

A NOTE ON A UNICITY THEOREM OF K. TOHGE

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Abstract

In this paper, we deal with the problem of uniqueness of meromorphic functions sharing three values CM, and get rid of the restriction on the hyper-orders in a unicity theorem of K. Tohge. An example is provided to show that the result in this paper is best possible.

1. Introduction and main results

Let f and g be two non-constant meromorphic functions in the complex plane. It is assumed that the reader is familiar with the standard notations of Nevanlinna's theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\bar{N}(r, f)$ and so on, which can be found in [1]. We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. The notation $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ ($r \rightarrow \infty, r \notin E$).

Let a be a complex number, we say that f and g share the value a CM provided $f - a$ and $g - a$ have the same zeros counting multiplicities (see [2]). We say that f and g share ∞ CM provided that $1/f$ and $1/g$ share 0 CM. In this paper, we also need the following definition.

DEFINITION. Let f be a non-constant meromorphic function. The hyper-order of f , denoted $\nu(f)$, is defined by

$$\nu(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

In 1988, K. Tohge [3] proved the following theorems:

THEOREM A. *Let f and g be two distinct transcendental meromorphic functions sharing 0, 1 and ∞ CM. If f' and g' share 0 CM, then f and g satisfy one of the following relations:*

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- (i) $f \cdot g \equiv 1$,
- (ii) $(f - 1)(g - 1) \equiv 1$,
- (iii) $f + g \equiv 1$,
- (iv) $f \equiv cg$,
- (v) $f - 1 \equiv c(g - 1)$,
- (vi) $[(c - 1)f + 1] \cdot [(c - 1)g - c] \equiv -c$,

where $c (\neq 0, 1)$ is a constant.

THEOREM B. *Let f and g be two distinct transcendental meromorphic functions sharing $0, 1$ and ∞ CM, and let $a (\neq 0)$ be a finite complex number. If f' and g' share a CM and $\max\{v(f), v(g)\} < 1$, then f and g satisfy one of the following relations:*

- (i) $f \cdot g \equiv 1$,
- (ii) $(f - 1)(g - 1) \equiv 1$,
- (iii) $[(c - 1)f + 1] \cdot [(c - 1)g - c] \equiv -c$, where $c (\neq 0, 1)$ is a constant.

Now it is natural to ask the following question:

QUESTION 1. What can be said if we get rid of the condition “ $\max\{v(f), v(g)\} < 1$ ” in Theorem B?

In this paper, we shall answer Question 1 and obtain a new result. Indeed, we shall prove the following theorem:

THEOREM 1. *Let f and g be two distinct transcendental meromorphic functions sharing $0, 1$ and ∞ CM, and let $a (\neq 0)$ be a finite complex number. If f' and g' share a CM, then f and g satisfy one of the following relations:*

- (i) $f = Ae^{a\omega z}$, $g = (1/A)e^{-a\omega z}$, where ω satisfying $\omega^2 = -1$, and $A (\neq 0)$ are constants;
- (ii) $f = 1 + Ae^{a\omega z}$, $g = 1 + (1/A)e^{-a\omega z}$, where ω satisfying $\omega^2 = -1$, and $A (\neq 0)$ are constants;
- (iii) $f(z) = 1/(c - 1)(Ae^{a(c-1)\omega z} - 1)$, $g(z) = c/(c - 1)(1 - (1/A)e^{-a(c-1)\omega z})$, where A, c and ω are constants satisfying $A \neq 0$, $c \neq 0, 1$ and $\omega^2 = 1/c$.

It is obvious that if f and g satisfy the relations (i), (ii) and (iii) of Theorem 1, then the order of f is equal to 1. By Theorem 1 we immediately deduce the following uniqueness theorem of meromorphic functions.

THEOREM 2. *Let f and g be two transcendental meromorphic functions sharing $0, 1$ and ∞ CM, and suppose that f' and g' share a CM, where $a (\neq 0)$ is a finite complex number. If the order of f is not equal to 1, then $f \equiv g$.*

2. Some lemmas

The following notations are used throughout this paper.

Let h be a non-constant meromorphic function, and let k be a positive

integer. We denote by $N_k(r, 1/(h-a))$ the counting function of a -points of h with multiplicity $\leq k$, and denote by $N_{(k)}(r, 1/(h-a))$ the counting function of a -points of h with multiplicity $\geq k$ (see [2]).

