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A NOTE ON THE PRODUCT OF MEROMORPHIC FUNCTIONS AND ITS DERIVATIVES

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Abstract

It is shown that if f is an even or odd transcendental meromorphic function and if c is any even meromorphic function which does not vanish identically and satisfies $T(r,c) = o(T(r,f))$ as $r \to +\infty$, then $ff' - c$ has infinitely many zeros.

1. Introduction and our main results

In 1959, W. K. Hayman [5] proved that

Theorem A. *If n is an integer greater than or equal to 3 and f is a transcendental meromorphic function, then $f^n f'$ takes every non-zero complex number infinitely many times.*

Later, he conjectured [6] that this remains valid for the cases $n = 1, 2$. In 1979, E. Mues [7] proved the case $n = 2$ and the conjecture was proven by A. Eremenko and W. Bergweiler [2] in 1995 and independently by H. H. Chen and M. L. Fang [3].

In 1994, Yik-Man Chiang asked W. Bergweiler whether $ff' - c$ has infinitely many zeros if f is a transcendental meromorphic function and if c is a meromorphic function which does not vanish identically and satisfies $T(r,c) = o(T(r,f))$ as $r \to +\infty$. In [8], Q. D. Zhang studied the value distribution of $\varphi(z)f(z)f'(z)$ and obtained the following theorem.

Theorem B. *If f is a transcendental meromorphic function and φ is a non-zero meromorphic function such that $T(r,\varphi) = S(r,f)$ as $r \to +\infty$, then*

$$T(r,f) < \frac{9}{2}\overline{N}(r,f) + \frac{9}{2}\overline{N}\left(r, \frac{1}{\varphi f f' - 1}\right) + S(r,f).$$

By this, we have

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Corollary A. *If* $\delta(\infty, f) > 7/9$, *then* $ff' - c$ *has infinitely many zeros.*

In 1995, W. Bergweiler and A. Eremenko [2] used some results from iteration theory to show that $ff' - c$ has infinitely many zeros if c is a non-zero constant. Later, W. Bergweiler [1] answered this question affirmatively in the case that f is of finite order and c is a polynomial. Actually, he showed that

Theorem C. *If* f *is a transcendental meromorphic function of finite order and* c *is a polynomial, then* $ff' - c$ *has infinitely many zeros.*

In the following discussion, we assume that f is a transcendental meromorphic function. From Corollary A, we can further assume that $\delta(\infty, f) \le 7/9$ and we shall show the following result.

Theorem. *Let* f *be a transcendental meromorphic function and* c *be a meromorphic function which does not vanish identically and satisfies* $T(r, c) = o(T(r, f))$ *as* $r \to +\infty$. *Then* $ff' - c$ *or* $ff' + c$ *has infinitely many zeros.*

Corollary. *Suppose that* f *is an even or odd transcendental meromorphic function and* c *is an even meromorphic function. Suppose further that there exists a sequence* $\{z_1, -z_1, z_2, -z_2, \dots,\}$ *which are zeros of* $\varphi ff' + 1$ *but not zeros or poles of* f. *Then* $ff' - c$ *has infinitely many zeros.*

Here, we assume that the readers are familiar with the basic concepts of the Nevanlinna value distribution theory and the notations $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $T(r, f)$ and etc., see e.g., [4].

2. A lemma

Lemma. *Suppose that* f *is a non-constant meromorphic function and that* φ *is a non-vanishing meromorphic function such that* $T(r, \varphi) = o(T(r, f))$ *as* $r \to +\infty$. *Then for any finite non-zero distinct complex numbers* a *and* b *and any positive integer* k *such that* $\varphi f^{(k)} \not\equiv$ *constant, we have*

$$T(r, f) < N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\varphi f^{(k)} - a}\right) + N\left(r, \frac{1}{\varphi f^{(k)} - b}\right) - N(r, f) - N\left(r, \frac{1}{(\varphi f^{(k)})'}\right) + S(r, f)$$

as $r \to +\infty$.

