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INVOLUTIONS FIXING THE DISJOINT UNION OF 3-REAL PROJECTIVE SPACE WITH DOLD MANIFOLD*

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Abstract

In this paper, we determine the existence of all involutions fixing a disjoint union of 3-real projective space $RP(3)$ with Dold manifold under the condition that the normal bundle to $RP(3)$ does not bound, and also study the representatives up to bordism of those involutions which exist.

§1. Introduction

Let (T, M) be an involution on a closed manifold, and let F denote the fixed point set of (T, M). When F is chosen as

$$\{\mathrm{pt}\} \sqcup S^m, \quad RP(2k), \quad RP(m) \sqcup RP(n), \quad \sqcup RP(2l+1)(l \text{ fixed}),$$
$$\sqcup_{i=1}^{p} RP(2l_i+1), \quad \sqcup_{i=1}^{r} (S^1)^{k_i},$$

and $(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_p}) \sqcup \{\mathrm{pt}\}$, respectively, the existence and the representative (up to bordism) of (T, M) have been studied in [2], [11], [9], [13], [4], [12] and [8]. The purpose of this paper is to determine the existence and the representative up to bordism of all involutions fixing a disjoint union $RP(3) \sqcup P(m, n)$, where $P(m, n)$ is the Dold manifold of dimension $m + 2n$ obtained from the product $S^m \times CP(n)$ of the m-sphere with the n-dimensional complex projective space by identifying (x, z) with $(-x, \bar{z})$ (here $(x, z) \in S^m \times CP(n)$). For this purpose, we first study the vector bundle over Dold manifold so that we can begin with our discussion on the existence of all involutions. The main method will be a formula given by Kosniowski and Stong in [5], and Lucas Theorem [10] will also be used. By setting an involution on Dold manifold $P(3, n+1)$, we partially give the representatives up to bordism of those involutions which exist.

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In particular, we will see that the case $F = RP(3) \sqcup P(m, n)$ possesses more complicated structure than those cases in [2], [4], [8], [9], [11], [12] and [13].

Throughout this paper, the coefficient group is Z_2. Let ω be the total Stiefel-Whitney class and ω_i the i-th Stiefel-Whitney class. Let $[N]$ denote the fundamental homology class of manifold N. Let $\sigma_i(x)$ denote the i-th elementary symmetric function $\Sigma x_1 \cdots x_i$.

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§2. The total Stiefel-Whitney class of vector bundle over Dold manifold $P(m, n)$

Following the notation of [14], let ξ be a 1-plane bundle over $P(m, n)$ and η a 2-plane bundle over $P(m, n)$. Let $c \in H^1(P(m, n); Z_2)$ be the generator and $d \in H^2(P(m, n); Z_2)$ the generator. Then $\omega\xi = 1 + c$, $\omega\eta = 1 + c + d$, $\xi \otimes \xi = 1$, $\xi \otimes \eta = \eta$ (see [14; Proposition (1.4)]).

LEMMA 2.1. *Let τ be the tangent bundle over $P(m, n)$. Then* $\omega\tau = (1 + c)^m(1 + c + d)^{n+1}$, *also denoted by* $\omega P(m, n)$.

Proof. From [14; Theorem (1.5)] we know

$$\tau \oplus \xi \oplus 2 = (m + 1)\xi \oplus (n + 1)\eta.$$

Hence it immediately follows that $\omega\tau = (1 + c)^m(1 + c + d)^{n+1}$. □

Let $\varphi(l) =$ the number of integers e with $0 < e \le l$ and $e \equiv 0, 1, 2, 4 \pmod 8$.

LEMMA 2.2. *Let λ be any a vector bundle over $P(m, n)$. If n is even and $m \equiv 0, 1, 3, 7 \pmod 8$, then*

$$\omega\lambda = (1 + c)^s(1 + c + d)^t$$

where s and t are nonnegative integers.

Proof. Let $\alpha = \xi - 1$, $\beta = \eta - 2$, $\gamma = \beta - \alpha$. According to [3; Theorem 5], when n is even and $m \equiv 0, 1, 3, 7 \pmod 8$, we know that $\tilde{KO}(P(m, n))$ is isomorphic to $Z_{2^{\varphi(m)}} + Z^{n/2}$ generated by $\alpha, \gamma, \gamma^2, \dots, \gamma^{n/2}$ such that $2^{\varphi(m)}\alpha = 0$, $\alpha^2 = -2\alpha$ and $\alpha\gamma = 0$. Therefore it suffices to compute $\omega\xi$ and $\omega\,(\underbrace{\eta \otimes \cdots \otimes \eta}_{p})$ for all $p \ge 0$. Using the formula in [7; p. 87] and $\omega\eta = 1 + c + d$, by direct computions, we can obtain that

$$\omega\,(\underbrace{\eta \otimes \cdots \otimes \eta}_{2l}) = (1 + c)^{2^{2l-1}}$$

and

$$\omega\underbrace{(\eta \otimes \cdots \otimes \eta)}_{2l+1} = (1+c+d)^{2^{2l}}.$$

Again since $\omega\xi = 1 + c$, it follows that $\omega\lambda$ can be as stated. □

§3. A formula and Lucas Theorem

In this section, we review a formula given by Kosniowski and Stong and Lucas Theorem, which will play the important roles in the following sections.

Let (T, M^n) be a smooth involution on a closed n-manifold and the fixed point data of (T, M^n) be $\mu \to F = \sqcup_k \mu^k \to F^{n-k}$. In [5], Kosniowski and Stong gave a formula for the calculation of the Stiefel-Whitney numbers of M^n in terms of the fixed point data $\mu \to F = \sqcup_k \mu^k \to F^{n-k}$. That is the following

Theorem 3.1 (Kosniowski and Stong). *If $f(x_1, \dots, x_n)$ is any symmetric polynomial over Z_2 in n variables of degree at most n, then*

$$f(x_1, \dots, x_n)[M^n] = \sum_k \frac{f(1+y_1, \dots, 1+y_k, z_1, \dots, z_{n-k})}{\prod_{i=1}^k (1+y_i)}[F^{n-k}]$$

where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_i(x)$, $\sigma_i(y)$, and $\sigma_i(z)$ by the Stiefel-Whitney classes $\omega_i(M^n)$, $\omega_i(\mu^k)$, and $\omega_i(F^{n-k})$ respectively, and taking the value of the resulting cohomology class on the fundamental homology class of M^n or F^{n-k}.

Besides, in their paper [5], Kosniowski and Stong also obtained the following

Proposition 3.2. *Let $\sqcup_k \mu^k \to F^{n-k}$ be a disjoint union of bundles over manifolds. A necessary and sufficient condition that the disjoint union is the fixed point data of an involution (T, M^n) is that*

$$\sum_k \frac{f(1+y, z)}{\prod(1+y)}[F^{n-k}] = 0$$

for all f of degree less than n.

By Proposition 3.2, it is easy to see the following result.

Corollary 3.3. *Let $\sqcup_k \mu^k \to F^{n-k}$ be a disjoint union of bundles over manifolds. For some positive integer m, if $\sqcup_k \mu^k \oplus mR \to F^{n-k}$ is the fixed point data of some involution, then so is $\sqcup_k \mu^k \oplus iR \to F^{n-k}$ for each $i < m$.*

Given any involution (T, M^n), as defined in [1], let $\Gamma^1(M^n)$ denote a $(n+1)$-manifold obtained from $S^1 \times M^n$ by identifying (z, x) with $(-z, Tx)$, and with an involution T_1 on $\Gamma^1(M^n)$ induced by $(z,x) \to (\bar{z},x)$. Let $(T_0,\Gamma^0(M^n)) = (T,M^n)$, and let $(T_i,\Gamma^i(M^n))$ be the i-th iteration of (T,M^n). Then a sequence of involutions $\{(T_i,\Gamma^i(M^n))\}$ are constructed. In addition, we also know from [1] that the normal bundle to the fixed point set of $(T_i,\Gamma^i(M^n))$ is

$$\sqcup_k(\mu^k \oplus iR \to F^{n-k}) \sqcup (\sqcup_{j=0}^{i-1}(i-j)R \to \Gamma^j(M^n))$$

where $\mu \to F = \sqcup_k(\mu^k - F^{n-k})$ is the original normal bundle to F in M^n. The following lemmas will be used in §§5, 6.