Let f and g share $0, 1$ and ∞ CM, we denote by $N_0(r)$ the counting function of the zeros of $f - g$ not containing the zeros of $f, 1/f$ and $f - 1$ (see [4] or [5]).

LEMMA 1 (see [2, Lemma 9.1]). *Let f and g be two non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\delta_1(0, f) + \delta_1(1, f) > \frac{3}{2},$$

where

$$\delta_1(0, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_1(r, 1/f)}{T(r, f)}, \quad \delta_1(1, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_1(r, 1/(f-1))}{T(r, f)},$$

then $f + g \equiv 1$.

LEMMA 2 (see [2, p. 369]). *Let F and G be two non-constant meromorphic functions, and let*

$$\phi \equiv \frac{F''}{F'} - \frac{G''}{G'}.$$

If z_∞ is a common simple pole of F and G , then $\phi(z_\infty) = 0$.

LEMMA 3 (see [4, Lemma 4]). *Let f and g be meromorphic functions sharing $0, 1, \infty$ CM. If $f \neq g$, then*

$$N_{(2)}\left(r, \frac{1}{f}\right) + N_{(2)}\left(r, \frac{1}{f-1}\right) + N_{(2)}(r, f) = S(r, f).$$

LEMMA 4 (see [5, Lemma 7] or [6, Lemma 3]). *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If f is a Möbius transformation of g , then f and g satisfy one of the following relations:*

- (i) $f \cdot g \equiv 1$,
- (ii) $(f-1)(g-1) \equiv 1$,
- (iii) $f + g \equiv 1$,
- (iv) $f \equiv cg$,
- (v) $f-1 \equiv c(g-1)$,
- (vi) $[(c-1)f+1] \cdot [(c-1)g-c] \equiv -c$,

where $c (\neq 0, 1)$ is a constant.

LEMMA 5 (see [8, p. 120]). *Let f_1, f_2, \dots, f_n be meromorphic functions linearly independent over the complex number field C such that*

$$\sum_{i=1}^n f_i \equiv 1.$$

Then

$$T(r, f_j) < \sum_{i=1}^n N\left(r, \frac{1}{f_i}\right) + (n-1) \sum_{\substack{i=1 \\ i \neq j}}^n \bar{N}(r, f_i) + S(r) \quad (1 \leq j \leq n),$$

where $T(r) = \max_{1 \leq i \leq n} \{T(r, f_i)\}$ and $S(r) = o(T(r))$ ($r \rightarrow \infty, r \notin E$).

LEMMA 6 (see [2, Theorem 1.62] or [10, Theorem 1]). *Let f_1, f_2, \dots, f_n be non-constant meromorphic functions, and let f_{n+1} ($\neq 0$) be a meromorphic function such that*

$$(2.1) \quad \sum_{i=1}^{n+1} f_i \equiv 1.$$

If there exists a subset $I \subseteq R^+$ satisfying $\text{mes} I = \infty$ such that

$$(2.2) \quad \sum_{i=1}^{n+1} N\left(r, \frac{1}{f_i}\right) + n \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \bar{N}(r, f_i) < (\lambda + o(1))T(r, f_j) \quad (r \rightarrow \infty, r \in I, j = 1, 2, \dots, n),$$

where $\lambda < 1$. Then $f_{n+1} \equiv 1$.

Remark. Lemma 6 plays an important role for the proof of Theorem 1. Now we give a simple proof of Lemma 6. Suppose that

$$(2.3) \quad \sum_{j=1}^n f_j \not\equiv 0.$$

Without loss of generality, let

$$(2.4) \quad \sum_{j=1}^n f_j \equiv \sum_{i=1}^k a_i f_i,$$

where f_1, f_2, \dots, f_k ($1 \leq k \leq n$) are linearly independent over the complex number field C , and a_1, a_2, \dots, a_k are nonzero constants. By (2.1) and (2.4), we have

$$(2.5) \quad \sum_{i=1}^k a_i f_i + f_{n+1} \equiv 1.$$

By Lemma 5, (2.2) and (2.5) we can easily verify that $f_1, f_2, \dots, f_k, f_{n+1}$ are linearly dependent over the complex number field C , hence we have

