COMPLETE MINIMAL SURFACES LYING IN SIMPLE SUBSETS OF R^3

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Abstract

In this paper, we prove the existence of orientable and nonorientable complete minimal surfaces of $\boldsymbol{R}^3$ lying in a solid cylinder, a ball or a halfspace, using the Runge's approximation theorem and the Enneper-Weierstrass representation of minimal surfaces.

From the point of global differential geometry, the complete surfaces are the interesting objects namely, those for which the geodesics are defined for all times. Equivalently, every divergent path must have infinite length. In this paper, we study the complete minimal surfaces in $\boldsymbol{R}^3$. One of the fundamental problems in this subject is to decide about the existence of a complete minimal surface that is contained in a simple set of $\boldsymbol{R}^3$ such as a halfspace, a slab, a solid cylinder or a ball. Notice that all the classical examples, the plane, the catenoid, the helicoid, Scherk's surface, Costa's surface etc ..., are not contained in any simple set. Therefore it is surprising in this respect that, Jorge and Xavier [J-X] constructed a complete minimal surface lying in a slab, which is defined by a minimal immersion $X : D \hookrightarrow \boldsymbol{R}^3$ defined on the unit disk in the plane. They used the Runge's approximation theorem, which is improving the Enneper-Weierstrass representation of X to find a way in D tending to the boundary, $|z| = 1$, but only such way is fairly long with respect to the induced metric by X although the Euclidean distance is short. Recently, Nadirashvili [N] used the Runge's theorem in a more elaborate way to construct a complete minimal surface of negative Gaussian curvature which is a subset of the unit ball. This example is also a disk type, topologically trivial, and hence there is no period problem.

Now that we have the complete minimal surfaces in a slab and in a ball, and it is tempting to ask whether there exists an unbounded example lying in a solid cylinder. The first goal of this paper is to answer the question in the affirmative by proving the following theorem:

Theorem 1. *There exists a complete orientable singly-periodic minimal surface in $\boldsymbol{R}^3$ which is contained in a solid cylinder.*

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We prove this theorem in Section 2, applying the method of Nadirashvili to a minimal surface defined on an annulus in the plane. There is no need to annihilate the period in this case. Now, the example in the theorem has the non-zero period vector and contains a fundamental region lying in a ball of $\boldsymbol{R}^3$. Note, it has the trivial structure topologically.

After that, in Section 3, we prove Lemma 1 which is the key lemma in this paper. In Section 4, using the z^2-type holomorphic maps and the Enneper-Weierstrass representation of a nonorientable minimal surface in $\boldsymbol{R}^3$ due to Meeks [M], we prove that:

Theorem 2. *There exists a complete nonorientable minimal surface lying in a ball of* $\boldsymbol{R}^3$. *Concretely, it is a Möbius strip topologically.*

Finally, in section 5, we consider the Enneper-Weierstrass representation of a minimal immersion which sends the concentric circles $\{z \in \boldsymbol{C} : |z| = c\}$, $0 < c < 1$, into horizontal planes of $\boldsymbol{R}^3$, and we prove the following theorem:

Theorem 3. *There exist orientable complete minimal surfaces of* $\boldsymbol{R}^3$ *lying in a halfspace*, $x_3 > 0$, *but not a slab, which are transverse to each horizontal plane. One of them is singly-periodic.*

We conclude this section by providing with some previous results in the subject. First, using the Runge's theorem, Rosenberg and Toubiana [R-T] have obtained a complete minimal surface, which is topologically a cylinder, transverse to the planes $x_3 = \text{constant}$, $|x_3|$ is bounded on the surface, and F. Lopez [L1] constructed a Möbius strip type example in a slab.

By the way, Brito [B] described a new technique, together with a power series containing Hadamard gaps, to construct disk type examples in a slab. Afterward, using the same method Costa and Simões [C-S] have constructed examples of genus k and N ends in a slab, for every $k = 1, 2, \dots$ and $1 \le N \le 3$.

While, using the Weierstrass' gap theory in the compact Riemann surface theory, F. Lopez [L2] have presented an analytically clear general construction method for hyperbolic minimal surfaces of arbitrary topology with a bounded coordinate function, which are some deformations of the given disk type examples.

On the contrary, there are many non-existence results under the certain extra conditions on the surface: Hoffman and Meeks [H-M] showed that a proper complete non-planar minimal surface in $\boldsymbol{R}^3$ can not be contained in a halfspace, and Xavier [X] proved that the convex hull of a complete non-planar minimal surface of bounded Gaussian curvature is $\boldsymbol{R}^3$.

1. Preliminaries

Let $\mathscr{M}$ be a set of connected open annuli in the plane with Jordan curve boundaries and containing the unit curve $\gamma := \{z \in \boldsymbol{C} : |z| = 1\}$. Let

$$X : M \hookrightarrow \boldsymbol{R}^3$$

be a minimal immersion defined on $M \in \mathscr{M}$. Then it is a conformal harmonic immersion, and so we can take a holomorphic map

$$\Phi^X = (\phi_1, \phi_2, \phi_3) =: 2\frac{\partial X}{\partial z} \in \boldsymbol{C}^3,$$

such that

$$\phi_1^2 + \phi_2^2 + \phi_3^2 \equiv 0 \tag{1.1}$$

$$0 < \|\Phi^X\|^2 = |\phi_1^2| + |\phi_2^2| + |\phi_3^2| < \infty. \tag{1.2}$$

On the other hand, if we have a holomorphic map Φ on M satisfying both (1.1) and (1.2), then we can define a minimal immersion X by

$$X(p) = \Re \int_{p_0}^{p} \Phi\, dz + X(p_0)$$

for some $p_0 \in M$. By the way, we can assume that $\phi_3 \not\equiv 0$ on M and define a holomorphic and a meromorphic function by

$$f = \phi_1 - i\phi_2, \quad g = \frac{\phi_3}{\phi_1 - i\phi_2} \not\equiv 0,$$

respectively, we call (f, g) *the Weierstrass data* of X. In particular, the meromorphic function $g : M \to \boldsymbol{C}$ is the stereographic projection of the Gauss map of X with respect to the north pole of S^2, just say it *the Gauss map* of X.

Using *the Enneper-Weierstrass representation*, we have

$$X(p) = \Re \int_{p_0}^{p} \left(\frac{1}{2} f(1 - g^2), \frac{i}{2} f(1 + g^2), fg\right) dz + X(p_0).$$

Now, we consider the several arguments:

(1) Let us denote $ds_X = \lambda_X |dz|$ the induced metric of M by X, where

$$\lambda_X^2 := 2\|\Phi^X\|^2 = \frac{1}{2}|f|^2(1 + |g|^2)^2$$

and let "dist_X" be the distance function of M with respect to ds_X. Then we can say that X is *complete* if $\mathrm{dist}_X(\gamma, \partial M)$ diverges.

