

UNICITY THEOREMS FOR MEROMORPHIC FUNCTIONS

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Abstract

This paper studies the problem of uniqueness of meromorphic functions. In this paper we will improve a result given by K. Tohge.

§1. Introduction

By a “meromorphic function” we will mean a meromorphic function in the complex plane. It is assumed that the reader is familiar with the notations of the Nevanlinna theory that can be found, for instance, in [2] or [4]. Let f and g be two non-constant meromorphic functions and a be a value in the extended complex plane. We say that f and g share a value a IM (ignoring multiplicity), if f and g have the same a -points, and also they share the value a CM (counting multiplicity), if f and g have the same a -points with the same multiplicity. Let k be a positive integer or ∞ , we denote by $\bar{E}_k(a, f)$ the set of a -points of f with multiplicity $\leq k$ (ignoring multiplicity), by $N_k(r, 1/(f - a))$ the counting function of a -points of f with multiplicity $\leq k$ and by $N_2(r, 1/(f - a))$ the counting function of a -points of f with multiplicity ≥ 2 (See [4]). Finally we say a is a Picard exceptional value of f , if $f(z) \neq a$.

In [3] K. Tohge proved the following:

THEOREM 1. *Let f and g be non-constant meromorphic functions that share three values $0, 1, \infty$ CM and f', g' share 0 CM. Then f and g satisfy one of the following:*

$$\begin{aligned}
 & \text{(i)} \quad f \equiv g, \\
 & \text{(ii)} \quad fg \equiv 1, \\
 & \text{(iii)} \quad (f - 1)(g - 1) \equiv 1, \\
 & \text{(iv)} \quad f + g \equiv 1,
 \end{aligned}
 \tag{1.1}$$

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- (v) $f \equiv cg$,
- (vi) $f - 1 \equiv c(g - 1)$,
- (vii) $[(c - 1)f + 1][(c - 1)g - c] \equiv -c$,

where $c (\neq 0, 1)$ is a constant.

In this paper, we prove the following theorem which is an improvement of Theorem 1.

THEOREM 2. *Let f and g be non-constant meromorphic functions that share two values $0, \infty$ CM and f', g' share the value 0 IM. If $\bar{E}_k(1, f) = \bar{E}_k(1, g)$, where k is a positive integer or ∞ , then f and g satisfy one of the identities in (1.1).*

§2. Some lemmas

LEMMA 1 (See [4]). *Let f and g be distinct non-constant entire functions and a_1, a_2 are distinct finite complex numbers. If a_1 is a Picard exceptional value of f, g and f, g share the value a_2 CM, then $f = e^\alpha + a_1$ and $g = (a_1 - a_2)^2 e^{-\alpha} + a_1$, where α is a non-constant entire function.*

LEMMA 2 (See [4]). *Let f and g be distinct non-constant meromorphic functions and a_1, a_2 be distinct finite complex numbers. If a_1, a_2 are Picard exceptional values of f, g and f, g share the value ∞ CM, then*

$$f = \frac{a_1 e^\alpha - a_2}{e^\alpha - 1}, \quad g = \frac{a_1 e^{-\alpha} - a_2}{e^{-\alpha} - 1},$$

where α is a non-constant entire function.

LEMMA 3 (See [1]). *Let f and g be non-constant meromorphic functions that share three values $0, 1, \infty$ CM. If f is a Möbius transformation of g , then f and g satisfy one of the identities in (1.1).*

Proof. Suppose $f \neq g$. Since f is a Möbius transformation of g ,

$$(2.1) \quad g = \frac{af + b}{cf + d},$$

where a, b, c, d are finite complex numbers and $ad - bc \neq 0$. There are three cases.

CASE I. If ∞ is a Picard exceptional value of f , then there are four subcases.

1. If 1 and 0 are Picard exceptional values of f , then this case is impossible due to the second fundamental theorem.

2. If 1 and 0 are not Picard exceptional values of f , then from (2.1) we get $b = 0$, $c + d = a$ and hence (2.1) becomes

$$(2.2) \quad c f g + d g = (c + d) f.$$

Since $b = 0$ and from $ad - bc \neq 0$ we find $ad \neq 0$ and hence (2.2) becomes

$$(2.3) \quad (A - 1) f g - A f + g \equiv 0,$$

where $A = a/d \neq 0$. If $A = 1$, then from (2.3) we find $f \equiv g$. This is a contradiction. Therefore $A \neq 1$. Thus from (2.3) we find $[(A - 1)f + 1] \cdot [(A - 1)g - A] \equiv -A$, which is (vii).

3. If 0 is a Picard exceptional value of f and 1 is not a Picard exceptional value of f , then by Lemma 1 we find $f = e^\alpha$ and $g = e^{-\alpha}$. From this we find $f g \equiv 1$, which is (ii).

4. If 1 is a Picard exceptional value of f and 0 is not a Picard exceptional value of f , then by Lemma 1 we find $f = e^\alpha + 1$ and $g = e^{-\alpha} + 1$. From this we find $(f - 1)(g - 1) \equiv 1$, which is (iii).

CASE II. If 0 is a Picard exceptional value of f , then there are two subcases.

1. If ∞ and 1 are not Picard exceptional values of f , then from (2.1) we find $c = 0$ and $a + b = d$. Again by (2.1) we get $f - 1 = A(g - 1)$ where $A (\neq 0, 1)$ is a constant, which is (vi).

2. If 1 is a Picard exceptional value of f and ∞ is not a Picard exceptional value of f , then by Lemma 2 we find $f = -1/(e^\alpha - 1)$ and $g = -1/(e^{-\alpha} - 1)$. From this we find $f + g \equiv 1$, which is (iv).