$$(2.6) \quad c_1 f_1 + c_2 f_2 + \dots + c_k f_k + c_{k+1} f_{n+1} = 0,$$

where $c_1, c_2, \dots, c_k, c_{k+1}$ are constants not all equal to zero. Noting that

f_1, f_2, \dots, f_k are linearly independent over the complex number field C , we can see that $c_{k+1} \neq 0$. From (2.6) we have

$$(2.7) \quad f_{n+1} = -\frac{c_1}{c_{k+1}}f_1 - \dots - \frac{c_k}{c_{k+1}}f_k,$$

substituting (2.7) into (2.5) we get

$$(2.8) \quad \sum_{i=1}^k \left(a_i - \frac{c_i}{c_{k+1}} \right) f_i \equiv 1.$$

From (2.8) we can see that $a_i - \frac{c_i}{c_{k+1}}$ ($i = 1, 2, \dots, k$) are not all equal to zero.

By Lemma 5, (2.2) and (2.8) we can have a contradiction. Thus, $\sum_{j=1}^n f_j \equiv 0$, and $f_{n+1} \equiv 1$, which proves Lemma 6.

In 1999, H. X. Yi [11] proved the following result, which is an extension of Lemma 6: Let f_1, f_2, \dots, f_n be non-constant meromorphic functions, and let $f_{n+1}, f_{n+2}, \dots, f_{n+m}$ be meromorphic functions such that

$$f_k \not\equiv 0 \quad (k = n+1, n+2, \dots, n+m)$$

and

$$\sum_{i=1}^{n+m} f_i \equiv A,$$

where A is a nonzero constant. If there exists a subset $I \subseteq R^+$ satisfying $\text{mes}I = \infty$ such that

$$\sum_{i=1}^{n+m} N\left(r, \frac{1}{f_i}\right) + (n+m-1) \sum_{\substack{i=1 \\ i \neq j}}^{n+m} \bar{N}(r, f_i) < (\lambda + o(1))T(r, f_j) \\ (r \rightarrow \infty, r \in I, j = 1, 2, \dots, n),$$

where $\lambda < 1$. Then there exist $t_i \in \{0, 1\}$ ($i = 1, 2, \dots, m$) such that

$$\sum_{i=1}^m t_i f_{n+i} \equiv A.$$

LEMMA 7 (see [5, Theorem 1]). *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\limsup_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} > \frac{1}{2},$$

then f is a Möbius transformation of g .

LEMMA 8 (see [5, Theorem 2]). *Let f and g be two non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$0 < \limsup_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2},$$

then f is not any Möbius transformation of g , and f and g satisfy one of the following relations:

$$\begin{aligned} \text{(i)} \quad & f \equiv \frac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}, \quad g \equiv \frac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}, \\ \text{(ii)} \quad & f \equiv \frac{e^{(k+1)\gamma} - 1}{e^{(k+1-s)\gamma} - 1}, \quad g \equiv \frac{e^{-(k+1)\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}, \\ \text{(iii)} \quad & f \equiv \frac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}, \quad g \equiv \frac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1}, \end{aligned}$$

where s and k (≥ 2) are positive integers such that $1 \leq s \leq k$, and s and $k+1$ are relatively prime, and γ is a non-constant entire function.

LEMMA 9 (see [4, Lemma 1]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, then there exist two entire functions α and β such that*

$$(2.9) \quad f \equiv \frac{e^\alpha - 1}{e^\beta - 1}, \quad g \equiv \frac{e^{-\alpha} - 1}{e^{-\beta} - 1},$$

where $e^\beta \neq 1$, $e^\alpha \neq 1$ and $e^{\beta-\alpha} \neq 1$, and

$$(2.10) \quad T(r, g) + T(r, e^\alpha) + T(r, e^\beta) = O(T(r, f)) \quad (r \notin E).$$

LEMMA 10 (see [12, Lemma 2.4]). *Let h be a non-constant meromorphic function and let α, β, γ be meromorphic functions such that $T(r, \alpha) + T(r, \beta) + T(r, \gamma) = S(r, h)$, and $\alpha \neq 0$ or $\gamma \neq 0$. Furthermore, let*

$$H = \alpha h^2 + \beta h + \gamma.$$

If $\bar{N}(r, h) = S(r, h)$, $\bar{N}(r, 1/h) = S(r, h)$ and $N_1(r, 1/H) = S(r, h)$, then $\beta^2 - 4\alpha\gamma \equiv 0$.