Lemma 3.4 (See [6]). *Let $\mu \to F = \sqcup_k(\mu^k \to F^{n-k})$ be the fixed data of an involution (T,M^n). Then a necessary and sufficient condition that $\mu \oplus mR \to F$ is the fixed data of some involution is that $\Gamma^j(M^n)$ bounds for each $j < m$.*

Lemma 3.5. *Let (T,M^n) be an involution and its fixed data be $\mu \to F = \sqcup_k\mu^k \to F^{n-k}$. If $\mu \oplus iR \to F$ is the fixed data of some involution, denoted by (T',M'), then (T',M') is bordant to $(T_i,\Gamma^i(M^n))$.*

Proof. First, by using Proposition 3.2, it is easy to show that $\sqcup_{j=0}^{i-1}(i-j)R \to \Gamma^j(M^n))$ is still the fixed data of some involution. Next, by [2; Theorem (23.1)], it immediately follows that $\Gamma^j(M^n)$ bounds for $0 \le j \le i-1$, and thus $\sqcup_k\mu^k \oplus iR \to F^{n-k}$ is bordant to $(\sqcup_k\mu^k \oplus iR \to F^{n-k}) \sqcup (\sqcup_{j=0}^{i-1}(i-j)R \to \Gamma^j(M^n))$. Furthermore, by [2; Theorem (25.2)], we obtain that (T',M') is bordant to $(T_i,\Gamma^i(M^n))$. This completes the proof. □

For any positive integer l, let $E(l)$ denote a set formed by all $i_1,\dots,i_\alpha$ in $2^{i_1}+\cdots+2^{i_\alpha}$, where $2^{i_1}+\cdots+2^{i_\alpha}$ is the dyadic decomposition of l. Then we have

Theorem 3.6 (Lucas) (see [10]). *Let p,q be two positive integers. Then $\binom{p}{q} \equiv 1 \pmod 2$ if and only if $E(q) \subset E(p)$.*

§4. The cases in which involutions do not exist

Throughout the following sections, we always suppose that (T,M^{m+2n+k}) is an involution on a closed $(m+2n+k)$-manifold with fixed point set $F = RP(3) \sqcup P(m,n)$ where $m,n > 0$. Let $v \to F = v_1 \to RP(3) \sqcup v_2 \to P(m,n)$ be the normal bundle of F in M^{m+2n+k}. First, by [2], [9] and [13], we have $\omega RP(3) = (1+a)^4 = 1$ and $\omega v_1 = (1+a)^h$ where h is nonnegative integer and $a \in H^1(RP(3);Z_2)$ is the generator. Let $c \in H^1(P(m,n);Z_2)$ be the generator and let $d \in H^2(P(m,n);Z_2)$ be the generator, by Lemma 2.1 and Lemma 2.2, it follows that $\omega P(m,n) = (1+c)^m(1+c+d)^{n+1}$ and for $m \equiv 0,1,3,7 \pmod 8$ and even $n, \omega v_2 = (1+c)^s(1+c+d)^t$.

Notice that if $v_1 \to RP(3)$ bounds, i.e., $\omega v_1 = 1$ or $(1+a)^2$, then $v_1 \to RP(3) \sqcup v_2 \to P(m,n)$ is bordant to $v_2 \to P(m,n)$. By [2; Theorem (25.2)], this means that the case $F = RP(3) \sqcup P(m,n)$ is identied with the case $F = P(m,n)$. Here we focus our attention on the case $F = RP(3) \sqcup P(m,n)$ with nonbounding normal bundle $\nu_1 \to RP(3)$. Hence in the following discussions, *we always suppose that* $\nu_1 \to RP(3)$ *does not bound, i.e.,* $\omega\nu_1 = 1 + a$ *or* $(1+a)^3$.

Now we begin on discussing the existence of (T, M^{m+2n+k}).

(I) The case $\chi(P(m,n)) = 1$, where $\chi(.)$ denotes the Euler characteristic number.

LEMMA 4.1. *There does not exist the involution* (T, M^{m+2n+k}) *with* $\chi(P(m,n)) = 1$.

Proof. Suppose that (T, M^{m+2n+k}) exists if $\chi(P(m,n)) = 1$. First, by [5; p. 313, Proposition], we have

$$\chi(M^{m+2n+k}) = \chi(RP(3)) + \chi(P(m,n)) = 0 + 1 = 1.$$

Hence it immedately follows that m, n, k must be all even. Next, using the fact (see [5; p. 317, Lemma]) that

$$\begin{aligned}&\sigma_i(1+y_1,\dots,1+y_e,z_1,\dots,z_{n-e})\\&\quad=\sum_{p+q\le i}\binom{e-p}{i-p-q}\sigma_p(y_1,\dots,y_e)\sigma_q(z_1,\dots,z_{n-e}),\end{aligned}$$

by direct computations, we have

$$\sigma_1(1+y,z)(\nu_1 \to RP(3)) = 1 + a$$

and

$$\sigma_1(1+y,z)(\nu_2 \to P(m,n)) = \varepsilon c$$

where $\varepsilon = 0$ or 1. We proceed as follows:

Case (i) When $\omega\nu_1 = 1 + a$, if $m = 4l + 2$, taking $f(x) = (\sigma_1(x))^{m+2}$. ($\deg f = m + 2 = 4l + 4 < m + 2n + k$) and using Theorem 3.1, then we have

$$\begin{aligned}0 &= f(x)[M^{m+2n+k}]\\&= \frac{(\sigma_1(1+y,z))^{4l+4}}{\prod(1+y)}[RP(3)] + \frac{(\sigma_1(1+y,z))^{4l+4}}{\prod(1+y)}[P(4l+2,n)]\\&= \frac{(1+a)^{4l+4}}{1+a}[RP(3)] + \frac{(\varepsilon c)^{4l+4}}{\omega\nu_2}[P(4l+2,n)]\\&= (1+a+a^2+a^3)[RP(3)] + 0\\&= 1.\end{aligned}$$

This leads to a contradiction. If $m = 4l$, first it is easy to see that k must be a positive even. Taking $(\sigma_1(x))^{m+4}$ of degree less than $m + 2n + k$ and using

Theorem 3.1, we have

$$\begin{aligned} 0 &= \frac{(1+a)^{4l+4}}{1+a}[RP(3)] + \frac{(\varepsilon c)^{4l+4}}{\omega v_2}[P(4l,n)] \\ &= 1+0 \\ &= 1. \end{aligned}$$

But this is impossible.

Case (ii) When $\omega v_1 = (1+a)^3$, if $m = 4l+2$ then we choose $(\sigma_1(x))^{m+4}$ and if $m = 4l$ then we choose $(\sigma_1(x))^{m+2}$, as in the case (i), we can obtain the contradictions and thus the case $\omega v_1 = (1+a)^3$ doesn't exist.

Combining the above discussions, the Lemma thus holds. □

(II) The case $\chi(P(m,n)) = 0$.

From $\chi(P(m,n)) = 0$, we have $(m+1)(n+1) \equiv 0 \pmod 2$. Therefore, in the following discussions we divide $\chi(P(m,n)) = 0$ into three cases: (i) m is even and n is odd; (ii) m, n are all odd; (iii) m is odd and n is even.

Notice that when $k = 0$, i.e., $\omega v_2 = 1$, it is easy to see that (T, M^{m+2n+k}) with $\chi(P(m,n)) = 0$ does not exist. Hence here we may assume that $k > 0$.

Lemma 4.2. *There does not exist the involution (T, M^{m+2n+k}) for which m is even and n is odd.*

Proof. Suppose that when m is even and n is odd, the involution (T, M^{m+2n+k}) exists. We proceed as follows:

If $\omega v_1 = 1+a$, using [5; p. 317, Lemma], then we have

$$\sigma_1(1+y,z)(v_1 \to RP(3)) = \binom{m+2n+k-3}{1} + a$$

and

$$\sigma_1(1+y,z)(v_2 \to P(m,n)) = \binom{k}{1} + \varepsilon c$$

where $\varepsilon = 0$ or 1. Consider the following:

(i) When $m = 4l+2$, taking $\left(\sigma_1(x) + \binom{k}{1}\right)^{m+2}$ of degree less than $m+2n+k$ and using Theorem 3.1, we have

$$0 = \frac{(1+a)^{4l+4}}{1+a}[RP(3)] + \frac{(\varepsilon c)^{4l+4}}{\omega v_2}[P(4l+2,n)] = 1+0 = 1.$$

This means that $m = 4l+2$ is impossible.

(ii) When either $m = 4l$, $n > 1$ or $m = 4l$, $n = 1$, $k > 2$, taking $(\sigma_1(x) + \binom{k}{1})^{m+4}$, in the same way as the above, we can show that either $m = 4l$, $n > 1$ or $m = 4l$, $n = 1$, $k > 2$ is impossible.

(iii) When $m = 4l$, $n = 1$, $k = 1$, taking symmetric polynoimal function 1, we have

$$0 = \frac{1}{1+a}[RP(3)] + \frac{1}{1+\sigma_1(y)}[P(4l,1)] = 1 + 0 = 1.$$

But this is a contradiction.

(iv) When $m = 4l$, $n = 1$, $k = 2$, if $\omega_1 v_2 = 0$, i.e., $\sigma_1(1+y,z)(v_2 \to P(m,n)) = \binom{k}{1} = 0$, taking $(\sigma_1(x))^{4l}$, then we can obtain a contradiction, and thus $\omega_1 v_2 = 0$ is impossible. Let $\omega_1 v_2 = c$. Using [5; p. 317, Lemma], we have

$$\sigma_2(1+y,z)(v_1 \to RP(3)) = 0$$

and

$$\sigma_2(1+y,z)(v_2 \to P(4l,1)) = 1 + c + c^2 + \sigma_2(y).$$

Taking $(\sigma_2(x) + \sigma_1^2(x))^2$ and using Theorem 3.1, it follows that

$$\begin{aligned}
0 &= \frac{(1+a)^4}{1+a}[RP(3)] + \frac{(1+c+\sigma_2(y))^2}{1+c+\sigma_2(y)}[P(4l,1)] \\
&= 1 + (1+c+\sigma_2(y))[P(4l,1)] \\
&= 1 + 0 \\
&= 1.
\end{aligned}$$

But this is a contradiction.