(2) We define *the period vector* of X by

$$Period(X) := \Re \int_{\Gamma} (\phi_1, \phi_2, \phi_3)\, dz \tag{1.3}$$

where Γ is a closed curve in M, which generates a translation of the image of M in $\boldsymbol{R}^3$ by

$$X(e^{2\pi i} z) = X(z) + Period(X).$$

(3) Let h be a holomorphic function on M, $h \neq 0$ in M, and set

$$\tilde{f}(z) = f(z)h(z), \quad \tilde{g}(z) = \frac{g(z)}{h(z)}.$$

Then it gives us another minimal immersion $\tilde{X} : M \hookrightarrow \boldsymbol{R}^3$, defined by

$$\tilde{X}(p) = \Re \int_{p_0}^{p} \left(\frac{1}{2} \tilde{f}(1 - \tilde{g}^2), \frac{i}{2} \tilde{f}(1 + \tilde{g}^2), \tilde{f}\tilde{g} \right) dz + \tilde{X}(p_0).$$

The period vector may vary in the deformation.

Finally, we state further notations which will be needed in later.

Notation 1. • Let $M \in \mathscr{M}$, then ∂M consists of two disjoint Jordan curves denoted by

$$\partial_I M := \partial M \cap \{|z| < 1\}, \quad \partial_O M := \partial M \cap \{|z| > 1\}.$$

Take a simple arc "$b(M)$" lying in M between $p \in \partial_I M$ and $q \in \partial_O M$, then we have a fundamental domain $F(M)$ of M with the cut $b(M)$, that is,

$$b(M) = \partial F(M) \backslash \partial M.$$

We call "$b(M)$" *the branch cut* of $F(M)$.

• Let "$\mathrm{dist}_{\boldsymbol{C}}$" and "$\mathrm{dist}_{\boldsymbol{R}^3}$" denote the standard Euclidean distance functions on the plane $\boldsymbol{C}$ and $\boldsymbol{R}^3$, respectively.

• If $E \in \boldsymbol{C}$, we define a subset $E_{1-\varepsilon} \subset E$ such that

$$\mathrm{dist}_{\boldsymbol{C}}(E, E_{1-\varepsilon}) = \varepsilon$$

• Let $B_r = \{x \in \boldsymbol{R}^3 : \|x\| < r\}$ be a ball of $\boldsymbol{R}^3$.

2. Proof of Theorem 1 (Examples in a solid cylinder)

In this section, we prove that Theorem 1 is the consequence of the following lemma which will be showed in the next section:

Lemma 1. *Let $X : M \hookrightarrow \boldsymbol{R}^3$, $M \in \mathscr{M}$, be a minimal immersion with $X(1) = 0$ and* $\mathrm{dist}_X(\gamma, \partial M) = \rho$ *for some $\rho \geq 1$. Suppose that there is a fundamental domain $F_X(M)$ of M with a branch cut "b_X" such that*

$$X(F_X(M)) \subset B_r$$

for some $r \geq 1$. Then for every $s, \delta > 0$ with $M_{1-\delta} \in \mathscr{M}$, there exists a minimal immersion

$$Y : \tilde{M} \hookrightarrow \boldsymbol{R}^3, \quad \tilde{M} \in \mathscr{M},$$

such that $Y(1) = 0$, $M_{1-\delta} \subset \tilde{M} \subset M$ and

$$\mathrm{dist}_Y(\gamma, \partial\tilde{M}) = \rho + s$$

$$\|\Phi^Y - \Phi^X\| \leq s^2/2\pi \quad on \ M_{1-\delta}$$

$$Y(F_Y(\tilde{M})) \subset B_{r+2s^2}$$

where $F_Y(\tilde{M})$ is a fundamental domain of $\tilde{M}$ with a branch cut "b_Y" such that

$$\mathrm{dist}_{\boldsymbol{C}}(b_X \cap \tilde{M}, b_Y) < 4\delta.$$

To the first, let a_n, $n = 1, 2, \dots$, be a sequence of positive constants specified later such that

$$a_1 < 1/5, \quad a_n > 2a_{n+1} \tag{2.1}$$

From the previous lemma, together with $\rho = \rho_n$, $r = r_n$, $s = 1/(n+1)$ and $\delta = a_n$, we have a sequence of minimal annuli

$$X_n : A_n \hookrightarrow \boldsymbol{R}^3, \quad A_n \in \mathscr{M},$$

$n = 1, 2, \dots,$ respectively, such that

$$\begin{cases} (A_n)_{1-a_n} \subset A_{n+1} \subset A_n \\ \|Period(X_1)\| = 1, \quad X_n(1) = 0 \\ \mathrm{dist}_{X_n}(\gamma, \partial A_n) = \rho_n, \quad \rho_n = 1 + 1/2 + \cdots + 1/n \\ X_n(F_n(A_n)) \subset B_{r_n}, \quad r_n = 2 + 2/2^2 + 2/3^2 + \cdots + 2/n^2 \\ \|\Phi^{n+1} - \Phi^n\| \le 1/(2\pi(n+1)^2) \quad \text{on } (A_n)_{1-a_n} \end{cases}$$

where $\Phi^n := 2\partial X_n/\partial z$ is a holomorphic map and $F_n(A_n)$ is a fundamental domain of A_n. Let us denote "b_n" the branch cut of $F_n(A_n)$, and let $b_1 = \{z \in A_1 : \arg z = 0\}$. By Lemma 1, we may assume that

$$\mathrm{dist}_{\boldsymbol{C}}(b_n \cap A_{n+1}, b_{n+1}) < 4a_n. \tag{2.2}$$

Observe that we can take an annulus $A \in \mathscr{M}$ as the limit of the decreasing sequence: $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$. That is,

$$A = \mathrm{Int}\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Denote $K_n = (A_n)_{1-2a_n}$, then $K_n \subset K_{n+1}$ by (2.1). Additionally, all of K_n's, $n = 1, 2, \dots,$ are contained in A.

Notice that for every compact subset K of A, there is an integer N such that $K \subset K_n$ for all $n \ge N$. Recall $\|\Phi^{n+1} - \Phi^n\| \le 1/(2\pi(n+1)^2)$ on K_n, and so $\{\Phi^n|_K\}_{n \in \boldsymbol{N}}$ is a normal family in Montel's sense. Hence we can find a subsequence of $\{\Phi^n|_K\}_{n \in \boldsymbol{N}}$ which is converging uniformly to a holomorphic map

$$\Phi : A \to \boldsymbol{C}^3$$

over all compact subsets of A as $n \to \infty$. Since Φ satisfies the conditions (1.1) and (1.2) clearly, we have a minimal immersion

$$\mathscr{X} : A \hookrightarrow \boldsymbol{R}^3$$

$\mathscr{X}(1) = 0$, defined by

$$\mathscr{X}(p) = \Re \int_1^p \Phi \, dz.$$

Then the following holds:

(a) Recall $\gamma \subset (A_n)_{1-a_n}$ for all $n \in \boldsymbol{N}$. Therefore we have

$$\begin{aligned}
&\|Period(X_n) - Period(X_1)\| \\
&\quad \leq \sum_{i=2}^{n} \|Period(X_i) - Period(X_{i-1})\| \\
&\quad = \sum_{i=2}^{n} \int_\gamma \|\Phi^i - \Phi^{i-1}\| \, |dz| \leq \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq \frac{3}{4}
\end{aligned}$$

and hence

$$\frac{1}{4} \leq \|Period(X_n)\| \leq \frac{7}{4}$$

for all $n \in \boldsymbol{N}$. Hence, $\mathscr{X}$ is singly-periodic.