3. Proof of Theorem 1

By the assumptions of Theorem 1, we have

$$(3.1) \quad \frac{f' - a}{g' - a} = e^\delta,$$

where δ is an entire function. Suppose that $e^\delta \equiv A$, where A is a nonzero constant. From (3.1) we get

$$(3.2) \quad f - Ag = (1 - A)az + C,$$

where C is a constant. Since $f \not\equiv g$, from (3.2) we know that

$$(3.3) \quad (1 - A)az + C \not\equiv 0.$$

By (3.2) and (3.3) we get

$$\delta_1(0, f) + \delta_1(1, f) = 2.$$

By Lemma 1, we have $f + g \equiv 1$ and $f' + g' \equiv 0$, which implies that a is a Picard value of f' and g' . This contradicts Hayman's inequality (see [1, Theorem 3.5]). Thus e^δ is not a constant, and hence

$$(3.4) \quad \delta' \not\equiv 0.$$

By logarithmic differentiation, from (3.1) we obtain

$$(3.5) \quad \delta' = \frac{f''}{f' - a} - \frac{g''}{g' - a}.$$

By Lemma 2, (3.4) and (3.5), we get

$$(3.6) \quad N_1(r, f) \leq N\left(r, \frac{1}{\delta'}\right) \leq T(r, \delta') + O(1) = S(r, f).$$

By Lemma 3, we have

$$(3.7) \quad N_2(r, f) = S(r, f).$$

By (3.6) and (3.7), we obtain

$$(3.8) \quad N(r, f) = S(r, f).$$

We discuss the following two cases.

CASE 1. Suppose that f is a Möbius transformation of g . By Lemma 4, we know that f and g satisfy one of the six relations in Lemma 4.

Assume that f and g satisfy the relation (i) in Lemma 4. Let $f = e^\alpha$, where α is a non-constant entire function. Then $g = e^{-\alpha}$. Substituting f and g into (3.1) we get

$$(3.9) \quad \frac{\alpha' e^{2\alpha} - a e^\alpha}{-\alpha' - a e^\alpha} = e^\delta.$$

By (3.9) we have

$$(3.10) \quad T(r, e^\delta) \geq T(r, e^\alpha) + S(r, f)$$

and

$$(3.11) \quad \frac{\alpha'}{a} e^\alpha + e^\delta + \frac{\alpha'}{a} e^{\delta-\alpha} \equiv 1.$$

By Lemma 6, (3.10) and (3.11) we obtain

$$(3.12) \quad \frac{\alpha'}{a} e^{\delta-\alpha} \equiv 1, \quad \frac{\alpha'}{a} e^{\alpha} + e^{\delta} \equiv 0.$$

From (3.12) we get $\alpha(z) = a\omega z + C$, where ω satisfying $\omega^2 = -1$, and C are constants. Thus $f(z) = Ae^{a\omega z}$ and $g(z) = (1/A)e^{-a\omega z}$, where A is a nonzero constant. From this we have the relation (i) in Theorem 1.

Assume that f and g satisfy the relation (ii) in Lemma 4. In the same manner as above, we can obtain $f(z) = 1 + Ae^{a\omega z}$ and $g(z) = 1 + (1/A)e^{-a\omega z}$, where ω satisfying $\omega^2 = -1$, and $A (\neq 0)$ are constants. From this we have the relation (ii) in Theorem 1.

Assume that f and g satisfy the relation (vi) in Lemma 4. In the same manner as above, we can obtain $f(z) = 1/(c-1)(Ae^{a(c-1)\omega z} - 1)$ and $g(z) = c/(c-1)(1 - (1/A)e^{-a(c-1)\omega z})$, where ω satisfying $\omega^2 = 1/c$, and $A (\neq 0)$ are constants. From this we have the relation (iii) in Theorem 1.