Proof. First of all, we have

$$m\left(r, \frac{1}{\varphi f}\right) \le m\left(r, \frac{1}{\varphi f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1). \tag{1}$$

From

$$m\left(r,\frac{1}{\varphi f}\right) = T(r,\varphi f) - N\left(r,\frac{1}{\varphi f}\right) + O(1),$$
$$m\left(r,\frac{1}{\varphi f^{(k)}}\right) = T(r,\varphi f^{(k)}) - N\left(r,\frac{1}{\varphi f^{(k)}}\right) + O(1),$$

and (1), we have

$$\begin{aligned}(2)\qquad T(r,\varphi f) \le N\left(r,\frac{1}{\varphi f}\right) + T(r,\varphi f^{(k)}) - N\left(r,\frac{1}{\varphi f^{(k)}}\right)\\ + m\left(r,\frac{f^{(k)}}{f}\right) + O(1).\end{aligned}$$

By the second fundamental theorem,

$$\begin{aligned}(3)\qquad T(r,\varphi f^{(k)}) < N\left(r,\frac{1}{\varphi f^{(k)}}\right) + N\left(r,\frac{1}{\varphi f^{(k)}-a}\right) + N\left(r,\frac{1}{\varphi f^{(k)}-b}\right)\\ - N_1(r) + S(r,\varphi f^{(k)})\end{aligned}$$

as $r \to +\infty$, where, as usual, $N_1(r)$ is defined as

$$N_1(r) = 2N(r,\varphi f^{(k)}) - N(r,(\varphi f^{(k)})') + N\left(r,\frac{1}{(\varphi f^{(k)})'}\right).$$

Let z_0 be a pole of order $p \ge 1$ of f. Then $f^{(k)}$ and $f^{(k+1)}$ have a pole of order $k+p$ and $k+p+1$ at z_0 respectively. Thus $2(k+p)-(k+p+1) = k+p-1 \ge p$ and

$$(4)\qquad N_1(r) \ge N(r,f) + N\left(r,\frac{1}{(\varphi f^{(k)})'}\right) + S(r,f).$$

It is clear that $S(r,f^{(k)}) = S(r,f)$ and $m\left(r,\frac{f^{(k)}}{f}\right) = S(r,f)$. Thus by (2), (3) and (4),

$$\begin{aligned}T(r,\varphi f) < N\left(r,\frac{1}{\varphi f}\right) + N\left(r,\frac{1}{\varphi f^{(k)}-a}\right) + N\left(r,\frac{1}{\varphi f^{(k)}-b}\right)\\ - N(r,f) - N\left(r,\frac{1}{(\varphi f^{(k)})'}\right) + S(r,f)\end{aligned}$$

as $r \to +\infty$. Since $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$, we have the desired result.

3. Proofs of the theorem and the corollary

Proof of the theorem. Let $\varphi = 1/c$, $F = (1/2)f^2$, $k = 1$, $a = 1$ and $b = -1$. Then by the above lemma, we have

$$2T(r,f) < 2N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\varphi ff'-1}\right) + N\left(r,\frac{1}{\varphi ff'+1}\right)$$
$$- 2N(r,f) - N\left(r,\frac{1}{(\varphi ff')'}\right) + S(r,f)$$

as $r \to +\infty$.

By the assumption that $\delta(\infty,f) \le 7/9$, we have

$$\frac{N\left(r,\frac{1}{\varphi ff'-1}\right) + N\left(r,\frac{1}{\varphi ff'+1}\right)}{2T(r,f)} > 0$$

as $r \to +\infty$. Hence, the result of the theorem follows.

Proof of the corollary. If

$$\frac{N\left(r,\frac{1}{\varphi ff'-1}\right)}{2T(r,f)} > 0$$

as $r \to +\infty$ outside a set of finite linear measure, then $\varphi ff' - 1$ has infinitely many zeros and thus $ff' - c$ has infinitely many zeros.

Let $z_0 \ne 0$ be a zero of $\varphi ff' + 1$. Since f is even or odd, f' is odd or even. Therefore, ff' is an odd function. Now we have $\varphi(-z_0)f(-z_0)f'(-z_0) - 1 = -\varphi(z_0)f(z_0)f'(z_0) - 1 = 0$ and thus $-z_0$ is a zero of $\varphi ff' - 1$. Hence, the desired result follows.

4. Remarks

1. In [8], Q. D. Zhang showed that

$$2T(r,f) < \overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\varphi ff'-1}\right) + S(r,f).$$

From this, it is immedicate that if $\delta(0,f) + (1/2)\delta(\infty,f) > 1/2$, then $ff' - c$ has infinitely many zeros, where $c = 1/\varphi$.

2. In [9], Z. F. Zhang and G. D. Song showed that if $a(z) \not\equiv 0$ and $T(r,a) = S(r,f)$ as $r \to +\infty$, n, k are positive integers, where $n \ge 2$, then $f(f^{(k)})^n - a(z)$ has infinitely many zeros.

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