If $\omega v_1(1+a)^3$, in the same way as the case $\omega v_1 = 1 + a$, take $\left(\sigma_1(x) + \binom{k}{1}\right)^{m+2}$, we may prove that $m = 4l$ is impossible; and take $\left(\sigma_1(x) + \binom{k}{1}\right)^{m+4}$, we may prove that either $m = 4l+2$, $n > 1$ or $m = 4l+2$, $n = 1$, $k > 2$ is impossible. When $m = 4l+2$, $n = 1$, and $k = 1$, taking $(\sigma_1(x))^2$, we have

$$\begin{aligned}
0 &= \frac{a^2}{(1+a)^3}[RP(3)] + \frac{(1+\sigma_1(y))^2}{1+\sigma_1(y)}[P(4l+2,1)] \\
&= 1 + (1+\sigma_1(y))[P(4l+2,1)] \\
&= 1 + 0 \\
&= 1.
\end{aligned}$$

So this case is impossible. When $m = 4l+2$, $n = 1$, and $k = 2$, since

$$\sigma_2(1+y,z)(v_1 \to RP(3)) = 1 + a^2$$

and

$$\sigma_2(1+y,z)(v_2 \to P(4l,1)) = 1 + \sigma_1(y) + \sigma_2(y),$$

choose $\sigma_2(x)$, it follows that

$$\begin{aligned} 0 &= \frac{1+a^2}{(1+a)^3}[RP(3)] + \frac{1+\sigma_1(y)+\sigma_2(y)}{1++\sigma_1(y)+\sigma_2(y)}[P(4l+2,1)] \\ &= 1+[P(4l+2,1)] \\ &= 1+0 \\ &= 1. \end{aligned}$$

This means that the case in which $m = 4l+2$, $n = 1$, and $k = 2$ is still impossible.

Together with the above discussions, we complete the proof of the Lemma 4.2. □

LEMMA 4.3. *There does not exist the involution (T, M^{m+2n+k}) for which m, n are all odd.*

Proof. Suppose that when m, n are all odd, the involution (T, M^{m+2n+k}) exists. Using [5; p. 317, Lemma], we have

$$\sigma_1(1+y,z)(\nu_1 \to RP(3)) = \binom{m+2n+k-3}{1} + a$$

and

$$\sigma_1(1+y,z)(\nu_2 \to P(m,n)) = \binom{k}{1} + c + \sigma_1(y).$$

If either $\omega_1\nu_2 = c$ (i.e., $\sigma_1(1+y,z)(\nu_2 \to P(m,n)) = \binom{k}{1}$) or $m = 1$, taking $\left(\sigma_1(x) + \binom{k}{1}\right)^3$ and using Theorem 3.1, then

$$0 = \frac{a^3}{\omega\nu_1}[RP(3)] + \frac{0}{\omega\nu_2}[P(m,n)] = 1+0 = 1.$$

But this is a contradiction. If $\omega_1\nu_2 = 0$, by direct computations, we have

$$\sigma_2(1+y,z)(\nu_1 \to RP(3)) = \binom{m+2n+k-3}{2} + \binom{m+2n+k-4}{1}\sigma_1(y) + \sigma_2(y)$$

and

$$\sigma_2(1+y,z)(\nu_2 \to P(m,n)) = \binom{k}{2} + \binom{k}{1}c + \sigma_2(y) + \binom{m+n+1}{2}c^2. \quad (4.1)$$

We proceed as follows:

(i) When $\sigma_2(y) + \binom{m+n+1}{2}c^2 = \varepsilon c^2 + d$ (here $\varepsilon = 0$ or 1) in (4.1) and $m \neq 3$, taking

$$f(x) = \left(\sigma_1(x) + \binom{k}{1}\right)^m \left(\sigma_2(x) + \binom{k}{2} + \binom{k-1}{1}\left(\sigma_1(x) + \binom{k}{1}\right)\right)^n$$

with $\deg f = m + 2n < m + 2n + k$, we have

$$\begin{aligned} 0 &= \frac{f(1+y,z)}{\omega v_1}[RP(3)] + \frac{c^m(c+\varepsilon c^2+d)^n}{\omega v_2}[P(m,n)] \\ &= \frac{f(1+y,z)}{\omega v_1}[RP(3)] + 1, \end{aligned}$$

but

$$\frac{f(1+y,z)}{\omega v_1}[RP(3)]$$
$$= \begin{cases} \dfrac{a^m(\sigma_2(y))^n}{\omega v_1}[RP(3)] = 0, & \text{if } m \equiv 1,5 \pmod 8 \ (m \neq 1); \\ \dfrac{a^m(1+\sigma_2(y))^n}{\omega v_1}[RP(3)] = 0, & \text{if } m \equiv 3,7 \pmod 8 \ (m \neq 3). \end{cases}$$

Obviously, this leads to a contradiction.

(ii) When $\sigma_2(y) + \binom{m+n+1}{2}c^2 = \varepsilon c^2 + d$ (here $\varepsilon \in Z_2$) in (4.1) and $m = 3$, it is obvious that k must be more than or equal to 2. If $k > 2$, taking

$$\left(\sigma_1(x) + \binom{k}{1}\right)^3 \left(\sigma_2(x) + \binom{k}{2} + \binom{k-1}{1}\left(\sigma_1(x) + \binom{k}{1}\right)\right)^{n+1}$$

of degree less than $m + 2n + k$, we can obtain the following contrary equation

$$\begin{aligned} 0 &= \frac{a^3(1+\sigma_2(y))^{n+1}}{\omega v_1}[RP(3)] + \frac{c^3(c+\varepsilon c^2+d)^{n+1}}{\omega v_2}[P(3,n)] \\ &= 1 + 0 \\ &= 1. \end{aligned}$$

Note that $\binom{k}{2} + \binom{2n+k}{2} = 1$. For $k = 2$, if $\omega v_1 = 1 + a$, taking $(1+\sigma_1(x))^4$, then we have the following contrary equation

$$0 = \frac{(1+a)^4}{1+a}[RP(3)] + \frac{(1+c)^4}{1+\binom{4+n}{2}c^2+\varepsilon c^2+d}[P(3,n)] = 1 + 0 = 1;$$

if $\omega v_1 = (1+a)^3$, taking $(1+\sigma_1(x))^2$, then we have

$$0 = \frac{(1+a)^2}{(1+a)^3}[RP(3)] + \frac{(1+c)^2}{1+\binom{4+n}{2}c^2+\varepsilon c^2+d}[P(3,n)] = 1 + 0 = 1,$$

but this is also a contradiction.

(iii) When $\sigma_2(y) + \binom{m+n+1}{2}c^2 = \varepsilon c^2$ in (4.1), if $m \equiv 1,5 \pmod 8$ $(m \neq 1)$, taking $\left(\sigma_1(x) + \binom{k}{1}\right)^2\left(\sigma_2(x) + \binom{k}{2} + \binom{k-1}{1}\left(\sigma_1(x) + \binom{k}{1}\right)\right) + \left(\sigma_1(x) + \binom{k}{1}\right)^3 + \varepsilon\big(\sigma_1(x)$

$+\binom{k}{1})^4$, we have

$$0 = \frac{a^2\sigma_2(y) + a^3 + \varepsilon a^4}{\omega v_1}[RP(3)] + \frac{c^2(c+\varepsilon c^2) + c^3 + \varepsilon c^4}{1 + \binom{m+n+1}{2}c^2 + \varepsilon c^2}[P(m,n)]$$

$$= 1 + 0$$

$$= 1.$$

This is impossible. If $m \equiv 3, 7 \pmod 8$, since $\binom{m+2n+k-3}{2} + \binom{k}{2} = 1$, taking $(\sigma_1(x) + \binom{k}{1})^3(\sigma_2(x) + \binom{k}{2} + \binom{k-1}{1}(\sigma_1(x) + \binom{k}{1})) + (\sigma_1(x) + \binom{k}{1})^4 + \varepsilon(\sigma_1(x) + \binom{k}{1})^5$, we have

$$0 = \frac{a^3(1+\sigma_2(y)) + a^4 + \varepsilon a^5}{\omega v_1}[RP(3)] + \frac{c^3(c+\varepsilon c^2) + c^4 + \varepsilon c^5}{1 + \binom{m+n+1}{2}c^2 + \varepsilon c^2}[P(m,n)]$$

$$= 1 + 0$$

$$= 1.$$

This contrary equation shows that $m \equiv 3, 7 \pmod 8$ is impossible.

Combining the above discussions, the Lemma 4.3 thus holds. □

LEMMA 4.4. *There does not exist the involution (T, M^{m+2n+k}) for which n is even and $m \equiv 1 \pmod 4$.*

Proof. Suppose that (T, M^{m+2n+k}) exists when n is even and $m \equiv 1 \pmod 4$. By direct computations, we have

$$\sigma_1(1+y,z)(v_1 \to RP(3)) = \binom{m+2n+k-3}{1} + a$$

and

$$\sigma_1(1+y,z)(v_2 \to P(m,n)) = \binom{k}{1} + \sigma_1(y).$$

Taking $(\sigma_1(x) + \binom{k}{1})^3$ and using Theorem 3.1, we have

$$0 = \frac{a^3}{\omega v_1}[RP(3)] + \frac{(\sigma_1(y))^3}{\omega v_2}[P(m,n)] = 1 + \frac{(\sigma_1(y))^3}{\omega v_2}[P(m,n)]$$

and thus $(\sigma_1(y))^3/(\omega v_2)[P(m,n)] = 1$. Furthermore, it follows that $\omega_1 v_2 = c$ and m is more than or equal to 5. Again using [5; p. 317, Lemma], we have

$$\sigma_2(1+y,z)(v_1 \to RP(3)) = \binom{k-2}{2} + \binom{k+1}{1}a + \sigma_2(y)$$

and

$$(4.2) \qquad \sigma_2(1+y,z)(v_2 \to P(m,n)) = \binom{k}{2} + \binom{k-1}{1}c + \sigma_2(y) + \sigma_2(z).$$

We proceed as follows:

(i) If $\sigma_2(y) + \sigma_2(z) = 0$ in (4.2), taking

$$\left(\sigma_1(x) + \binom{k}{1}\right)^3 \left(\sigma_2(x) + \binom{k}{2} + \binom{k+1}{1}\left(\sigma_1(x) + \binom{k}{1}\right)\right),$$

we can obtain the following contrary equation

$$0 = \frac{a^3(1+\sigma_2(y))}{\omega v_1}[RP(3)] + \frac{c^3 \times 0}{\omega v_2}[P(m,n)] = 1 + 0 = 1.$$

(ii) If $\sigma_2(y) + \sigma_2(z) = \varepsilon c^2 + d$ (here $\varepsilon \in Z_2$) in (4.2), choose

$$\left(\sigma_1(x) + \binom{k}{1}\right)^m \left(\sigma_2(x) + \binom{k}{2} + \binom{k+1}{1}\left(\sigma_1(x) + \binom{k}{1}\right)\right)^n$$

of degree less than $m + 2n + k$, we have

$$0 = \frac{a^m(1+\sigma_2(y))^n}{\omega v_1}[RP(3)] + \frac{c^m(\varepsilon c^2 + d)^n}{\omega v_2}[P(m,n)] = 0 + 1 = 1.$$

This leads to a contradiction.