Assume that f and g satisfy the relation (iii) in Lemma 4. Since f and g share 0, 1 and ∞ CM, from the relation (iii) in Lemma 4, we know that 0 and 1 are Picard values of f . Thus $N(r, f) = T(r, f) + S(r, f)$, which contradicts (3.8).

Assume that f and g satisfy the relations (iv) and (v) in Lemma 4. In the same manner as above, we can obtain contradictions.

CASE 2. Suppose that f is not any Möbius transformation of g . By Lemma 7, we consider the following two subcases.

SUBCASE 2.1. Assume that

$$0 < \limsup_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2}.$$

By Lemma 8, we know that f and g satisfy one of the three relations in Lemma 8.

Assume that f and g satisfy the relation (i) in Lemma 8. Then we have $N(r, f) = T(r, f) + S(r, f)$, which contradicts (3.8).

Assume that f and g satisfy the relation (ii) in Lemma 8. By (3.8) we know that $k = s$. Thus,

$$(3.13) \quad f = e^{k\gamma} + e^{(k-1)\gamma} + \cdots + 1, \quad g = e^{-k\gamma} + e^{-(k-1)\gamma} + \cdots + 1.$$

By (3.13) we obtain

$$(3.14) \quad T(r, f) = kT(r, e^\gamma) + S(r, f), \quad T(r, g) = kT(r, e^\gamma) + S(r, f).$$

Substituting (3.13) into (3.1) we get

$$(3.15) \quad \frac{k\gamma' e^{2k\gamma} + (k-1)\gamma' e^{(2k-1)\gamma} + \cdots + \gamma' e^{(k+1)\gamma} - ae^{k\gamma}}{-k\gamma' - (k-1)\gamma' e^\gamma - \cdots - \gamma' e^{(k-1)\gamma} - ae^{k\gamma}} = e^\delta.$$

By (3.14) and (3.15) we have

$$(3.16) \quad T(r, e^\delta) \geq kT(r, e^\gamma) + S(r, f)$$

and

$$(3.17) \quad \begin{aligned} & \frac{k\gamma'}{a} e^{k\gamma} + \frac{(k-1)\gamma'}{a} e^{(k-1)\gamma} + \dots + \frac{\gamma'}{a} e^\gamma + e^\delta \\ & + \frac{\gamma'}{a} e^{\delta-\gamma} + \frac{2\gamma'}{a} e^{\delta-2\gamma} + \dots + \frac{k\gamma'}{a} e^{\delta-k\gamma} \equiv 1. \end{aligned}$$

By Lemma 6, (3.14), (3.16) and (3.17) we obtain

$$\frac{k\gamma'}{a} e^{\delta-k\gamma} \equiv 1$$

and hence

$$(3.18) \quad e^\delta \equiv \frac{a}{k\gamma'} e^{k\gamma}.$$

Substituting (3.18) into (3.17) we get

$$(3.19) \quad \left(\frac{k\gamma'}{a} + \frac{a}{k\gamma'} \right) e^{k\gamma} + \left(\frac{(k-1)\gamma'}{a} + \frac{1}{k} \right) e^{(k-1)\gamma} + \dots + \left(\frac{\gamma'}{a} + \frac{k-1}{k} \right) e^\gamma \equiv 0.$$

From (3.19) we obtain

$$(3.20) \quad \frac{k\gamma'}{a} + \frac{a}{k\gamma'} \equiv 0, \quad \frac{(k-1)\gamma'}{a} + \frac{1}{k} \equiv 0, \quad \frac{\gamma'}{a} + \frac{k-1}{k} \equiv 0.$$

From (3.20) we have a contradiction.

Assume that f and g satisfy the relation (iii) in Lemma 8. By (3.8) we know that $s = k$. Thus,

$$f = -e^{k\gamma} - e^{(k-1)\gamma} - \dots - e^\gamma, \quad g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \dots - e^{-\gamma}.$$

In the same manner as above, we can obtain a contradiction.