(b) Let us denote $\mathscr{F}(A_n)$ the canonical fundamental domain of A_n:

$$\mathscr{F}(A_n) = \{z \in A_n : 0 \leq \arg z < 2\pi\}$$

which has the branch cut "$b_1 \cap A_n = \{z \in A_n : \arg z = 0\}$". Together with (2.1) and (2.2), we have

$$\begin{aligned}
\mathrm{dist}_{\boldsymbol{C}}(b_n, b_1 \cap A_n) &< 4(a_1 + a_2 + \cdots + a_{n-1}) \\
&< 4a_1\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) < \pi.
\end{aligned}$$

It implies that $\mathscr{F}(A_n)$ is contained in the union of three fundamental domains:

$$F_n(A_n) \cup e^{2\pi i} F_n(A_n) \cup e^{-2\pi i} F_n(A_n).$$

Since $r_n = 2 + 2/2^2 + 2/3^2 + \cdots + 2/n^2 \leq 4$ for all $n \in \boldsymbol{Z}$, we have

$$X_n(\mathscr{F}(A_n)) \subset B_{r_n + \|Period(X_n)\|} \subset B_6.$$

Recall $\mathscr{F}(A) \subset \mathscr{F}(A_n)$ for all $n \in \boldsymbol{N}$, and so we have shown that:

$$\mathscr{X}(\mathscr{F}(A)) \subset B_6.$$

Moreover, $\mathscr{X}(A)$ is contained in a solid cylinder of $\boldsymbol{R}^3$ with the axis line of the direction $Period(\mathscr{X}) \neq 0$ from (a).

(c) Recall we can choose a_n, $n = 1, 2, \ldots$, satisfying that,

$$\mathrm{dist}_{X_n}(\gamma, \partial K_n) \geq \frac{2}{3}\rho_n. \tag{2.3}$$

Let d_n, $n = 1, 2, \ldots$, be another sequence of constants, such that

$$\frac{1}{2} < d_n < 1$$

$$\|\Phi^{n+1}\| \geq d_n \|\Phi^n\| \quad \text{on } K_n$$

$$d_1 d_2 \cdots d_n \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Then, together with (2.3), we have for all $m > n$

$$\begin{aligned} \operatorname{dist}_{X_m}(\gamma, \partial K_n) &\geq d_{m-1} \operatorname{dist}_{X_{m-1}}(\gamma, \partial K_n) \\ &\geq \frac{1}{2} \operatorname{dist}_{X_n}(\gamma, \partial K_n) \geq \frac{1}{2}\frac{2}{3}\rho_n. \end{aligned}$$

It follows that,

$$\operatorname{dist}_{\mathscr{X}}(\gamma, \partial A) \geq \operatorname{dist}_{\mathscr{X}}(\gamma, \partial K_n) \geq \frac{1}{3}\rho_n.$$

Recall $\rho_n = 1 + 1/2 + \cdots + 1/n$ tends to infinity as $n \to \infty$.

As a result, $\mathscr{X}(A)$ is the complete singly-periodic minimal surface of $\boldsymbol{R}^3$ lying in a solid cylinder. Hence the proof of the theorem is finished.

3. Proof of Lemma 1

In this section we prove the previous lemma in Section 2. We will use the method of Nadirashvili in [N]. Notice that, however, the property (16) of [N] is not induced by the condition (10). So, we will also use the argument of Collin and Rosenberg [C-R] who filled the gap.

To the first, denote

$$\mu = \sup_M \|\Phi^X\| + 1, \quad v = \inf_M \|\Phi^X\|.$$

Let $N > 10$ be a sufficiently large number which will be specified later with $M_{1-\delta} \subset M_{1-2/N}$. And let

$$U = U_O \cup U_I := M \backslash M_{1-2/N}$$

where U_O and U_I are the outer and the inner components, respectively. Let $r_i = 1 - i/N^3$, $i = 0, 1, \ldots, 2N^2$, and denote

$$E_i = U_O \cap (M_{r_{2i}} \backslash M_{r_{2i+1}}), \quad \tilde{E}_i = U_O \cap (M_{r_{2i+1}} \backslash M_{r_{2i+2}})$$

$$E = \bigcup_{i=0}^{N^2-1} E_i, \quad \tilde{E} = \bigcup_{i=0}^{N^2-1} \tilde{E}_i.$$

If we denote $S_1 = \partial E$, then it consists of $(2N^2 + 1)$-number of closed simple curves. Let $l_1, l_2, \ldots, l_{k_1}$ be the transversal lines of U_O for some integer k_1, and denote

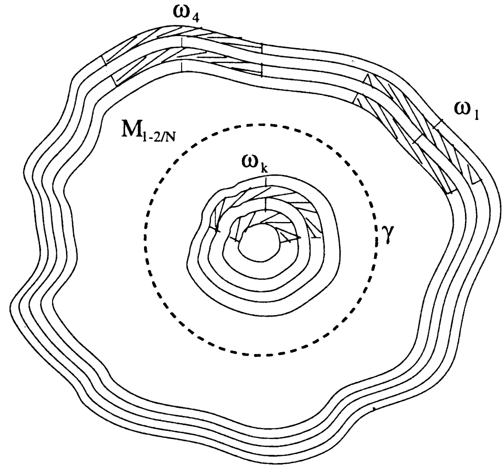

FIGURE 1

$$L_1 = E \cap \bigcup_{i=1}^{[k_1/2]} l_{2i}, \quad \tilde{L}_1 = \tilde{E} \cap \bigcup_{i=1}^{[k_1/2]} l_{2i-1}$$

$$H_1 = L_1 \cup \tilde{L}_1 \cup S_1.$$

We denote H_1' the open $1/8N^3$-neighborhood of H_1, then $U_O \backslash H_1'$ consists of $2N^2k_1$-number of compact subsets. Let ω_i, $i = 1, \ldots, k_1$, be the union of segments $U_0 \cap l_i$ and those components which have nonempty intersection with $U_O \cap l_i$, respectively.

Similarly, repeat this processing on the inner component U_I, together with the k_2-number of transversal lines, to take the compact subsets $\omega_{k_1+j} \subset U_I$, $j = 1, \ldots, k_2$. We may assume that

$$\frac{1}{N} \leq \operatorname{diam}_{\mathbf{C}}(\omega_i') \leq \frac{7}{N} \tag{3.1}$$

where ω_i' is an open $1/8N^3$-neighborhood of ω_i and $i = 1, \ldots, k$ with $k = k_1 + k_2$. (see Figure 1).

PROPOSITION 1 ([N]). *If $ds = \lambda|dz|$ is a metric on M such that*

$$\begin{cases} \lambda \geq 1 & \text{on } M \\ \lambda \geq N^4 & \text{on } \bigcup_{i=1}^{k} \omega_i' \end{cases}$$

then for all smooth curves σ connecting γ and ∂M, the arc length of σ with respect to ds is larger than N.