(iii) If $\sigma_2(y) + \sigma_2(z) = c^2$ in (4.2), choose

$$\left(\sigma_1(x) + \binom{k}{1}\right)^3$$
$$\times \left(\sigma_2(x) + \binom{k}{2} + \binom{k+1}{1}\left(\sigma_1(x) + \binom{k}{1}\right)\right) + \left(\sigma_1(x) + \binom{k}{1}\right)^5,$$

we have

$$\begin{aligned} 0 &= \frac{a^3(1+\sigma_2(y)) + a^5}{\omega v_1}[RP(3)] + \frac{c^3c^2 + c^5}{\omega v_2}[P(m,n)] \\ &= 1 + 0 \\ &= 1, \end{aligned}$$

but this is impossible.

Combining the above discussions, we complete the proof of the Lemma 4.4. □

Recall that for even n and $m \equiv 3 \pmod 4$, $\omega v_2 = (1+c)^s(1+c+d)^t$. Let $2^N > \max\{m,n\}$, $2^M > m$. We have

LEMMA 4.5. *Suppose that* (T, M^{m+2n+k}) *exists for which* n *is even,* $m \equiv 3 \pmod 4$ *and* $\omega v_1 = 1 + a$. *Then*

$$E(n) \subset E(2^{N+1} - t) \quad \text{and} \quad E(m) \subset E(2^{M+1} + 2^{N+1} - s - t - n).$$

Proof. Choose symmetric polynomial function 1, we have

$$\begin{aligned} 0 &= \frac{1}{1+a}[RP(3)] + \frac{1}{(1+c)^s(1+c+d)^t}[P(m,n)] \\ &= 1 + (1+c)^{2^{M+1}-s}(1+c+d)^{2^{N+1}-t}[P(m,n)] \\ &= 1 + \binom{2^{N+1} - t}{n}\binom{2^{M+1} + 2^{N+1} - s - t - n}{m}, \end{aligned}$$

and thus the result holds by Lucas Theorem. □

In virtue of $\omega v_2 = (1+c)^s(1+c+d)^t$ with even n and $m \equiv 3 \pmod 4$, we may assume that $E(s) \subset \{i | 2^i \le m\}$, $E(t) \subset \{j | 2^j \le n\}$ because $c^x = 0$, $d^y = 0$ for $x > m$, $y > n$, and $(1+c)^{2^i} = 1$, $(1+d)^{2^j} = 1$ for $2^i > m$, $2^j > n$.

LEMMA 4.6. *Let* n *is even and* t *is odd. If* $E(n) \subset E(2^{N+1} - t)$, *then* $t < n$.

Proof. First, we prove that $E(n) \cap E(t) = \Phi$ (empty set). Since $E(2^{N+1} - 1) = \{0, 1, \ldots, N\}$, it follows that each element in $E(t-1)$ doesn't belong to $E(2^{N+1} - t)$. Hence $E(n) \cap E(t-1) = \Phi$. Since t is odd and n is even, we have $E(t) = E(t-1) \cup \{0\}$ and $0 \notin E(n)$. Therefore we can obtain $E(n) \cap E(t) = \Phi$. Next, let $u = \max\{\alpha | \alpha \in E(t)\}$, $v = \max\{\beta | \beta \in E(n)\}$. It is obvious that $u \ne v$ since $E(n) \cap E(t) = \Phi$. Because $E(t) \subset \{j | 2^j \le n\}$, we have $u < v$ and thus $E(t) \subset E(2^v - 1)$. Furthermore, we have $n \ge 2^v > t$. This completes the proof. □

LEMMA 4.7. *If* n *is even,* $n \equiv 3 \pmod 4$ *and* $\omega v_1 = 1 + a$, *then* (T, M^{m+2n+k}) *doesn't exist except for the following cases: either*

(I) $\omega v_2 = (1+c)^2(1+c+d)$, $m = 3$, $n = 4l + 2$, $4 \le k \le 6$;

or

(II) $\omega v_2 = 1 + c + d$, $m = 3$, $n = 4l$, $2 \le k \le 4A - 3$

where $A = \min\{\alpha | \alpha > 0$ *and* $E(\alpha) \subset E(2L)\}$.

Proof. Suppose that (T, M^{m+2n+k}) exists under the condition that n is even, $m \equiv 3 \pmod 4$ and $\omega v_1 = 1 + a$. By direct computations, we have

$$\sigma_1(1+y, z)(v_1 \to RP(3)) = \binom{m + 2n + k - 3}{1} + a$$

and

$$\sigma_1(1+y, z)(v_2 \to P(m,n)) = \binom{k}{1} + c.$$

Taking $\left(\sigma_1(x)+\binom{k}{1}\right)^m$, then

$$\begin{aligned} 0 &= \frac{a^m}{1+a}[RP(3)] + \frac{c^m}{(1+c)^s(1+c+d)^t}[P(m,n)] \\ &= \frac{a^m}{1+a}[RP(3)] + \binom{2^{N+1}-t}{n}. \end{aligned}$$

By Lemma 4.5, we have $a^m/(1+a)[RP(3)] = 1$, and thus m must be 3. Since

$$\sigma_2(1+y,z)(v_1 \to RP(3)) = \binom{2n+k}{2} + \binom{2n+k-1}{1}a$$

and

$$\begin{aligned} \sigma_2(1+y,z)(v_2 \to P(3,n)) = &\binom{k}{2} + \binom{k-1}{1}c + \binom{n+4}{2}c^2 \\ &+ \binom{s+t}{2}c^2 + \binom{t}{1}d + d, \end{aligned}$$

take $\left(\sigma_1(x)+\binom{k}{1}\right)^3\left(\sigma_2(x)+\binom{k}{2}\right)^n$, we may conclude that

$$\begin{aligned} 0 &= \frac{0}{1+a}[RP(3)] + \frac{c^3\left(\binom{t}{1}d+d\right)^n}{(1+c)^s(1+c+d)^t}[P(m,n)] \\ &= 0 + \binom{t}{1} + 1. \end{aligned}$$

and thus $\binom{t}{1} = 0$ is impossible. Let $\binom{t}{1} = 1$, i.e., t is odd, by Lemma 4.5, we know that $s+t$ is odd, and thus s must be even. Furthermore, we can obtain $s = 0$ or 2 since $E(s) \subset \{i \mid 2^i \leq m\}$ and $m = 3$. On the other hand, since t is odd and n is even, by Lemma 4.6, we have $t < n$. Therefore, it follows that the highest degree term in $\omega v_2 = (1+c)^s(1+c+d)^t$ is $c^s d^t \neq 0$, and thus $k \geq s+2t$. We proceed as follows:

If $\binom{n+4}{2} + \binom{s+t}{2} = 1$, choose

$$\left(\sigma_1(x)+\binom{k}{1}\right)\left(\sigma_2(x)+\binom{k}{2}+\binom{k-1}{1}\left(\sigma_1(x)+\binom{k}{1}\right)\right),$$

by Theorem 3.1, we may obtain a contradiction.

If $\binom{n+4}{2} + \binom{s+t}{2} = 0$, we have

$$\sigma_4(1+y,z)(v_1 \to RP(3)) = \binom{2n+k}{4} + \binom{2n+k-1}{3}a$$

and

$$\sigma_4(1+y,z)(v_2 \to P(3,n))$$
$$= \begin{cases} \binom{k}{4} + \binom{k-1}{3} c + d + cd + c^2 + \binom{t}{2} d^2, & \text{if } \binom{n+4}{2} = \binom{s+t}{2} = 1; \\ \binom{k}{4} + \binom{k-1}{3} c + d + cd + d^2 + \binom{t}{2} d^2, & \text{if } \binom{n+4}{2} = \binom{s+t}{2} = 0. \end{cases}$$

Consider the following

(i) When $\binom{n+4}{2} = \binom{s+t}{2} = 1$, i.e., $n = 4l+2$, since $E(n) \cap E(t) = \Phi$ (see the proof of Lemma 4.6), it follows that $1 \notin E(t)$, i.e., $\binom{t}{2} = 0$. Furthermore, from $\binom{s+t}{2} = 1$ we have $s = 2$. If $t > 1$, choose

$$\left(\sigma_1(x) + \binom{k}{1}\right)^3 \left(\sigma_4(x) + \binom{2n+k}{4}\right)^{t-1},$$

by Theorem 3.1 we have

$$\begin{aligned} 0 &= \frac{0}{1+a}[RP(3)] + \frac{c^3(1+d)^{t-1}}{(1+c)^2(1+c+d)^t}[P(3,4l+2)] \\ &= 0 + \frac{c^3}{1+d}[P(3,4l+2)] \\ &= 1. \end{aligned}$$

