^2(z) + (2z - 3)\psi(z).$$

If $\Delta \not\equiv 0$, then

$$(3.33) \quad \bar{N}\left(r, \frac{1}{f' - 1}\right) \leq N\left(r, \frac{1}{\Delta}\right) + O(\log r) \leq T(r, \Delta) + S(r, f) \leq S(r, f).$$

Thus by (3.8) and (3.33) we get a contradiction: $T(r, f) = S(r, f)$.

Hence, $\Delta \equiv 0$, that is

$$(3.34) \quad 2(z - 1)\psi'(z) + \psi^3(z) + 2(z - 1)\psi^2(z) + (2z - 3)\psi(z) \equiv 0.$$

Obviously, by (3.34), ψ is an entire function. We claim that ψ is not transcendental. Indeed, if ψ is transcendental, then by (3.34) we have

$$\begin{aligned} 3T(r, \psi) &= 3m(r, \psi) = m(r, \psi^3) \\ &= m(r, 2(z - 1)\psi^2 + (2z - 3)\psi + 2(z - 1)\psi') \\ &\leq m(r, \psi) + m\left(r, 2(z - 1)\psi + (2z - 3) + 2(z - 1)\frac{\psi'}{\psi}\right) \\ &\leq 2m(r, \psi) + S(r, \psi) = 2T(r, \psi) + S(r, \psi). \end{aligned}$$

Thus we get a contradiction: $T(r, \psi) = S(r, \psi)$. Hence ψ is a polynomial. Next, by simple computation, we deduce that either $\psi \equiv 0$ or $\psi \equiv -1$.

If $\psi \equiv 0$, Then by (3.10) we get

$$(z - 1)f''(z) \equiv z[f'(z) - 1],$$

which means $H \equiv 0$, a contradiction.

If $\psi \equiv -1$, then by (3.18) we know $\phi \equiv 0$. Thus by (3.9) we have

$$\frac{f'(z) - 1}{f(z) - z} \equiv \frac{f''(z) - 1}{f'(z) - z}.$$

Next we can easily deduce that $f \equiv f'$.

Now we prove that f can not be a polynomial.

By simple computation, f can not be a polynomial with $\deg f \leq 2$. Next we prove that f can not be a polynomial with $\deg f \geq 3$. Suppose that there exists such polynomial f with $f(z) = z \Leftrightarrow f'(z) = z$ and $f'(z) = z \Rightarrow f''(z) = z$, and $d = \deg f \geq 3$. Let z_1, z_2, \dots, z_n be the fixed points of f . Then we have

$$(3.35) \quad f(z) = z + A(z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2} \cdots (z - z_n)^{\alpha_n},$$

$$(3.36) \quad f'(z) = z + B(z - z_1)^{\beta_1}(z - z_2)^{\beta_2} \cdots (z - z_n)^{\beta_n},$$

and

$$(3.37) \quad f''(z) = z + C(z - z_1)^{\gamma_1}(z - z_2)^{\gamma_2} \cdots (z - z_n)^{\gamma_n}p(z),$$

where $p (\neq 0)$ is a polynomial, and A, B, C are three non-zero constants and $\{\alpha_j\}, \{\beta_j\}, \{\gamma_j\}$ ($j = 1, 2, \dots, n$) are positive integers satisfying

$$(3.38) \quad \sum_{j=1}^n \alpha_j = d, \quad \sum_{j=1}^n \beta_j = d - 1, \quad \sum_{j=1}^n \gamma_j + \deg p = d - 2.$$

From (3.35) and (3.36) we obtain

$$(3.39) \quad 1 + A \sum_{i=1}^n \alpha_i (z - z_i)^{\alpha_i - 1} \prod_{j \neq i} (z - z_j)^{\alpha_j} \equiv z + B \prod_{j=1}^n (z - z_j)^{\beta_j}.$$

If $\alpha_j \geq 2$, then by (3.39) we get $z_j = 1$. Similarly, we know that if $\beta_j \geq 2$, then $z_j = 1$. Without loss of generality, we assume that $j = 1$. Thus by (3.35)–(3.36) and (3.38) we have

$$(3.40) \quad f(z) = z + A(z - 1)^{\alpha_1}(z - z_2) \cdots (z - z_n),$$

$$(3.41) \quad f'(z) = z + B(z - 1)^{\alpha_1 - 1}(z - z_2) \cdots (z - z_n).$$

If $\alpha_1 \geq 3$, then by (3.40) we get $f(1) = 1$ and $f''(1) = 0$, which contradicts $f(z) = z \Rightarrow f''(z) = z$. Thus $\alpha_1 = 2$. Hence by (3.37)–(3.38), and (3.41) we have

$$(3.42) \quad f'(z) = z + B(z - 1)(z - z_2) \cdots (z - z_n).$$

$$(3.43) \quad f''(z) = z + C(z - 1)(z - z_2) \cdots (z - z_n).$$

Thus by (3.42) and (3.43) we get a contradiction: $\deg f' = \deg f''$. The proof of Theorem 3 is complete.

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