SUBCASE 2.2. Assume that

$$\limsup_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} = 0.$$

Thus,

$$(3.21) \quad N_0(r) = S(r, f).$$

Noting that f and g share 0, 1 and ∞ CM, by Lemma 9 we have (2.9) and (2.10). If $e^\beta \equiv C$, where C is a nonzero constant. From (2.9) we obtain

$$\frac{(f-1)g}{f(g-1)} \equiv C.$$

From this we get that f is a Möbius transformation of g , which is a contradiction. Thus, e^β is not a constant. From (2.9) we obtain

$$(3.22) \quad f - g = \frac{(e^\alpha - 1)(1 - e^{\beta-\alpha})}{e^\beta - 1}.$$

We use $N_0^*(r)$ to denote the counting function of the common zeros of $e^\alpha - 1$ and $e^\beta - 1$. From (3.22), the following formula is obviously

$$N_0(r) = N_0^*(r) + S(r, f).$$

From this and (3.21) we have

$$(3.23) \quad N_0^*(r) = S(r, f).$$

By (2.9) and (3.23), we have

$$(3.24) \quad N(r, f) = N\left(r, \frac{1}{e^\beta - 1}\right) + S(r, f).$$

By (3.8) and (3.24) we get

$$(3.25) \quad T(r, e^\beta) = S(r, f).$$

From (2.9) and (3.25) we have

$$(3.26) \quad T(r, f) = T(r, e^\alpha) + S(r, f), \quad T(r, g) = T(r, f) + S(r, f).$$

Substituting (2.9) into (3.1) we get

$$(3.27) \quad \frac{(\alpha' e^\beta - \beta' e^\beta - \alpha') e^{2\alpha} + (\beta' e^\beta - a e^{2\beta} + 2a e^\beta - a) e^\alpha}{(\alpha' e^{2\beta} + \beta' e^\beta - \alpha' e^\beta) - (\beta' e^\beta + a e^{2\beta} - 2a e^\beta + a) e^\alpha} = e^\delta.$$

It is obvious that

$$(3.28) \quad \beta' e^\beta - a e^{2\beta} + 2a e^\beta - a \neq 0, \quad \beta' e^\beta + a e^{2\beta} - 2a e^\beta + a \neq 0.$$

If $\alpha' e^\beta - \beta' e^\beta - \alpha' \equiv 0$, then

$$(3.29) \quad \alpha' = \frac{\beta' e^\beta}{e^\beta - 1}.$$

By integration, from (3.29) we obtain

$$(3.30) \quad e^\alpha = C(e^\beta - 1),$$

where C is a nonzero constant, which is a contradiction. Thus

$$(3.31) \quad \alpha' e^\beta - \beta' e^\beta - \alpha' \neq 0.$$

In the same manner as above, we have

$$(3.32) \quad \alpha' e^{2\beta} + \beta' e^\beta - \alpha' e^\beta \neq 0.$$

By (3.25), (3.26), (3.27), (3.28), (3.31) and (3.32) we have

$$(3.33) \quad T(r, e^\delta) \geq T(r, e^\alpha) + S(r, f)$$

and

$$(3.34) \quad -\frac{\alpha'e^\beta - \beta'e^\beta - \alpha'}{\beta'e^\beta - ae^{2\beta} + 2ae^\beta - a}e^\alpha - \frac{\beta'e^\beta + ae^{2\beta} - 2ae^\beta + a}{\beta'e^\beta - ae^{2\beta} + 2ae^\beta - a}e^\delta \\ + \frac{\alpha'e^{2\beta} + \beta'e^\beta - \alpha'e^\beta}{\beta'e^\beta - ae^{2\beta} + 2ae^\beta - a}e^{\delta-\alpha} \equiv 1.$$

By Lemma 6, (3.25), (3.26), (3.33) and (3.34) we obtain

$$(3.35) \quad \frac{\alpha'e^{2\beta} + \beta'e^\beta - \alpha'e^\beta}{\beta'e^\beta - ae^{2\beta} + 2ae^\beta - a}e^{\delta-\alpha} \equiv 1,$$

and

$$(3.36) \quad \frac{\alpha'e^\beta - \beta'e^\beta - \alpha'}{\beta'e^\beta - ae^{2\beta} + 2ae^\beta - a}e^\alpha + \frac{\beta'e^\beta + ae^{2\beta} - 2ae^\beta + a}{\beta'e^\beta - ae^{2\beta} + 2ae^\beta - a}e^\delta \equiv 0.$$