This shows that for $t > 1$, the involution (T, M^{7+8l+k}) doesn't exist. Hence, in the next discussions, it suffices to check the following case:

$$s = 2, \quad t = 1, \quad m = 3, \quad n = 4l+2, \quad k \geq 4.$$

By direct computations, we have

$$\sigma_{8l+6}(1+y,z)(v_1 \to RP(3)) = \binom{8l+4+k}{8l+6} + \binom{8l+3+k}{8l+5} a$$

and

$$\begin{aligned} &\sigma_{8l+6}(1+y,z)(v_2 \to P(3,4l+2)) \\ &\quad = d^{4l+2} + cd^{4l+2} + c^3 c^{4l+1} + \left(\binom{k-1}{2} + \binom{k-4}{2}\right) c^2 d^{4l+1} \\ &\qquad + \text{terms of degree less than } 8l+4. \end{aligned}$$

If $k > 6$, choose

$$f(x) = \sigma_{8l+6}(x)\left(\sigma_4(x) + \binom{8l+4+k}{4} + \binom{k-1}{3}\left(\sigma_1(x) + \binom{k}{1}\right)\right)$$
$$\times\left(\sigma_1(x) + \binom{k}{1}\right)^3$$

with $\deg f = 8l+13 < m+2n+k$, since

$$f(1+y,z)(v_1 \to RP(3)) = \left(\binom{8l+4+k}{8l+6} + \binom{8l+3+k}{8l+5}a\right) \times 0 \times a^3 = 0$$

and

$$f(1+y,z)(v_2 \to P(3,4l+2))$$
$$= (d^{4l+2} + \text{terms of degree less than } 8l+4)(1+d)c^3,$$

by Theorem 3.1, then

$$0 = \frac{0}{1+a}[RP(3)]$$
$$+\frac{c^3(1+d)(d^{4l+2} + \text{terms of degree less than } 8l+4)}{(1+c)^2(1+c+d)}[P(3,4l+2)]$$
$$= 0 + (c^3d^{4l+2} + \text{terms of degree less than } 8l+7)[P(3,4l+2)]$$
$$= 1,$$

but this is a contradiction.

(ii) When $\binom{n+4}{2} = \binom{s+t}{2} = 0$, i.e., $n = 4l$, taking

$$\left(\sigma_1(x) + \binom{k}{1}\right)^3\left(\sigma_4(x) + \binom{k}{4} + 1\right)^t,$$

by Theorem 3.1, we may obtain that $\binom{t}{2} = 1$ is impossible. Let $\binom{t}{2} = 0$, since $\binom{s+t}{2} = 0$, then $s = 0$. Taking

$$\left(\sigma_1(x) + \binom{k}{1}\right)^3\left(\sigma_4(x) + \binom{k}{4}\right)^{t-1},$$

by Theorem 3.1, we may easily show that $t > 1$ is impossible. Now we need only to check the following case:

$$s = 0, \quad t = 1, \quad m = 3, \quad n = 4l, \quad k \geq 2.$$

Let $A = \min\{\alpha|\alpha > 0 \text{ and } E(\alpha) \subset E(2l)\}$. If $k+3 > 4A$, taking

$$\left(\sigma_4(x) + \binom{k}{4} + \binom{k-1}{3}\left(\sigma_1(x) + \binom{k}{1}\right)\right)^{2l+A}$$

and using Theorem 3.1, then

$$
\begin{aligned}
0 &= \frac{0}{1+a}[RP(3)] + \frac{(d+cd+d^2)^{2l+A}}{1+c+d}[P(3,4l)] \\
&= 0 + d^{2l+A}(1+c+d)^{2l+A-1}[P(3,4l)] \\
&= d^{2l+A}\left(\binom{2l+A-1}{2l-A} d^{2l-A}(1+c)^{2A-1}\right)[P(3,4l)] \\
&= \binom{2l+A-1}{2A-1}\binom{2A-1}{3}.
\end{aligned}
$$

From the definition of A, it is easy to see that $A = 2^i$ for some i. Therefore we may conclude that $E(3) = \{0,1\} \subset E(2A-1)$ and

$$E(2A-1) = E(A) \cup E(A-1) \subset E(2l) \cup E(A-1) = E(2l+A-1).$$

Furthermore, by Lucas Theorem, we have

$$\binom{2l+A-1}{2A-1}\binom{2A-1}{3} = 1.$$

But this leads to a contradiction.

Combining the above, consequently we complete the proof. □

Now consider the case in which n is even, $m \equiv 3 \pmod 4$ and $\omega v_1 = (1+a)^3$.

Lemma 4.8. *Suppose (T, M^{m+2n+k}) exists under the condition that n is even, $m \equiv 3 \pmod 4$ and $\omega v_1 = (1+a)^3$. Then*

$$E(n) \subset E(2^{N+1} - t)$$

and $s+t$ is odd.

Proof. Choose symmetric polynomial function $\left(\sigma_1(x) + \binom{k}{1}\right)^2$, we have

$$
\begin{aligned}
0 &= \frac{a^2}{(1+a)^3}[RP(3)] + \frac{(\sigma_1(y))^2}{(1+c)^s(1+c+d)^t}[P(m,n)] \\
&= 1 + \frac{(\sigma_1(y))^2}{(1+c)^s(1+c+d)^t}[P(m,n)]
\end{aligned}
$$

and thus $(\sigma_1(y))^2/((1+c)^s(1+c+d)^t)[P(m,n)] = 1$. This means that $\omega_1 v_2 \neq 0$. Let $\omega_1 v_2 = c$. Then we have

$$\begin{aligned}1 &= \frac{(\sigma_1(y))^2}{(1+c)^s(1+c+d)^t}[P(m,n)] \\ &= \frac{c^2}{(1+c)^s(1+c+d)^t}[P(m,n)] \\ &= c^2(1+c)^{2^{M+1}-s}(1+c+d)^{2^{N+1}-t}[P(m,n)] \\ &= \binom{2^{N+1}-t}{n}\binom{2^{M+1}+2^{N+1}-s-t-n}{m-2},\end{aligned}$$

and thus the result holds by Lucas Theorem. □

Note that for the case in which n is even, $m \equiv 3 \pmod 4$ and $\omega v_1 = (1+a)^3$, it is easy to see that Lemma 4.6 still holds.

LEMMA 4.9. *If n is even, $m \equiv 3 \pmod 4$ and $\omega v_1 = (1+a)^3$, then (T, M^{m+2n+k}) doesn't exist except for the following cases: either*

(I) $\omega v_2 = 1+c+d,\ m=3,\ n=4l+2,\ 2 \le k \le 4$;

or

(II) $\omega v_2 = (1+c)^2(1+c+d),\ m=3,\ n=4l,\ 4 \le k \le 4A-1$ *where* $A = \min\{\alpha | \alpha > 0$ *and* $E(\alpha) \subset E(2l)\}$.

Proof. Suppose (T, M^{m+2n+k}) exists when n is even, $m \equiv 3 \pmod 4$ and $\omega v_1 = (1+a)^3$. Choose $\left(\sigma_1(x) + \binom{k}{1}\right)^m$, it follows that

$$\begin{aligned}0 &= \frac{a^m}{(1+a)^3}[RP(3)] + \frac{c^m}{(1+c)^s(1+c+d)^t}[P(m,n)] \\ &= \frac{a^m}{(1+a)^3}[RP(3)] + \binom{2^{N+1}-t}{n} \\ &= \frac{a^m}{(1+a)^3}[RP(3)] + 1 \text{ (by Lemma 4.8)}\end{aligned}$$

and thus $m = 2$ or 3. Furthermore, we have that $m=3$ since $m \equiv 3 \pmod 4$. By direct computations, we have

$$\sigma_2(1+y,z)(v_1 \to RP(3)) = \binom{2n+k}{2} + \binom{2n+k-1}{1}a + a^2$$

and

$$\begin{aligned}\sigma_2(1+y,z)(v_2 \to P(3,n)) &= \binom{k}{2} + \binom{k-1}{1}c \\ &\quad + \binom{n+4}{2}c^2 + \binom{s+t}{2}c^2 + \binom{t}{1}d + d.\end{aligned}$$

As in the proof of Lemma 4.7, choose $\left(\sigma_1(x)+\binom{k}{1}\right)^3\left(\sigma_2(x)+\binom{k}{2}\right)^n$, we may obtain that $\binom{t}{1}=0$ is impossible. So $\binom{t}{1}=1$. Furthermore, by Lemma 4.8 we have that s must be even, and thus $s=0$ or 2. Again by Lemma 4.8 and Lemma 4.6, it follows that $t<n$. Hence $k\geq s+2t$. We proceed as follows:

If $\binom{n+4}{2}+\binom{s+t}{2}=0$, take

$$\left(\sigma_1(x)+\binom{k}{1}\right)\left(\sigma_2(x)+\binom{k}{2}+\binom{k-1}{1}\left(\sigma_1(x)+\binom{k}{1}\right)\right),$$

we may conclude that this case is impossible.

