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GROWTH OF SOLUTIONS OF AN *n*-th ORDER LINEAR DIFFERENTIAL EQUATION WITH ENTIRE COEFFICIENTS

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Abstract

We consider a differential equation $f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$, where $A_0(z), \dots, A_{n-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. Suppose that there exist a positive number μ, and a sequence $(z_j)_{j \in N}$ with $\lim_{j\to+\infty} z_j = \infty$, and also two real numbers α, β $(0 \le \beta < \alpha)$ such that $|A_0(z_j)| \ge e^{\alpha|z_j|^\mu}$ and $|A_k(z_j)| \le e^{\beta|z_j|^\mu}$ as $j \to +\infty$ $(k = 1, \dots, n-1)$. We prove that all solutions $f \not\equiv 0$ of this equation are of infinite order. This result is a generalization of one theorem of Gundersen ([3], p. 418).

1. Introduction

For $n \ge 2$, we consider a linear differential equation

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where $A_0(z), \dots, A_{n-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. Let $\rho(f)$ denote the order of an entire function f, that is,

$$\rho(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [6]), and $M(r,f) = \max_{|z|=r}|f(z)|$.

It is well-known that all solutions of (1.1) are entire functions, and if some of the coefficients of (1.1) are transcendental, then (1.1) has at least one solution with order $\rho(f) = +\infty$.

The question which arises is: What conditions on $A_0(z), \dots, A_{n-1}(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order?

In this paper we prove three results concerning this question. According to [5], [7, pp. 199–209], [8, pp. 106–108], [9, pp. 65–67], we know that if

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$A_0(z), \ldots, A_{n-1}(z)$ are polynomials with $A_0(z) \not\equiv 0$, then every solution f of (1.1) is an entire function with finite rational order.

In the study of the differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.2}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions, Gundersen proved the following result:

Theorem 1.1 ([3, p. 418]). *Let $A(z)$ and $B(z) \not\equiv 0$ be entire functions, and let α, β, θ_1 and θ_2 be real numbers with $\alpha > 0$, $\beta > 0$ and $\theta_1 < \theta_2$. If*

$$|B(z)| \geq \exp\{(1 + o(1))\alpha|z|^{\beta}\} \tag{1.3}$$

and

$$|A(z)| \leq \exp\{o(1)|z|^{\beta}\} \tag{1.4}$$

as $z \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$, then every solution $f \not\equiv 0$ of (1.2) *has infinite order.*

Remark. Theorem 1.1 was recently extented for an n-th order linear differential equation (see [1]).

In the same paper, Gundersen also proved the following:

Theorem 1.2 ([3, p. 417]). *Let $A(z)$ and $B(z)$ be entire functions such that either* (i) $\rho(A) < \rho(B)$ *or* (ii) *A is a polynomial and B is transcendental. Then every solution $f \not\equiv 0$ of* (1.2) *has infinite order.*

2. Statement and proof of results

In this paper we prove the following results:

Theorem 2.1. *Suppose that there exist a positive number μ, and a sequence of points $(z_j)_{j \in N}$ with $\lim_{j \to +\infty} z_j = \infty$, and two real numbers α, β $(0 \leq \beta < \alpha)$ such that*

$$|A_0(z_j)| \geq e^{\alpha|z_j|^{\mu}} \tag{2.1}$$

and

$$|A_k(z_j)| \leq e^{\beta|z_j|^{\mu}} \quad (k = 1, \ldots, n-1), \tag{2.2}$$

as $j \to +\infty$. Then every solution $f \not\equiv 0$ of (1.1) *has infinite order.*

From Theorem 2.1, we deduce the following two results:

Corollary 2.2. *Suppose that*

$$\max\{\rho(A_k) : k = 1, \ldots, n-1\} < \rho(A_0). \tag{2.3}$$

Then every solution $f \not\equiv 0$ *of* (1.1) *has infinite order.*

Corollary 2.3. *Suppose that* $A_1(z), \ldots, A_{n-1}(z)$ *are polynomials and* $A_0(z)$ *is transcendental. Then every solution* $f \not\equiv 0$ *of* (1.1) *has infinite order.*

Corollary 2.2 and Corollary 2.3 were proved by Z.-x. Chen and S.-a. Gao in [2]. In this paper, we give another proof.

In the proof of Theorem 2.1, we need the following lemma:

Lemma 2.4 ([4, p. 89]). *Let* w *be a transcendental entire function of finite order* ρ. *Let* $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers satisfying* $k_i > j_i \geq 0$ *for* $i = 1, \ldots, m$, *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E \subset [0, 2\pi)$ *with linear measure zero such that, if* $\psi_0 \in [0, 2\pi) - E$, *then there is a constant* $R_0 = R_0(\psi_0) > 1$ *such that for all* z *satisfying* $\arg z = \psi_0$ *with* $|z| \geq R_0$ *and for all* $(k, j) \in \Gamma$, *the following estimate hold*:

$$\left|\frac{w^{(k)}(z)}{w^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Proof of Theorem 2.1. Suppose that $f \not\equiv 0$ is a solution of (1.1) with $\rho(f) < +\infty$. We can write (1.1) as

$$\frac{1}{A_0(z)} \frac{f^{(n)}}{f} + \sum_{k=1}^{n-1} \frac{A_k(z)}{A_0(z)} \frac{f^{(k)}}{f} = -1. \tag{2.4}$$

Then by Lemma 2.4, there exists a set $E \subset [0, 2\pi)$ with linear measure zero such that, if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z_j satisfying $\arg z_j = \psi_0$with $|z_j| \geq R_0$ and for all $k = 1, 2, \ldots, n$, we have

$$\left|\frac{f^{(k)}(z_j)}{f(z_j)}\right| \leq |z_j|^{kc} \quad (k = 1, \ldots, n, \quad c = \rho - 1 + \varepsilon). \tag{2.5}$$