Proof. Let σ_i be a segment of σ, which meets the subset $M_{r_{2i}} \backslash M_{r_{2i+1}}$ for some $i = 0, 1, \dots, N^2 - 1$. If σ_i transverses ω'_j for some $j = 1, \dots, k$, then we can show that:

$$\int_{\sigma_i} ds \geq N^4 \frac{1}{2N^3} = \frac{N}{2}.$$

If not, that is, σ_i does not meet every ω'_j, $j = 1, \dots, k$, except a very small area, then the Euclidean length of σ_i is more than $2/N$, by (3.1). Hence the arc length of σ_i is at least $1/N$. Therefore, since $\sigma = \sigma_0 \cup \sigma_1 \cup \cdots \cup \sigma_{N^2-1}$, we have

$$\int_{\sigma} ds \geq N^2 \frac{1}{N} = N. \qquad \square$$

Proposition 2 ([N]). *For all constants $T > 0$, where $i = 1, \dots, k$, there is a holomorphic function h_i defined on M, $h_i(z) \neq 0$ in M, such that*

$$|h_i - 1| < \frac{1}{T} \quad \text{on } M \backslash \omega'_i$$

$$|h_i - T| < \frac{1}{T} \quad \text{on } \omega_i.$$

Proof. Denote the Riemann sphere by $S^2 = \boldsymbol{C} \cup \{\infty\}$. Observe that the complement of the union of two compact subsets $M \backslash \omega'_i$ and ω_i in S^2 is either connected or composing of two components. By virtue of the Runge's theorem, for every $\tilde{\varepsilon} > 0$ there exists a holomorphic or meromorphic function $\tilde{h}_i$ on the plane, with only one pole at zero, such that

$$|\tilde{h}_i| < \tilde{\varepsilon} \quad \text{on } M \backslash \omega'_i$$

$$|\tilde{h}_i - \ln T| < \tilde{\varepsilon} \quad \text{on } \omega_i.$$

Let us define

$$h_i(z) := \exp(\tilde{h}_i(z))$$

then the restriction of h_i on M is a holomorphic function, because that $0 \notin M$. Together with a sufficiently small $\tilde{\varepsilon} > 0$, we evidently have

$$|h_i - 1| < \frac{1}{T} \quad \text{on } M \backslash \omega'_i$$

$$|h_i - T| < \frac{1}{T} \quad \text{on } \omega_i$$

as required. $\square$

Now, we prove the following assertion, which plays the crucial role of this proof of Lemma 1:

Assertion 1. *There is a sequence of minimal immersions*

$$Y_0 = X, Y_1, \dots, Y_k : M \hookrightarrow \boldsymbol{R}^3$$

such that all of $\mathscr{H}_1, \mathscr{H}_2, \dots, \mathscr{H}_k$ *hold*:

$$(\mathscr{H}_i)\begin{cases} \|\Phi^i - \Phi^{i-1}\| < \dfrac{\varepsilon}{Nk} & \text{on } M\backslash\omega_i' \\ \|\Phi^i\| \geq \dfrac{v}{2\sqrt{N}} & \text{on } \omega_i' \\ \|\Phi^i\| \geq \dfrac{v}{2}N^{3.5} & \text{on } \omega_i \end{cases}$$

where $\Phi^i := 2\partial Y_i/\partial z$ *and* $\varepsilon > 0$ *is a sufficienlty small constant.*

Proof. First, we assume that $\Phi^0, \dots, \Phi^{i-1}$ are already defined such that $\mathscr{H}_1, \dots, \mathscr{H}_{i-1}$ are all true. Since $\omega_i' \subset M\backslash(\omega_1' \cup \cdots \cup \omega_{i-1}')$, we have

$$\|\Phi^{i-1} - \Phi^0\| \leq \sum_{j=1}^{i-1} \|\Phi^j - \Phi^{j-1}\| \leq \frac{\varepsilon}{N} \quad \text{on } \omega_i'.$$

It follows, together with large N, that

$$\frac{5v}{8} \leq \|\Phi^{i-1}\| \leq \mu \quad \text{on } \omega_i'. \tag{3.2}$$

Let $G_{i-1} : M \to S^2$ be the Gauss map of Y_{i-1}. Then by (3.1) and (3.2), we can say that

$$\operatorname{diam}_{S^2}(G_{i-1}(\tilde{\omega}_i')) \leq \frac{7\mu}{N}, \quad \operatorname{diam}_{\boldsymbol{R}^3}(Y_{i-1}(\tilde{\omega}_i')) \leq \frac{7\mu}{N} \tag{3.3}$$

where $\tilde{\omega}_i'$ is a fundamental domain of ω_i' such that

$$\tilde{\omega}_i' \cap F_X(M) \neq \phi. \tag{3.4}$$

Now observe that, after a rotation, we may assume the following; if $\operatorname{dist}_{\boldsymbol{R}^3}(0, Y_{i-1}(\tilde{\omega}_i')) \geq 1/\sqrt{N}$, then

$$\angle(\pm\vec{e}_3, Y_{i-1}(\tilde{\omega}_i')) \leq \frac{7\mu}{\sqrt{N}} \tag{3.5}$$

and

$$\operatorname{dist}_{S^2}(\pm\vec{e}_3, G_{i-1}(\omega_i')) \geq \frac{1}{\sqrt{N}}, \tag{3.6}$$

where $\vec{e}_3 = (0,0,1) \in \boldsymbol{R}^3$. (see Figure 2).

Let (f_{i-1}, g_{i-1}) be the Weierstrass data of the minimal immersion Y_{i-1}, and set

$$f_i(z) = f_{i-1}(z)h_i(z), \quad g_i(z) = \frac{g_{i-1}(z)}{h_i(z)}$$

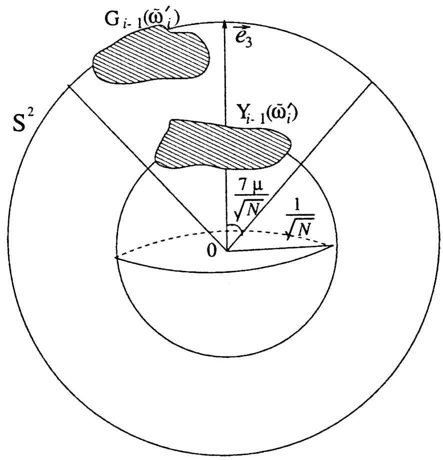

FIGURE 2

where h_i is the holomorphic function in the Proposition 2. Now we have another minimal surface $Y_i(z) = \Re \int^z \Phi^i(\zeta)\, d\zeta$ such that:

$$\Phi^i = \left(\frac{1}{2} f_i(1 - g_i^2), \frac{\sqrt{-1}}{2} f_i(1 + g_i^2), f_i g_i\right).$$

Observe that the following holds:

- On the domain $M \backslash \omega_i'$, for a sufficiently large T,

$$\begin{aligned}\|\Phi^i - \Phi^{i-1}\| &= \frac{1}{2}|f_{i-1}|\,|h_i - 1| + \frac{1}{2}|f_{i-1}g_{i-1}^2|\left|1 - \frac{1}{h_i}\right| \\ &\le \frac{1}{2T}\sup_M |f_{i-1}| + \frac{1}{2(T-1)}\sup_M |f_{i-1}g_{i-1}^2| \\ &\le \frac{\varepsilon}{Nk}.\end{aligned}$$