If $\binom{n+4}{2}+\binom{s+t}{2}=1$, consider

$$\begin{aligned}\sigma_4(1+y,z)(v_1\to RP(3))&=\binom{2n+k}{4}+\binom{2n+k-1}{3}a\\&\quad+\binom{2n+k-2}{2}a^2+\binom{2n+k-3}{1}a^3\end{aligned}$$

and

$\sigma_4(1+y,z)(v_2\to P(3,n))$

$$=\begin{cases}\binom{k}{4}+\binom{k-1}{3}c+\binom{k}{2}c^2+d+cd+\binom{k-1}{1}c^3+\binom{t}{2}d^2,\\ \qquad\qquad\qquad\qquad if\binom{n+4}{2}=1,\binom{s+t}{2}=0;\\ \binom{k}{4}+\binom{k-1}{3}c+\binom{k-2}{2}c^2+d+cd+\binom{k-3}{1}c^3+d^2+\binom{t}{2}d^2,\\ \qquad\qquad\qquad\qquad if\binom{n+4}{2}=0,\binom{s+t}{2}=1.\end{cases}$$

Now consider the following cases:

When $\binom{n+4}{2}=1$ (i.e. $n=4l+2$) and $\binom{s+t}{2}=0$, by the proof of Lemma 4.6 we have $E(n)\cap E(t)=\Phi$. Thus $1\notin E(t)$, i.e. $\binom{t}{2}=0$. Furthermore, since $\binom{s+t}{2}=0$, if follows that $s=0$. If $t>1$, take

$$\left(\sigma_1(x)+\binom{k}{1}\right)^3\left(\sigma_4(x)+\binom{2n+k}{4}\right)^{t-1},$$

it is easy to see that $t>1$ is impossible. Moreover, it needs to check the following case:

$$s=0,\quad t=1,\quad m=3,\quad n=4l+2,\quad k\geq 2.$$

By computations, we have

$$\sigma_{8l+6}(1+y,z)(v_1 \to RP(3)) = \binom{8l+4+k}{8l+6} + \binom{8l+3+k}{8l+5}a + \binom{8l+2+k}{8l+4}a^2 + \binom{8l+1+k}{8l+3}a^3$$

and

$$\begin{aligned}\sigma_{8l+6}(1+y,z)(v_2 \to P(3,4l+2)) &= d^{4l+2} + cd^{4l+2} + \left(\binom{k-1}{2} + \binom{k-2}{2}\right)c^2d^{4l+1} \\ &\quad + \text{terms of degree less than } 8l+4.\end{aligned}$$

If $k > 4$, take $\sigma_{8l+6}(x)\left(\sigma_4(x) + \binom{8l+4+k}{4} + \binom{k-1}{3}\left(\sigma_1(x) + \binom{k}{1}\right) + \binom{k-2}{2}\left(\sigma_1(x) + \binom{k}{1}\right)^2 + \binom{k-3}{1}\left(\sigma_1(x) + \binom{k}{1}\right)^3\right)\left(\sigma_1(x) + \binom{k}{1}\right)$ of degree $8l+11$ less than $m+2n+k$, by Theorem 3.1, then

$$\begin{aligned}0 &= \frac{0}{1+a}[RP(3)] \\ &\quad + \frac{c(1+c^2+d+cd)\left(d^{4l+2} + cd^{4l+2} + \left(\binom{k-1}{2} + \binom{k-2}{2}\right)c^2d^{4l+1}\right)}{1+c+d} \\ &\quad \times [P(3,4l+2)] \\ &\quad + \frac{c(1+c^2+d+cd)(\text{terms of degree less than } 8l+4)}{1+c+d} \\ &\quad \times [P(3,4l+2)] \\ &= \frac{c(1+c)(1+c+d)\left(d^{4l+2} + cd^{4l+2} + \left(\binom{k-1}{2} + \binom{k-2}{2}\right)c^2d^{4l+1}\right)}{1+c+d} \\ &\quad \times [P(3,4l+2)] \\ &\quad + \frac{c(1+c)(1+c+d)(\text{terms of degree less than } 8l+4)}{1+c+d}[P(3,4l+2)] \\ &= c(1+c)\left(d^{4l+2} + cd^{4l+2} + \left(\binom{k-1}{2} + \binom{k-2}{2}\right)c^2d^{4l+1}\right)[P(3,4l+2)] \\ &\quad + c(1+c)(\text{terms of degree less than } 8l+4)[P(3,4l+2)] \\ &= 1,\end{aligned}$$

but this is a contradiction.

When $\binom{n+4}{2}=0$ (i.e. $n=4l$) and $\binom{s+t}{2}=1$, take

$$\left(\sigma_1(x)+\binom{k}{1}\right)^3\left(\sigma_4(x)+\binom{k}{4}+1\right)^t,$$

it follows that $\binom{t}{2}=1$ is impossible. Let $\binom{t}{2}=0$. Since $\binom{s+t}{2}=1$, we have $s=2$. If $t>1$, choose

$$\left(\sigma_1(x)+\binom{k}{1}\right)^3\left(\sigma_4(x)+\binom{k}{4}\right)^{t-1},$$

then we conclude that this case is impossible. Furthermore, it needs merely to check the following case:

$$s=2,\quad t=1,\quad m=3,\quad n=4l,\quad k\geq 4.$$

If $k+1>4A$, choose $(\sigma_1(x)+\binom{k}{1}+1)^2(\sigma_4(x)+\binom{k}{4}+\binom{k-1}{3}(\sigma_1(x)+\binom{k}{1})+\binom{k-2}{2}(\sigma_1(x)+\binom{k}{1})^2+\binom{k-3}{1}(\sigma_1(x)+\binom{k}{1})^3)^{2l+A}$, then we have

$$\begin{aligned}0&=\frac{0}{1+a}[RP(3)]+\frac{(1+c)^2(d+cd+d^2)^{2l+A}}{(1+c)^2(1+c+d)}[P(3,4l)]\\&=0+d^{2l+A}(1+c+d)^{2l+A-1}[P(3,4l)]\\&=\binom{2l+A-1}{2A-1}\binom{2A-1}{3}.\end{aligned}$$

We know from the proof of Lemma 4.7 that

$$\binom{2l+A-1}{2A-1}\binom{2A-1}{3}=1.$$

Therefore, it follows that $k+1>4A$ is impossible.

Together with the above discussions, this completes the proof. □

Combining Lemmas 4.1, 4.2, 4.3, 4.4, 4.7, 4.9, we have

Proposition 4.10. *There doesn't exist the involution (T,M^{m+2n+k}) such that $v_1\to RP(3)$ doesn't bound except for the following four cases:*

(I) $\omega v_1=1+a$, $\omega v_2=(1+c)^2(1+c+d)$, $m=3$, $n=4l+2$, $4\leq k\leq 6$;

(II) $\omega v_1=1+a$, $\omega v_2=1+c+d$, $m=3$, $n=4l$, $2\leq k\leq 4A-3$;

(III) $\omega v_1=(1+a)^3$, $\omega v_2=1+c+d$, $m=3$, $n=4l+2$, $2\leq k\leq 4$;

(IV) $\omega v_1=(1+a)^3$, $\omega v_2=(1+c)^2(1+c+d)$, $m=3$, $n=4l$, $4\leq k\leq 4A-1$ *where* $A=\min\{\alpha|\alpha>0$ *and* $E(\alpha)\subset E(2l)\}$.

Remark 4.1. It should be point out that if n is even and $m=3$, even though $v_1\to RP(3)$ bounds, we may easily prove that (T,M^{m+2n+k}) doesn't exist.

§5. The cases in which involutions exist

In this section, we will prove that those involutions (T, M^{m+2n+k}) of the four cases stated in Proposition 4.10 must exist. Our main result is stated as follows:

THEOREM 5.1. *Suppose that (T, M^{m+2n+k}) is an involution on a closed $(m+2n+k)$-manifold with the fixed point set $F = RP(3) \sqcup P(m,n)$ $(m,n>0)$, and with nonbounding normal bundle to $RP(3)$. Then there only exist those involutions (T, M^{m+2n+k}) satisfying the following four cases respectively*:

(I) $\omega v_1 = 1+a$, $\omega v_2 = (1+c)^2(1+c+d)$, $m=3$, $n=4l+2$, $4 \le k \le 6$;
(II) $\omega v_1 = 1+a$, $\omega v_2 = 1+c+d$, $m=3$, $n=4l$, $2 \le k \le 4A-3$;
(III) $\omega v_1 = (1+a)^3$, $\omega v_2 = 1+c+d$, $m=3$, $n=4l+2$, $2 \le k \le 4$;
(IV) $\omega v_1 = (1+a)^3$, $\omega v_2 = (1+c)^2(1+c+d)$, $m=3$, $n=4l$, $4 \le k \le 4A-1$ *where* $A = \min\{\alpha | E(\alpha) \subset E(2l)$ *and* $\alpha > 0\}$.

For convenience, in the following proofs, let $\sigma_i(1+y,z) = \sigma_\imath, \sigma_\imath^{(1)} = \sigma_i(1+y,z)(v_1 \to RP(3))$, $\sigma_\imath^{(2)} = \sigma_i(1+y,z)(v_2 \to P(m,n))$, and $\sigma_\imath = 0$ means $\sigma_\imath^{(1)} = 0$, $\sigma_\imath^{(2)} = 0$.

As the sake of analogue of proof way, we will only prove the cases (I), (II) of Theorem 5.1. In fact, the proof method for the cases (I), (II) in Theorem 5.1 is not only the same as that for the cases (III), (IV), but also the involutions of the cases (I), (II) (i.e., case $\omega v_1 = 1+a$) possess completely the analogous structures as those of the cases (III), (IV), (i.e., cases $\omega v_1 = (1+a)^3$). This is just seen from the results of Theorem 5.1 and the proofs of Lemmas 4.5, 4.6, 4.7, 4.8, and 4.9.

Recall that ξ is a 1-plane bundle over $P(m,n)$ and η is a 2-plane bundle over $P(m,n)$ (see §2). Let $\imath$ be a canonical line bundle over $RP(3)$.