Then by (2.1), (2.2) and (2.5), we obtain that

$$\left|\frac{A_k(z_j)}{A_0(z_j)}\right| \left|\frac{f^{(k)}(z_j)}{f(z_j)}\right| \leq \frac{1}{e^{(\alpha-\beta)|z_j|^{\mu}}} |z_j|^{kc} \quad (k = 1, \ldots, n-1, \quad c = \rho - 1 + \varepsilon). \tag{2.6}$$

Since

$$\lim_{j \to +\infty} \frac{1}{e^{(\alpha-\beta)|z_j|^{\mu}}} |z_j|^{kc} = 0 \quad (k = 1, \ldots, n-1, \quad c = \rho - 1 + \varepsilon),$$

we see

$$\lim_{j \to +\infty} \left|\frac{A_k(z_j)}{A_0(z_j)}\right| \left|\frac{f^{(k)}(z_j)}{f(z_j)}\right| = 0 \quad (k = 1, \ldots, n-1).$$

We have also from (2.1) and (2.5) that

$$\left|\frac{1}{A_0(z_j)}\right| \left|\frac{f^{(n)}(z_j)}{f(z_j)}\right| \le \frac{1}{e^{\alpha|z_j|^{\mu}}} |z_j|^{nc} \quad (c = \rho - 1 + \varepsilon), \tag{2.7}$$

which implies

$$\lim_{j\to+\infty} \left|\frac{1}{A_0(z_j)}\right| \left|\frac{f^{(n)}(z_j)}{f(z_j)}\right| = 0. \tag{2.8}$$

Letting $j \to +\infty$ in the relation

$$\frac{1}{A_0(z_j)} \frac{f^{(n)}(z_j)}{f(z_j)} + \sum_{k=1}^{n-1} \frac{A_k(z_j)}{A_0(z_j)} \frac{f^{(k)}(z_j)}{f(z_j)} = -1,$$

we get a contradiction. Thus every solution $f \not\equiv 0$ of (1.1) has infinite order.

Next, we give two examples that illustrates Theorem 2.1.

Example 1. Consider the differential equation

$$f'' - f' - e^{2z} f = 0. \tag{2.9}$$

In this equation, for $z = z_j = j \to \infty$, we have

$$|A_0(z_j)| = |-e^{2z_j}| = e^{2j} \ge e^{2|z_j|},$$
$$|A_1(z_j)| = 1 \le e^{0.|z_j|}.$$

It is easy to see that the conditions (2.1) and (2.2) in Theorem 2.1 are satisfied. The two linearly independent functions $f_1(z) = e^{e^z}$ and $f_2(z) = e^{-e^z}$ are solutions of (2.9) with $\rho(f_1) = \rho(f_2) = +\infty$.

Example 2. Consider the differential equation

$$f''' - (3 + 6e^z) f'' + (2 + 6e^z + 11e^{2z}) f' - 6e^{3z} f = 0. \tag{2.10}$$

In this equation, for $z = z_j = (1 + i) j \to \infty$, we have

$$|A_0(z_j)| = |-6e^{3z_j}| = 6e^{3j} > e^{3j} = e^{3(\sqrt{2}/2)|z_j|},$$
$$|A_1(z_j)| = |2 + 6e^{z_j} + 11e^{2z_j}| \le 19e^{2j} < e^{(5/2)j} = e^{5(\sqrt{2}/4)|z_j|}$$

and

$$|A_2(z_j)| = |-(3 + 6e^{z_j})| \le 9e^j < e^{(5/2)j} = e^{5(\sqrt{2}/4)|z_j|}.$$

Hence the conditions (2.1) and (2.2) of Theorem 2.1 are verified. The three linearly independent functions $f_1(z) = e^{e^z}$, $f_2(z) = e^{2e^z}$ and $f_3(z) = e^{3e^z}$ are solutions of (2.10) with $\rho(f_1) = \rho(f_2) = \rho(f_3) = +\infty$.

We now give a generalization of Example 2.

Example 3. Consider the differential equation

$$f^{(n)} + P_{n-1}(e^z)f^{(n-1)} + \cdots + P_1(e^z)f' + \beta e^{\alpha z} f = 0, \tag{2.11}$$

where $\alpha \in \boldsymbol{R}$ with $\alpha > 0$, $\beta \in \boldsymbol{C}^\star$, and $P_1, \dots, P_{n-1}$ are polynomials such that $\max_{1 \le k \le n-1} \deg(P_k) < \alpha$. It follows from Theorem 2.1 that every solution $f \not\equiv 0$ of (2.11) has infinite order.

Proof of Corollary 2.2. Let $\max_{1 \le k \le n}\{\rho(A_k)\} = b < \rho(A_0) = a$. Then for a given ε, $0 < \varepsilon < (a-b)/2$, we have

$$|A_k(z)| < e^{|z|^{b+\varepsilon}} \quad (k = 1, \dots, n-1) \tag{2.12}$$

and

$$|A_0(z)| > e^{|z|^{a-\varepsilon}} \tag{2.13}$$

for sufficiently large $|z|$. Then by (2.13), there exists a number $\alpha > 1$ such that

$$|A_0(z)| > e^{\alpha |z|^{b+\varepsilon}} \tag{2.14}$$

for sufficiently large $|z|$. By making use of (2.12), (2.14) and Theorem 2.1, we get our result.

Proof of Corollary 2.3. As in the proof of Corollary 2.2, we obtain immediately Corollary 2.3.

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