- Recall $g_{i-1} : \boldsymbol{M} \to \boldsymbol{C}$ is a stereographic projection of G_{i-1} with respect to the north pole of S^2, and hence by (3.6) we have

$$\frac{2}{\sqrt{N}} \le |g_{i-1}| \le \frac{\sqrt{N}}{2}$$

on the domain ω_i' by (3.6). It follows that on ω_i', we also have

$$\begin{aligned}\|\Phi^i\| &= \frac{1}{2}|f_{i-1}|\,|h_i|\left(1 + \frac{|g_{i-1}|^2}{|h_i|^2}\right) \ge |f_{i-1}|\,|g_{i-1}| \\ &= \frac{|g_{i-1}|}{1 + |g_{i-1}|^2}\|\Phi^{i-1}\| \ge \frac{4}{5\sqrt{N}}\|\Phi^{i-1}\| \ge \frac{\nu}{2\sqrt{N}}.\end{aligned}$$

• Similarly, on ω_i

$$\begin{aligned}\|\Phi^i\| &\geq \frac{1}{2}|f_{i-1}|\,|h_i| \geq \frac{1}{2}|f_{i-1}|(T-1)\\ &= \frac{\|\Phi^{i-1}\|}{1+|g_{i-1}|^2}(T-1) \geq \frac{4}{N+4}\|\Phi^{i-1}\|(T-1)\\ &\geq \frac{5v}{2(N+4)}(T-1) \geq \frac{v}{2}N^{3.5}\end{aligned}$$

for large N.

Until now, we have shown that $\mathscr{H}_i$ also holds. By induction, we finish the proof of Assertion 1. □

Now let us define a new minimal immersion by:

$$Y := Y_k - Y_k(1)$$

then $\Phi^Y := \Phi^k$, $Y(1) = 0$, $\|Y - Y_k\| \leq \varepsilon/N$. Observe that Assertion 1 leads us that

$$\|\Phi^k - \Phi^0\| \leq \sum_{i=1}^{k}\|\Phi^i - \Phi^{i-1}\| \leq \frac{\varepsilon}{N} \quad \text{on } M\Big\backslash\bigcup_{i=1}^{k}\omega_i'. \tag{3.7}$$

Additionally, for large N,

$$\|\Phi^Y - \Phi^X\| \leq s^2/2\pi \quad \text{on } M_{1-\delta} \tag{3.8}$$

since $M_{1-\delta} \subset M\backslash(\omega_1' \cup \cdots \cup \omega_k')$.

By the way, sine $\omega_i' \subset M\backslash\bigcup_{j=i+1}^{k}\omega_j'$, we have

$$\|\Phi^k\| \geq \|\Phi^i\| - \|\Phi^k - \Phi^i\| \geq \frac{v}{2\sqrt{N}} - \frac{\varepsilon}{N} \quad \text{on } \omega_i'$$

$$\|\Phi^k\| \geq \frac{v}{2}N^{3.5} - \frac{\varepsilon}{N} \quad \text{on } \omega_i.$$

It follows that,

$$\begin{cases} \|\Phi^k\| \geq \dfrac{v}{3\sqrt{N}} & \text{on } M \\ \|\Phi^k\| \geq N^4\dfrac{v}{3\sqrt{N}} & \text{on } \omega_i \cup \cdots \cup \omega_k. \end{cases} \tag{3.9}$$

Thus, by Proposition 1, we can show that

$$\operatorname{dist}_Y(\gamma, \partial M) \geq N\frac{v}{3\sqrt{N}} \geq \rho + s$$

for a sufficiently large N.

Now take a fundamental domain M_F of M, defined by

$$M_F := \left(F_X(M) \Big\backslash \bigcup_{i=1}^{k} \omega_i'\right) \cup \bigcup_{i=1}^{k} \tilde{\omega}_i'$$

see (3.4). Then, from (3.3), we can say that

$$\|X(z)\| < r + \frac{7\mu}{N} \quad \text{for all } z \in M_F. \tag{3.10}$$

Let $\tilde{M}$ denote a subset of M such that,

$$\operatorname{dist}_Y(\gamma, \partial\tilde{M}) = \rho + s \tag{3.11}$$

then $\partial\tilde{M}$ is the union of smooth curves in M, since the Gaussian curvature of a minimal surface is nonpositive. Observe $\tilde{M} \in \mathscr{M}$. Set a fundamental domain of $\tilde{M}$ by

$$F_Y(\tilde{M}) := \tilde{M} \cap M_F$$

then

$$\operatorname{dist}_{\boldsymbol{C}}(b_X \cap \tilde{M}, b_Y) \leq \frac{7}{N} < 4\delta \tag{3.12}$$

where "b_X", "b_Y" are the branch cuts of $F_X(M)$ and $F_Y(\tilde{M})$, respectively.

Now we show the following assertion. Recall, together with (3.8), (3.11) and (3.12), it leads us to prove that the restriction of Y on $\tilde{M}$, denote by Y again:

$$Y : \tilde{M} \hookrightarrow \boldsymbol{R}^3$$

is the required map. Hence we completes the analysis of the proof of Lemma 1:

Assertion 2. $Y(F_Y(\tilde{M})) \subset B_{r+2s^2}$.

Proof. We consider the two cases:

Case 1. Let $z \in \partial(F_Y(\tilde{M})) \backslash \bigcup_{i=1}^{k} \tilde{\omega}_i'$, then by (3.7) and (3.10),

$$\|Y(z)\| \leq \|X(z)\| + \frac{1}{N} < r + \frac{7\mu}{N} + \frac{1}{N} < r + 2s^2 \tag{3.13}$$

for a sufficiently large N.

Case 2. Let $z \in \partial(F_Y(\tilde{M})) \cap \tilde{\omega}_i'$, for some $i = 1, \ldots, k$. Then, it is clear that

$$\|Y(z) - Y_i(z)\| \leq \frac{\varepsilon}{N} \tag{3.14}$$

for $\tilde{\omega}_i' \subset M_F \backslash \bigcup_{j \neq i} \tilde{\omega}_j'$. Take a geodesic curve η in $\tilde{M}$ connecting from z to the curve γ, such that

$$\int_\eta ds_Y = \rho + s.$$

Let η meet $\partial\tilde{\omega}_i'$ at a point $\bar{z} \in M_F \backslash \bigcup_{j=1}^k \tilde{\omega}_j'$, then

$$\|Y_{i-1}(\bar{z})\| \le r + \frac{c_1}{N}, \quad c_1 > 0. \tag{3.15}$$

Recall the Euclidean distance between γ and $\partial M_{1-2/N}$ is less than $(3\sqrt{N}/v)(\rho + s)$ by (3.9), and hence by (3.7),

$$\begin{aligned} \mathrm{dist}_Y(\gamma, \bar{z}) &\ge \mathrm{dist}_Y(\gamma, \partial M_{1-2/N}) \\ &\ge \mathrm{dist}_X(\gamma, \partial M_{1-2/N}) - \frac{\varepsilon}{N}\frac{3\sqrt{N}}{v}(\rho + s) \\ &\ge \rho - \frac{2\mu}{N} - \frac{3\varepsilon}{v\sqrt{N}}(\rho + s). \end{aligned}$$