Proof of Theorem 5.1 (I). For $4 \le k \le 6$, since

$$\imath \oplus (8l+3+k)R \to RP(3) \sqcup 2\xi \oplus \eta \oplus (k-4)R \to P(3, 4l+2)$$

is bordant to $v_1^{8l+4+k} \to RP(3) \sqcup v_2^k \to P(3,4l+2)$ with $\omega v_1 = 1+a$ and $\omega v_2 = (1+c)^2(1+c+d)$, by Corollary 3.3 and Lemmas 3.4, 3.5 it suffices to show that the involution (T, M^{8l+12}) corresponding to the case $k=5$ of Theorem 5.1 (II) exists, and M^{8l+12} bounds. Moreover, by Theorem 3.1 and Proposition 3.2 it needs merely to prove that $v_1^{8l+9} \to RP(3) \sqcup v_2^5 \to P(3,4l+2)$ with $\omega v_1 = 1+a$ and $\omega v_2 = (1+c)^2(1+c+d)$ satisfies the following equation

$$\frac{f(1+y,z)}{1+a}[RP(3)] + \frac{f(1+y,z)}{(1+c)^2(1+c+d)}[P(3,4l+2)] = 0$$

for all symmetric polynomial $f(x)$ of degree less than or equal to $8l+12$. For this, we first compute $\sigma_\imath$ for $0 \le i \le 8l+12$. We proceed as follows.

Using [5; p. 317, Lemma], it follows that

$$\sigma_i^{(1)} = \binom{8l+9}{i} + \binom{8l+8}{i-1}a$$

and

$$\sigma_i^{(2)} = \sigma_i(z) + (1+c)\sigma_{i-1}(z) + (c^2+d)\sigma_{i-2}(z) + (c^2+d+c^3)\sigma_{i-3}(z) + (1+c^2+d+c^2d)\sigma_{i-4}(z) + (1+c)^2(1+c+d)\sigma_{i-5}(z).$$

From $\omega P(3,4l+2) = (1+c)^3(1+c+d)^{4l+3}$ we have that for $0 \le j \le l$,

$$\omega_{8j}P(3,4l+2) = \binom{l}{j}d^{4j} + \binom{l}{j-1}c^2d^{4j-1}, \quad \omega_{8j+1}P(3,4l+2) = \binom{l}{j-1}c^3d^{4j-1},$$

$$\omega_{8j+2}P(3,4l+2) = \binom{l}{j}(c^2+d)d^{4j}, \quad \omega_{8j+3}P(3,4l+2) = \binom{l}{j}cd^{4j+1},$$

$$\omega_{8j+4}P(3,4l+2) = \binom{l}{j}d^{4j+2}, \quad \omega_{8j+5}P(3,4l+2) = 0,$$

$$\omega_{8j+6}P(3,4l+2) = \binom{l}{j}d^{4j+3}, \quad \omega_{8j+7}P(3,4l+2) = \binom{l}{j}cd^{4j+3}.$$

Note that $\omega_{8l+6}P(3,4l+2) = 0$ and $\omega_{8l+7}P(3,4l+2) = 0$. Furthermore, by direct computations, we obtain the following table.

TABLE 1

	$\sigma_{8h+p}^{(1)}$	$\sigma_{8h+p}^{(2)}$
$p=0$	$\binom{l+1}{h}$	$\binom{l+1}{h}d^{4h} + \binom{l}{h-1}(1+c+d)(c+d+cd+c^3)d^{4h-3}$
$p=1$	$\binom{l+1}{h}(1+a)$	$\binom{l+1}{h}(1+c)d^{4h} + \binom{l}{h-1}(1+c)^2(1+c+d)d^{4h-2}$
$p=2$	0	$\binom{l}{h-1}(1+c)(1+c+d)d^{4h-1}$
$p=3$	0	$\binom{l}{h-1}(1+c)(1+c+d)d^{4h-1}$
$p=4$	0	$\binom{l}{h}(1+c)(1+c+d)d^{4h}$
$p=5$	0	$\binom{l}{h}(1+c)^2(1+c+d)d^{4h} + \binom{l}{h-1}(c^2+c^3)(1+c+d)d^{4h-1}$
$p=6$	0	$\binom{l}{h}(1+c+d)(d+cd+c^2+c^3)d^{4h} + \binom{l}{h-1}c^3(1+c+d)d^{4h-1}$
$p=7$	0	$\binom{l}{h}(1+c+d)(c^2+d+cd)d^{4h}$

Next, we look at the results of σ_i from the above table. For each σ_i with $i \not\equiv 0, 1 \pmod 8$, σ_i possesses the property that $\sigma_i^{(1)} = 0$ and $\sigma_i^{(2)}$ always contains a factor $(1+c+d)$. Thus for any a symmetric polynomial function f which can be expressed as a sum of those monomials $\sigma_{i_1}^{a_1}\sigma_{i_2}^{a_2}\cdots\sigma_{i_r}^{a_r}$ of degree less than or equal to $8l+12$, if each monomial $\sigma_{i_1}^{a_1}\sigma_{i_2}^{a_2}\cdots\sigma_{i_r}^{a_r}$ at least contains a elementary symmetric polynomial σ_i with $i \not\equiv 0,1 \pmod 8$ as its factors, then it is easy to see that

$$\frac{\sigma_{i_1}^{a_1}\sigma_{i_2}^{a_2}\cdots\sigma_{i_r}^{a_r}(1+y,z)}{(1+a)}[RP(3)] + \frac{\sigma_{i_1}^{a_1}\sigma_{i_2}^{a_2}\cdots\sigma_{i_r}^{a_r}(1+y,z)}{(1+c)^2(1+c+d)}[P(3,4l+2)] = 0$$

and thus

$$\frac{f(1+y,z)}{1+a}[RP(3)] + \frac{f(1+y,z)}{(1+c)^2(1+c+d)}[P(3,4l+2)] = 0.$$

For σ_i with $i \equiv 0,1 \pmod 8$, since $\sigma_1^{(1)} = 1+a$ and $\sigma_1^{(2)} = 1+c$, it follows that $\sigma_{8h+1}^{(1)} + \sigma_1^{(1)}\sigma_{8h}^{(1)} = 0$ and

$$\sigma_{8h+1}^{(2)} + \sigma_1^{(2)}\sigma_{8h}^{(2)} = \binom{l}{h-1}(1+c+d)(c+c^2+c^3)d^{4h-3}.$$

In particular, we see that $\sigma_{8h+1}^{(2)} + \sigma_1^{(2)}\sigma_{8h}^{(2)}$ contains the factor $(1+c+d)$. Therefore, in further discussions, it needs only to consider σ_1 and σ_{8h} with $E(h) \subset E(l+1)$. Consider any symmetric polynomial function f' which can be expressed as a sum of those monomials $\sigma_1^u\sigma_{8h_1}^{v_1}\sigma_{8h_2}^{v_2}\cdots\sigma_{8h_r}^{v_r}$ of degree less than or equal to $8l+12$ where $E(h_w) \subset E(l+1)$ for $w = 1,\dots,r$. For each monomial $\sigma_1^u\sigma_{8h_1}^{v_1}\sigma_{8h_2}^{v_2}\cdots\sigma_{8h_r}^{v_r}$, if $u \equiv 0 \pmod 4$ then

$$\begin{aligned}
&\frac{\sigma_1^u\sigma_{8h_1}^{v_1}\sigma_{8h_2}^{v_2}\cdots\sigma_{8h_r}^{v_r}(1+y,z)}{1+a}[RP(3)] + \frac{\sigma_1^u\sigma_{8h_1}^{v_1}\sigma_{8h_2}^{v_2}\cdots\sigma_{8h_r}^{v_r}(1+y,z)}{(1+c)^2(1+c+d)}[P(3,4l+2)] \\
&= 1 + \frac{d^{4\sum_{w=1}^r h_w}}{(1+c)^2(1+c+d)}[P(3,4l+2)] \\
&\quad + \frac{\text{lower degree's terms of containing the factor } (1+c+d)}{(1+c)^2(1+c+d)}[P(3,4l+2)] \\
&= 1 + \frac{d^{4\sum_{w=1}^r h_w}}{(1+c)^2(1+c+d)}[P(3,4l+2)] + 0 \\
&= 1 + \binom{2^{N+1}-1}{4l+2-4\sum_{w=1}^r h_w}\binom{2^{M+1}+2^{N+1}-4l+4\sum_{w=1}^r h_w - 5}{3} \\
&= 0
\end{aligned}$$

where $2^N > \max\{3, 4l+2\}$, $2^M > 3$. If $u \equiv 1,2,3 \pmod 4$, then

$$\frac{\sigma_1^u\sigma_{8h_1}^{v_1}\sigma_{8h_2}^{v_2}\cdots\sigma_{8h_r}^{v_r}(1+y,z)}{1+a}[RP(3)]+\frac{\sigma_1^u\sigma_{8h_1}^{v_1}\sigma_{8h_2}^{v_2}\cdots\sigma_{8h_r}^{v_r}(1+y,z)}{(1+c)^2(1+c+d)}[P(3,4l+2)]$$

$$=0+\frac{(1+c)^u d^{4\sum_{w=1}^r h_w}}{(1+c)^2(1+c+d)}[P(3,4l+2)]$$

$$+\frac{(1+c)^u(\text{lower degree's terms of containing the factor }(1+c+d))}{(1+c)^2(1+c+d)}$$

$$\times [P(3,4l+2)]$$

$$=\frac{(1+c)^u d^{4\sum_{w=1}^r h_w}}{(1+c)^2(1+c+d)}[P(3,4l+2)]+0$$

$$=\binom{2^{N+1}-1}{4l+2-4\sum_{w=1}^r h_w}\binom{2^{M+1}+2^{N+1}-4l+4\sum_{w=1}^r h_w-5+u}{3}$$

$$=0.\quad (\text{since } u\equiv 1,2,3 \pmod 4)$$

So we can still obtain that

$$\frac{f'(1+y,z)}{1+a}[RP(3)]+\frac{f'(1+y,z)}{(1+c)^2(1+c+d)}[P(3,4l+2)]=0.$$

Combining the above discussions, we complete the proof. □

Proof of Theorem 5.1 (II). First, it is easy to see that for $2\le k\le 4A-3$, the vector bundle $\iota\oplus(8l+k-1)R\to RP(3)\sqcup\eta\oplus(k-2)R\to P(3,4l)$ is bordant to $v_1^{8l+k}\to RP(3)\sqcup v_2^k\to P(3,4l)$ with $\omega v_1=1+a$ and $\omega v_2=1+c+d$ where R denotes the trivial line bundle. Hence by Corollary 3.3, we need merely to check the case $k=4A-3$ of Theorem 5.1 (II). Let $k=4A-3$. Now, by means of Proposition 3.2, we show that $v_1\to RP(3)\sqcup v_2\to P(3,4l)$ with $\omega v_1=1+a$ and $\omega v_2=1+c+d$ is the fixed point data of involution (T,M^{8l+4A}). We proceed as follows:

Step (i). The computation of σ_ι for $0\le i\le 8l+4A$. Using [5; p. 317, Lemma], we have

$$\sigma_\iota^{(1)}=\binom{8l+4A-3}{i}+\binom{8l+4A-4}{i-1}a$$

and

$$\sigma_\iota^{(2)}=\sum_{q\le\iota}\binom{4A-3}{i-q}\sigma_q(z)+c\sum_{1+q\le\iota}\binom{4A-4}{i-q-1}\sigma_q(z)+d\sum_{2+q\le\iota}\binom{4A-5}{i-q-2}\sigma_q(z).$$

From

$$\omega P(3,4l)=(1+c)^3(1+c+d)^{4l+1},$$

it follows that

$$\omega_{2j}P(3,4l) = d^j, \quad \omega_{2j+2}P(3,4l) = d^{j+1}, \quad \omega_{2j+3}P(3,4l) = cd^{j+1},$$
$$\omega_{2j+4}P(3,4l) = c^2d^{j+1}, \quad \omega_{2j+5}P(3,4l) = c^3d^{j+1}, \quad \omega_h P(3,4l) = 0$$

where $E(j) \subset E(4l)$, $h \neq 2j, 2j+2, 2j+3, 2j+4, 2j+5$.

Let $E(r) \subset E(4l)$. First, we compute σ_{2r+p} for $p \leq 4A - 1$. The results are stated as follows.

Table 2

	$\sigma^{(1)}_{2r+p}$	$\sigma^{(2)}_{2r+p}$
$p = 8H$	1	$d^r(1 + c(1+c)^3 d(1+c+d))$, if $H \neq 0$; d^r, if $H = 0$ and $E(2r-4A) \not\subset E(8l)$; $d^r + cd^{r-2A+1}(1+c)^3(1+c+d)$, if $H = 0$ and $E(2r-4A) \subset E(8l)$.
$p = 8H+1$	$1+a$	$d^r(1+c+c^2d+c^2d^2+c^3d^2)$, if $H \neq 0$; $d^r(1+c)$, if $H = 0$ and $E(2r-4A) \not\subset E(8l)$; $(1+c)d^r + c^2d^{2r-2A+1}(1+d+cd)$, if $H = 0$ and $E(2r-4A) \subset E(8l)$.
$p = 8H+2$	0	$d^r(c^3d + c^3d^2)$, if $H \neq 0$; 0, if $H = 0$ and $E(2r-4A) \not\subset E(8l)$; $c^3d^{r-2A+1} + c^3d^{r-2A+2}$, if $H = 0$ and $E(2r-4A) \subset E(8l)$.
$p = 8H+3$	0	0
$p = 8H+4$	1	$d^r(1+cd+d+d^2)$
$p = 8H+5$	$1+a$	$d^r(1+c+d+d^2+c^2d+cd^2)$
$p = 8H+6$	0	$d^r(d+c^3d+d^2+cd^2+c^2d^2)$
$p = 8H+7$	0	$d^r(d+d^2+cd^2+c^2d^2+c^3d^2)$

Next, we compute σ_{2r+p} for $p \geq 4A$. Let

$$u = \min\{w \mid E(2r + 4Aw) \not\subset E(8l) \text{ and } w \geq 1\}.$$

When $4A \leq p \leq 4Au - 1$ $(u > 1)$, since $E(2r + 4Aw) \subset E(8l)$ for each $w \leq u - 1$, thus the computation of σ_{2r+p} for $4Aw \leq p \leq 4A - 1$ can be completely iterated as the computation procedure of the case $0 \leq p \leq 4A - 1$, and results can be obtain as long as r in the results of Table 2 is replaced by $r + 2Aw$.

When $p \geq 4Au$, if $2r + 4Au = 8l + 4A$, we have $\sigma_{8l+4A} = 0$. If $2r + 4Au < 8l + 4A$, let $v = \min E(2r + 4Au)$, then we have

$$\sigma^{(1)}_{2r+4Au} = 0, \quad \sigma^{(2)}_{2r+4Au} = \begin{cases} d^r(cd + cd^2 + c^2d^2 + c^3d^2), & \text{if } u = 1; \\ 0, & \text{if } u > 1. \end{cases} \tag{5.1}$$

$$(5.2)\qquad \sigma^{(1)}_{2r+4Au+1}=0,\quad \sigma^{(2)}_{2r+4Au+1}=\begin{cases} d^r(c^2d+c^2d^2+c^3d^2), & \text{if } u=1;\\ 0, & \text{if } u>1.\end{cases}$$

$$(5.3)\qquad \sigma^{(1)}_{2r+4Au+2}=0,\quad \sigma^{(2)}_{2r+4Au+2}=\begin{cases} d^r(c^3d+c^3d^2), & \text{if } u=1;\\ 0, & \text{if } u>1.\end{cases}$$

$$(5.4)\qquad \sigma_{2r+4Au+I}=0,\quad \text{if } 3\le I\le 2^v-1.$$

In the next computations, when p exceeds $4Au+2^v-1$, since either $E(2r+4Au+2^v)\subset E(8l)$ or $E(2r+4Au+2^v)\not\subset E(8l)$ always holds, hence whichever of both happens, we always can iterate the above computation procedure. In particular, the results of all $\sigma_\iota\neq 0$ must belong to those forms showed in table 2, (5.1), (5.2), (5.3).

Step (ii). From the computation results of all σ_ι $(i=0,1,\dots,8l+4A)$, it is easy to see that only using $\sigma_0,\sigma_1,\sigma_4,\sigma_{4A}$, each σ_i $(i\neq 0,1,4,4A)$ having the property $\sigma_\iota\neq 0$ can be changed into σ'_ι of degree i such that $\sigma'_\iota=0$. This means that for any symmetric polynomial function $f(x)$ of degree less than or equal to $8l+4A$, f can be generated by $\sigma_0,\sigma_1,\sigma_4,\sigma_{4A}$. In other words, f can be expressed as the sum of those expressions $\sigma_0^{t_1}\sigma_1^{t_2}\sigma_4^{t_3}\sigma_{4A}^{t_4}$. Furthermore, we can easily obtain that for any symmetric polynomial function $f(x)$ of degree less than $8l+4A$,

$$\frac{f(1+y,z)}{1+a}[RP(3)]+\frac{f(1+y,z)}{1+c+d}[P(3,4l)]=0,$$

and thus $v_1\to RP(3)\sqcup v_2\to P(3,4l)$ with $\omega v_1=1+a$ and $\omega v_2=1+c+d$ is the fixed point data of (T,M^{8l+4A}) by Proposition 3.2. This completes the proof. □

§6. The representative of involution up to bordism

In this section, we discuss the representatives up to bordism of those involutions stated in Theorem 5.1.

Lemma 6.1. *There exists an involution G on $P(3,n+1)$ such that the fixed point set of $(G,P(3,n+1))$ is $RP(3)\sqcup P(3,n)$.*

Proof. First, setting an involution g on $S^3\times CP(n+1)$ by

$$g:((x_0,x_1,x_2,x_3),(z_0,z_1,\dots,z_n))\to((x_0,x_1,x_2,x_3),(-z_0,z_1,\dots,z_n)).$$

It is obvious that the fixed point set of $(g,S^3\times CP(n+1))$ is $S^3\times CP(0)\sqcup S^3\times CP(n)$. Next, we can at once obtain an involution G on $P(3,n+1)$ induced by $(g,S^3\times CP(n+1))$, and easily see that the fixed point set of $(G,P(3,n+1))$ is exactly $RP(3)\sqcup P(3,n)$. This completes the proof. □

By the Theorem 5.1, Remark 4.1 and Lemma 3.5, it immediately follows that

THEOREM 6.2. (i) *The involution* (T, M^{3+8l+k}) *stated in Theorem* 5.1 (II) *is bordant to* $(G_{k-2}, \Gamma^{k-2}(P(3, 4l+1)))$ *where* $2 \leq k \leq 4A - 3$.

(ii) *The involution* (T, M^{7+8l+k}) *stated in Theorem* 5.1 (III) *is bordant to* $(G_{k-2}, \Gamma^{k-2}(P(3, 4l+3)))$ *where* $2 \leq k \leq 4$.

As for the representatives up to bordism of those involutions stated in Theorem 5.1 (I), (IV), we have done many tries, but nothing conclusive.

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