From (3.35) and (3.36) we get

$$(3.37) \quad \frac{(-\alpha'e^\beta + \beta'e^\beta + \alpha')(\alpha'e^{2\beta} + \beta'e^\beta - \alpha'e^\beta)}{(\beta'e^\beta + ae^{2\beta} - 2ae^\beta + a)(\beta'e^\beta - ae^{2\beta} + 2ae^\beta - a)} \equiv 1.$$

From (3.37) we obtain

$$(3.38) \quad e^\beta \left(\alpha' - \frac{\beta'}{2} \right)^2 \equiv a^2 e^{2\beta} + \left(\frac{(\beta')^2}{4} - 2a^2 \right) e^\beta + a^2.$$

Set

$$(3.39) \quad H = a^2 e^{2\beta} + \left(\frac{(\beta')^2}{4} - 2a^2 \right) e^\beta + a^2,$$

then

$$(3.40) \quad H = e^\beta \left(\alpha' - \frac{\beta'}{2} \right)^2.$$

Applying Lemma 10 to H , from (3.39) and (3.40) we have

$$(3.41) \quad \left(\frac{(\beta')^2}{4} - 2a^2 \right)^2 - 4a^4 \equiv 0.$$

From (3.41) we get

$$(3.42) \quad \beta' = 4a\omega,$$

and hence

$$(3.43) \quad e^\beta = Ae^{4a\omega z},$$

where ω satisfying $\omega^2 = 1$, and A are nonzero constants. Substituting (3.42) and (3.43) into (3.38), we have

$$\alpha' = 2a\omega + a\omega_1 \left(B_1 e^{2a\omega z} + \frac{1}{B_1} e^{-2a\omega z} \right),$$

where B_1 and ω_1 are constants satisfying $B_1^2 = A$ and $\omega_1^2 = 1$. Thus,

$$(3.44) \quad \alpha = 2a\omega z + \frac{\omega_1}{2\omega} \left(B_1 e^{2a\omega z} - \frac{1}{B_1} e^{-2a\omega z} \right) + C,$$

where C is a constant. Set $B = \frac{\omega_1 B_1}{\omega}$, then $B^2 = A$ and

$$(3.45) \quad \alpha = 2a\omega z + \frac{1}{2} \left(B e^{2a\omega z} - \frac{1}{B} e^{-2a\omega z} \right) + C.$$

From (3.45) we have

$$(3.46) \quad \alpha' = 2a\omega + Ba\omega \cdot e^{2a\omega z} + \frac{a\omega}{B} \cdot e^{-2a\omega z}.$$

Noting that $B^2 = A$, from (3.42), (3.43) and (3.46) we get

$$(3.47) \quad \alpha' e^{2\beta} + \beta' e^\beta - \alpha' e^\beta = B^5 a\omega \cdot e^{10a\omega z} + 2B^4 a\omega \cdot e^{8a\omega z} + 2B^2 a\omega \cdot e^{4a\omega z} - Ba\omega \cdot e^{2a\omega z}$$

and

$$(3.48) \quad \beta' e^\beta - a e^{2\beta} + 2a e^\beta - a = (4a\omega + 2a)B^2 \cdot e^{4a\omega z} - aB^4 \cdot e^{8a\omega z} - a.$$

Substituting (3.47) and (3.48) into (3.35), we deduce

$$(3.49) \quad \frac{e^{8a\omega z} - (4\omega + 2)/B^2 \cdot e^{4a\omega z} + 1/B^4}{e^{8a\omega z} + 2/B \cdot e^{6a\omega z} + 2/B^3 \cdot e^{2a\omega z} - 1/B^4} \equiv -B\omega e^{\delta - \alpha + 2a\omega z}.$$

Let

$$(3.50) \quad P_1(\chi) = \chi^8 - \frac{4\omega + 2}{B^2} \cdot \chi^4 + \frac{1}{B^4}, \quad P_2(\chi) = \chi^8 + \frac{2}{B} \cdot \chi^6 + \frac{2}{B^3} \cdot \chi^2 - \frac{1}{B^4}.$$

From (3.50) we can easily see that every root of $P_j(\chi) = 0$ ($j = 1, 2$) is not equal to zero, and that there is at least one root of $P_1(\chi) = 0$ that is not any root of $P_2(\chi) = 0$. Thus, from (3.49) we can have a contradiction.

Theorem 1 is thus completely proved.

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