It follows, together with (3.7) and (3.9) again, that

$$\mathrm{dist}_Y(z, \bar{z}) \le \int_\eta ds_Y - \mathrm{dist}_Y(\gamma, \bar{z}) \le s + \frac{2\mu}{N} + \frac{3\varepsilon}{v\sqrt{N}}(\rho + s).$$

From (3.14), it implies that

$$\mathrm{dist}_{Y_i}(z, \bar{z}) \le s + \frac{c_2}{N}, \quad c_2 > 0. \tag{3.16}$$

Now, suppose that $\mathrm{dist}_{\boldsymbol{R}^3}(0, Y_{i-1}(\tilde{\omega}_i')) \le 1/\sqrt{N}$, then

$$\|Y_{i-1}(\bar{z})\| \le \frac{1}{\sqrt{N}} + \frac{7\mu}{N}$$

by (3.3). From (3.14), (3.16), we can say that

$$\begin{aligned} \|Y(z)\| &\le \|Y_i(z)\| + \frac{1}{N} \le \|Y_i(\bar{z})\| + s + \frac{c_2 + 1}{N} \\ &\le \|Y_{i-1}(\bar{z})\| + \frac{\varepsilon}{N} + s + \frac{c_2 + 1}{N} \\ &\le r + 2s^2. \end{aligned} \tag{3.17}$$

On the other hand, if $\mathrm{dist}_{\boldsymbol{R}^3}(0, Y_{i-1}(\tilde{\omega}_i')) \ge 1/\sqrt{N}$. Then by (3.5),

$$\|\pi(Y_{i-1}(\bar{z}))\| \le \frac{7\mu}{\sqrt{N}}\|Y_{i-1}(\bar{z})\| \le \frac{7\mu}{\sqrt{N}}\left(r + \frac{c_1}{N}\right)$$

where $\pi : \boldsymbol{R}^3 \to \boldsymbol{R}^2$ is the orthogonal projection along the (x_1, x_2)-plane. Since $Y_i^3 \equiv Y_{i-1}^3$, we have

$$\|\pi(Y_i(\bar{z}))\| \le \|\pi(Y_{i-1}(\bar{z}))\| + \|Y_i(\bar{z}) - Y_{i-1}(\bar{z})\| \le \frac{c_4}{\sqrt{N}}$$

for some $c_4 > 0$, as well as, by (3.16)

$$\|\pi(Y_i(z))\| < s + \frac{c_5}{\sqrt{N}}, \quad c_5 > 0.$$

By the Pythagorean theorem, together with the fact:

$$\|Y_i^3(z)\| = \|Y_{i-1}^3(z)\| \le r + \frac{\varepsilon}{N}$$

it implies that

$$\|Y_i(z)\|^2 \le \left(s + \frac{c_5}{\sqrt{N}}\right)^2 + \left(r + \frac{\varepsilon}{N}\right)^2.$$

Since $r \ge 1$, by (3.14) again, we have for large N,

$$\begin{aligned} \|Y(z)\| &\le \sqrt{r^2 + s^2} + s^2 \\ &\le \sqrt{r^2 + 2rs^2 + s^4} + s^2 = r + 2s^2. \end{aligned} \tag{3.18}$$

As a result of (3.13), (3.17) and (3.18), we have shown that

$$Y(\partial F_Y(\tilde{M})) \subset B_{r+2s^2}.$$

By virtue of the maximum principle of the minimal surface, it implies that

$$Y(F_Y(\tilde{M})) \subset B_{r+2s^2}$$

as desired. □

4. Proof of Theorem 2 (Nonorientable examples in a ball)

Recall, independently, Martin and Morales [M-M] have also generalized the technique of Nadirashvili in a minimal immersion on an annulus. On the contrary, their interest is on the construction of a bounded complete example with non-trivial topological structure. To annihilate the period, they used the z^2-type holomorphic maps. In this section, we prove the existence of a nonorientable bounded complete minimal surface, using the z^2-type holomorphic maps again and the Enneper-Weierstrass representation of nonorientable minimal surface in $\boldsymbol{R}^3$ due to Meeks [M].

First, we have some notations:

Notation 2. • Let $I : \boldsymbol{C} \to \boldsymbol{C}$ be the inversion defined by $I(z) = -1/\bar{z}$.

• Let $\mathcal{N} := \{M \in \mathcal{M} : I(M) = M\}$ be the set of annuli invariant under the inversion.

• We say that a holomorphic map $\Phi : M \hookrightarrow \boldsymbol{C}^3$, $M \in \mathcal{N}$, is *z^2-type*, if there is a holomorphic map Ψ such that:

$$\Phi(z) = \Psi(z^2) \quad \text{for all } z \in M.$$

If $X : M \hookrightarrow \boldsymbol{R}^3$, $M \in \mathcal{N}$, is a minimal immersion and $\Phi^X := 2\partial X/\partial z$ is z^2-type, then $Period(X) = 0$ and $X(z) + X(-z)$ is constant. We denote it by $S(X) := X(z) + X(-z)$.

• Let us denote the subset $E^{1-\varepsilon}$ of $E \subset \boldsymbol{C}$ such that

$$E^{1-\varepsilon} = I(E^{1-\varepsilon}), \quad \mathrm{dist}_{\boldsymbol{C}}(\partial E_O, \partial E_O^{1-\varepsilon}) = \varepsilon$$

where ∂E_O and $\partial E_O^{1-\varepsilon}$ denote the outer components of ∂E and $\partial E^{1-\varepsilon}$, that is, both are contained in $\{|z| > 1\}$, respectively.

Proposition 3 ([M]). *Let $X : M \hookrightarrow \boldsymbol{R}^3$, $M \in \mathcal{N}$, be a minimal immersion with $Period(X) = 0$. Then it is the double covering of a nonorientable minimal surface if and only if*

$$g(I(z)) = I(g(z)) \tag{4.1}$$

$$(zg(z))^2 = -\frac{\bar{f}(I(z))}{f(z)} \tag{4.2}$$

where (f, g) is the Weierstrass data of X and I is the inversion. The nonorientable surface is, concretely, the Möbius strip $M/\{1, I\}$.

Now, using Lemma 2 in the end of this section, we have a sequence of double coverings of minimal Möbius strips, $X_n : A_n \hookrightarrow \boldsymbol{R}^3$, $A_n \in \mathcal{N}$, where $n = 1, 2, \ldots,$ such that,

(1) $A_n^{1-a_n} \subset A_{n+1} \subset A_n$

(2) $\Phi^n := 2\dfrac{\partial X_n}{\partial z}$ is z^2-type, $\quad S(X_n) = 0$

(3) $\mathrm{dist}_{X_n}(\gamma, \partial A_n) = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$

(4) $X_n(A_n) \subset B_{r_n}, \quad r_n = 2 + \dfrac{2}{2^2} + \cdots + \dfrac{2}{n^2} \leq 4$

(5) $\|X_{n+1} - X_n\| \leq \dfrac{1}{(n+1)^2}$ on $A_n^{1-a_n}$

where $a_n > 0$ is specified later with $a_n \geq 2a_{n+1}$.

Define $A = \mathrm{Int}(\bigcap_{n=1}^{\infty} A_n)$, then $A \in \mathcal{N}$. Notice that $\{X_n|_A\}$ is a Cauchy sequence on every compact subsets of A, and hence we have the minimal surface

$$\mathcal{X} : A \hookrightarrow \boldsymbol{R}^3$$

as $n \to \infty$. Observe that $\mathcal{X}$ is also a double covering of a minimal Möbius strip, such that

(a) $Period(\mathcal{X}) = 0$

(b) $\mathcal{X}(A) \subset B_4$

(c) $\mathrm{dist}_{\mathcal{X}}(\gamma, \partial A) = \infty$.

Hence $\mathscr{X}$ defines a complete nonorientable minimal surface lying in a ball of $\boldsymbol{R}^3$, and the analysis of the proof of Theorem 2 completes.

Lemma 2. *Let $X : M \hookrightarrow \boldsymbol{R}^3$, $M \in \mathscr{M}$, be a double covering of a minimal surface such that $X(1) = 0$, Φ^X is z^2-type, $S(X) = 0$, $\mathrm{dist}_X(\gamma, \partial M) = \rho$ and $X(M) \subset B_r$ for some $\rho, r \geq 1$. Then for every $\delta, s > 0$ with $M^{1-2\delta} \in \mathscr{N}$, there exists a double covering of a nonorientable minimal immersion*

$$Y : \tilde{M} \hookrightarrow \boldsymbol{R}^3, \quad \tilde{M} \in \mathscr{N}$$

such that $Y(1) = 0$, Φ^Y is z^2-type, $S(Y) = 0$, $M^{1-\delta} \subset \tilde{M} \subset M$ and

$$\mathrm{dist}_Y(\gamma, \partial\tilde{M}) = \rho + s, \quad Y(\tilde{M}) \subset B_{r+2s^2}$$

$$\|Y - X\| \leq s^2 \quad \text{on} \ M^{1-\delta}.$$

Proof. Let us denote the outer subset of M by:

$$M_O := \{z \in M : |z| > 1\}.$$

Similar to the previous section, we can take the disjoint compact subsets

$$\omega_1, \omega_2, \ldots, \omega_{2k} \subset M_O \cap (M \backslash M^{1-1/N})$$

for some $k \in \boldsymbol{N}$, which satisfies (3.1) and Proposition 1. To prove this lemma, we assume that

$$\omega_{2i} = -\omega_{2i-1}, \quad i = 1, 2, \ldots, k$$

and let

$$\Omega_i := \omega_{2i-1} \cup \omega_{2i}, \quad \Omega_i' := \omega_{2i-1}' \cup \omega_{2i}'.$$

Now, we take a similar modification of a minimal surface to that of the previous section with respect to Ω_i. To precise, suppose that there are double coverings of minimal Möbius strips, $Y_0 = X$, $Y_1, \ldots, Y_{i-1} : M \hookrightarrow \boldsymbol{R}^3$, such that all of $\tilde{\mathscr{H}}_1, \ldots, \tilde{\mathscr{H}}_{i-1}$ hold:

$$(\tilde{\mathscr{H}}_i) \begin{cases} \Phi^i := 2\dfrac{\partial Y_i}{\partial z} \ \text{is} \ z^2\text{-type} \\[2ex] \|\Phi^i - \Phi^{i-1}\| \leq \dfrac{1}{Nk} \quad \text{on} \ M_O \backslash \Omega_i'. \\[2ex] \|\Phi^i\| \geq \dfrac{\nu}{2\sqrt{N}} \quad \text{on} \ \Omega_i', \quad \|\Phi^i\| \geq \dfrac{\nu}{2} N^{3.5} \quad \text{on} \ \Omega_i \end{cases}$$

where $\nu \leq \|\Phi^X\| \leq \mu - 1$ and large N. Then observe,

$$\|S(Y_{i-1})\| = \|Y_{i-1}(1) + Y_{i-1}(-1)\| \leq \frac{1}{N}.$$

After the rotation, we may assume that:

$$\text{(4.3)} \qquad \text{if } \operatorname{dist}_{\boldsymbol{R}^3}(0, Y_{i-1}(\Omega_i')) \geq \frac{1}{\sqrt{N}}, \quad \text{then } \angle(\pm\vec{e}_3, Y_{i-1}(\Omega_i')) \leq \frac{8\mu}{\sqrt{N}}$$

$$\text{(4.4)} \qquad \text{as well as } \operatorname{dist}_{S^2}(\pm\vec{e}_3, G_{i-1}(\Omega_i')) \geq \frac{1}{\sqrt{N}}.$$

Note, the similar conditions (3.6) and (3.5) of them are the crucial role of the proof of Assertion 1 and Assertion 2 in the previous section.

By the Runge's theorem again, there is a holomorphic function H_i on M such that

$$|H_i(z^2)| < \varepsilon \quad \text{on } M \backslash \omega_i'$$

$$|H_i(z^2) - \log T| < \varepsilon \quad \text{on } \omega_i$$

for all $T > 0$ and $\varepsilon > 0$. Set

$$h_i(z) = \frac{\exp(H_i(z^2) + H_i(-z^2))}{\exp(\overline{H_i}(I(z^2)) + \overline{H_i}(-I(z^2)))}$$

then it is a z^2-type holomorphic function on M, never vanishing and

$$h_i(I(z)) = -I(h_i(z))$$

$$|h_i - 1| < \frac{1}{T} \quad \text{on } M_O \backslash \Omega_i'$$

$$|h_i - T| < \frac{1}{T} \quad \text{on } \Omega_i$$

with the sufficiently small ε. Now let (f_{i-1}, g_{i-1}) and (f_i, g_i) be the Weierstrass data of Y_{i-1} and Y_i, respectively, such that $f_i = f_{i-1}$ and $g_i = g_{i-1}/h_i$. Then the holomorphic map Φ^i of Y_i is also z^2-type and $Period(Y_i) = 0$. Notice that

$$\begin{aligned} g_i(I(z)) &= \frac{g_{i-1}(I(z))}{h_i(I(z))} = \frac{I(g_{i-1}(z))}{-I(h_i(z))} = \frac{-\overline{h_i}(z)}{\overline{g_{i-1}}(z)} \\ &= I(g_i(z)) \\ (zg_i(z))^2 &= \frac{(zg_{i-1}(z))^2}{(h_i(z))^2} = \frac{-\overline{f_{i-1}}(I(z))}{f_{i-1}(z)} \frac{\overline{h_i}(I(z))}{h_i(z)} \\ &= \frac{-\overline{f_i}(I(z))}{f_i(z)} \end{aligned}$$

which follows, by (4.1) and (4.2), that Y_i is also a double covering of a non-orientable minimal surface. Similar to Assertion 1 in the previous section, together with a sufficiently large T, we can show that $\mathscr{H}_i$ also holds. Define

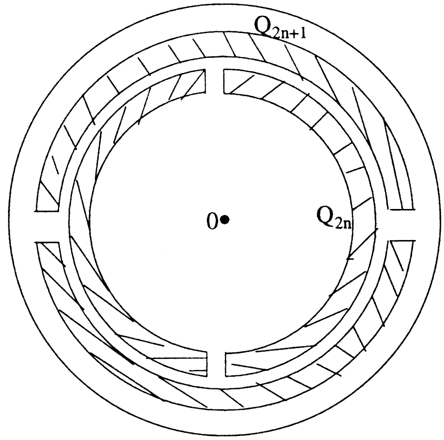

FIGURE 3

$$Y := Y_k - Y_k(1) - \frac{S(Y_k)}{2}$$

and $\tilde{M} \in \mathcal{N}$ such that

$$\mathrm{dist}_Y(\gamma, \partial\tilde{M}) = \rho + s.$$

Then $M^{1-\delta} \subset \tilde{M}$, $\tilde{M} \in \mathcal{M}$ and $\|Y - Y_k\|$ is very small. Repeat the processing of Assertion 1 and Assertion 2, together with (4.3), (4.4), then we can show that

$$Y : \tilde{M} \hookrightarrow \boldsymbol{R}^3$$

is the desired minimal surface. □

5. Proof of Theorem 3 (Examples in a halfspace)

In this section, we construct complete minimal surfaces of $\boldsymbol{R}^3$ lying in a halfspace, $x_3 > 0$, but not a slab, which are transverse to every horizontal plane, similar to [R-T], lying in a slab.

Let $D := \{z \in \boldsymbol{C} : |z| < 1\}$ and $D^* := D \backslash \{0\}$. And let Q_n be a compact sliced annulus contained in $\{t_{n-1} < |z| < t_n\}$, $0 < t_1 < t_2 < \cdots < 1$, where deleting two antipodal pieces centered at the imaginary axe when n is even, and the real axe when n is odd. (see Figure 3). Moreover, $\{Q_n\}$ converges to the boundary circle $|z| = 1$ as $n \to \infty$. Denote $c_n = -\ln s_n$, $n = 1, 2, \ldots$, where s_n is the width of Q_n. Then, by the Runge's theorem again, we can take a holomorphic function h on D such that

$$|h - c_n| < 1 \quad \text{on } Q_n \tag{5.1}$$

for all $n = 1, 2, \ldots$, respectively.

Let us define a minimal surface $X : D^* \hookrightarrow \boldsymbol{R}^3$, $X(1/2) = 0$, which has the Weierstrass data (f, g) and sends the concentric circles $\{|z| = c\}$, $0 < c < 1$, into horizontal planes of $\boldsymbol{R}^3$. Then the third coordinate function X^3 is harmonic on D^* and $X^3|_{\{|z|=c\}} = \text{constant}$. By the uniqueness of solutions to the Dirichlet problem, we have $X^3(z) = a \log|z| + b$ for some real constants a, b. Let $a = -1$ and $b = 0$, then $g(z)f(z) = 2\partial X^3/\partial z = -1/z$ and hence

$$X(p) = \Re \int_{1/2}^{p} \left(\frac{-1}{2z}\left(\frac{1}{g} - g\right), \frac{-i}{2z}\left(\frac{1}{g} + g\right), \frac{-1}{z} \right) dz$$

by the Enneper-Weierstrass representation. Moreover, we define the induced metric ds of X by

$$ds = \frac{1}{|z|}\left(\frac{1}{|g|} + |g|\right)|dz|.$$

Now let us take such minimal surfaces X_1 and X_2, with the Gauss maps g_1 and g_2 defined by:

$$g_1(z) = \frac{1}{z} \exp h(z^2), \quad g_2(z) = \exp h(z)$$

respectively, where h is given in (5.1). Observe that it leads us to prove Theorem 3 by following:

(a) Recall g_1 and g_2 are all holomorphic on D^* and never vanishing, and hence $X_1(D^*)$ and $X_2(D^*)$ are transverse to every horizontal plane.

(b) It is clear that $X_1^3 = X_2^3 = -\log|z| > 0$ on D^*, and hence $X_1(D^*)$ and $X_2(D^*)$ are contained in a halfspace $x_3 > 0$ of $\boldsymbol{R}^3$ but not a slab.

(c) Denote $\Phi^{x_j} := 2\partial X_j/\partial z = (\phi_1^{X_j}, \phi_2^{X_j}, \phi_3^{X_j})$. Then both $\phi_1^{X_1}$ and $\phi_2^{X_1}$ are z^2-type, and $\phi_3^{X_1} = -1/z$ has no real residue. It follows that:

$$Period(X_1) = 0$$

and X_1 is well-defined, not periodic. On the other hand, we compute that:

$$\begin{aligned}
&\Re \int_\gamma \phi_1^{X_2}\, dz + i\Re \int_\gamma \phi_2^{X_2}\, dz \\
&\quad = \overline{\int_\gamma \frac{-1}{2zg_2}\, dz} - \int_\gamma \frac{-g_2}{2z}\, dz = \pi i\left(\overline{\mathrm{Res}_0\ \frac{1}{zg_2}} + \mathrm{Res}_0\ \frac{g_2}{z}\right) \\
&\quad = \pi i(\overline{\exp(-h(0))} + \exp h(0)) \neq 0
\end{aligned}$$

where $\gamma = \{|z| = 1\}$. It follows that

$$Period(X_2) \neq 0$$

clearly, and hence X_2 is singly-periodic.

(d) Now let $\beta \in D^*$ be a piecewise differentiable curve. We call it a *divergent curve*, if either it has infinite Euclidean length or it has finite Euclidean length but tends to the origin or the boundary curve $|z| = 1$. In order that X_1 and X_2 are *complete*, each divergent curve has the infinite arc length with respect to ds_1, ds_2, respectively. We consider the following three cases of the divergent curve β:

• Let β has infinite Euclidean length. Recall $|z| < 1$ on D^*, and so

$$L_j(\beta) := \int_\beta ds_{X_j} = \int_\beta \frac{1}{|z|}\left(\frac{1}{|g_j(z)|} + |g_j(z)|\right)|dz|$$
$$\geq 2\int_\beta |dz| = \infty, \quad j = 1, 2.$$

• If β tends to the origin of the plane, then we have

$$L_j(\beta) \geq 2\int_\beta \frac{1}{|z|}|dz| = \infty, \quad j = 1, 2.$$

• Let β tend to the boundary curve $|z| = 1$ with the finite Euclidean length, and let $\tilde{\beta} := \{z^2 | z \in \beta\}$. By the hypothsis of Q_n's, both β and $\tilde{\beta}$ must cross all but a finite number of Q_{2n} or all but a finite number of Q_{2n-1}. Since

$$|g_1(z)| = \frac{1}{|z|}|e^{c_n}|\,|e^{h(z^2)-c_n}| \geq e^{c_n - 1} \quad \text{for all } z^2 \in Q_n,$$

the either case, we have a number N such that:

$$L_1(\beta) \geq \int_\beta \frac{1}{|z|}|g_1(z)|\,|dz| \geq \sum_{n>N,\ \text{even or odd}} r_n e^{c_n - 1} = \infty.$$

Similarly, since

$$|g_2(z)| = |e^{c_n}|\,|e^{h(z)-c_n}| \geq e^{c_n - 1} \quad \text{for all } z \in Q_n,$$

and hence $L_2(\beta) = \infty$.

Therefore X_1 and X_2 are all complete, and we have shown Theorem 3.

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