CASE III. If 1 is a Picard exceptional value of f , then there is only one subcase: If $0, \infty$ are not Picard exceptional values of f , then by (2.1) we find $b = c = 0$ and hence (2.1) becomes $f \equiv A g$, which is (v). \square

§3. Proof of Theorem 2

From the conditions of Theorem 2 we find

$$(3.1) \quad f = e^\alpha g,$$

where α is an entire function. If $e^\alpha \equiv c$, where c is a nonzero constant, then from this and (3.1) we deduce (v). We now suppose e^α is non-constant and hence $\alpha' \neq 0$. Again from (3.1) we have

$$(3.2) \quad f' = g' e^\alpha + g \alpha' e^\alpha.$$

Let z_0 be a zero for f' of order $p \geq 1$, then the Taylor expansion of f' about z_0 is

$$(3.3) \quad f'(z) = a_p(z - z_0)^p + \cdots, \quad a_p \neq 0.$$

Since f' and g' share the value 0 IM, therefore

$$(3.4) \quad g'(z) = b_q(z - z_0)^q + \cdots, \quad b_q \neq 0.$$

Without loss of generality, we can assume that $p \leq q$. From (3.2), (3.3) and (3.4) we find

$$g(z)\alpha'(z) = (z - z_0)^p[a_p e^{-\alpha} - b_q(z - z_0)^{q-p} + O(z - z_0)].$$

From this we find either $\alpha'(z_0) = 0$ or $g(z_0) = 0$. If $g(z_0) = 0$ then $p = q$. Thus we find

$$(3.5) \quad \bar{N}\left(r, \frac{1}{f'}\right) - N'_E(r, 0) \leq N\left(r, \frac{1}{\alpha'}\right) \leq T(r, \alpha') + O(1) \leq S(r, f) + S(r, g),$$

where $N'_E(r, 0)$ denotes the counting function of zeros of f' and g' with same multiplicity, each zero being counted only once. Similarly with respect to g' we find

$$(3.6) \quad \bar{N}\left(r, \frac{1}{g'}\right) - N'_E(r, 0) = S(r, f) + S(r, g).$$

Let z_1 be a zero for $f - 1$ of order $p \geq 2$. Then z_1 is also a zero for f' and hence for g' . From (3.2) we find $\alpha'(z_1) = 0$. From this we find

$$(3.7) \quad \bar{N}_2\left(r, \frac{1}{f-1}\right) \leq N\left(r, \frac{1}{\alpha'}\right) \leq T(r, \alpha') + O(1) \leq S(r, f) + S(r, g).$$

Similarly with respect to g we find

$$(3.8) \quad \bar{N}_2\left(r, \frac{1}{g-1}\right) = S(r, f) + S(r, g).$$

We denote by $N_1(r, 1)$ the counting function of common simple 1-points of f and g . Noting $\bar{E}_k(1, f) = \bar{E}_k(1, g)$, from (3.7) and (3.8), we have

$$(3.9) \quad \bar{N}\left(r, \frac{1}{f-1}\right) - N_1(r, 1) = S(r, f) + S(r, g),$$

and

$$(3.10) \quad \bar{N}\left(r, \frac{1}{g-1}\right) - N_1(r, 1) = S(r, f) + S(r, g).$$

Set

$$(3.11) \quad \Delta_1 = \frac{f''}{f'} - \frac{g''}{g'}.$$

From the fundamental estimate of logarithmic derivative it follows that

$$(3.12) \quad m(r, \Delta_1) = S(r, f) + S(r, g).$$

From (3.11) we find

$$(3.13) \quad N(r, \Delta_1) \leq \bar{N}\left(r, \frac{1}{f'}\right) - N'_E(r, 0) + \bar{N}\left(r, \frac{1}{g'}\right) - N'_E(r, 0).$$

From (3.5), (3.6), (3.12) and (3.13) we find

$$(3.14) \quad T(r, \Delta_1) = S(r, f) + S(r, g).$$

Let z_∞ be a simple pole of f , then from (3.11) we find

$$(3.15) \quad \Delta_1(z_\infty) = 0.$$

If $\Delta_1 \equiv 0$, then from (3.11) we find

$$(3.16) \quad f = ag + b,$$

where $a (\neq 0)$, b are constants. If $b = 0$, then from (3.16) we deduce (v). We now suppose $b \neq 0$. Since $\bar{E}_k(1, f) = \bar{E}_k(1, g)$, therefore, if $\bar{N}_k(r, 1/(f-1)) \not\equiv 0$, then from (3.16) we find $a + b = 1$. From this and (3.16) we deduce (vi). We now suppose

$$(3.17) \quad \bar{N}_k\left(r, \frac{1}{f-1}\right) = \bar{N}_k\left(r, \frac{1}{g-1}\right) \equiv 0.$$

From (3.1) and (3.16) we find

$$(3.18) \quad f - 1 = \frac{(b-1)e^\alpha + a}{e^\alpha - a} \quad \text{and} \quad g - 1 = \frac{a + b - e^\alpha}{e^\alpha - a}.$$

From this and (3.17) we find $b = 1$ and $a = -1$. From this and (3.18) we deduce (iv). We now suppose $\Delta_1 \not\equiv 0$. From (3.14) and (3.15) we find

$$(3.19) \quad N_{11}(r, f) \leq N\left(r, \frac{1}{\Delta_1}\right) \leq T(r, \Delta_1) + O(1) \leq S(r, f) + S(r, g).$$

Set

$$(3.20) \quad \Delta_2 = \frac{f'}{f-1} - \frac{f'}{f} - \frac{g'}{g-1} + \frac{g'}{g}.$$

From the fundamental estimate of logarithmic derivative it follows that

$$(3.21) \quad m(r, \Delta_2) = S(r, f) + S(r, g).$$

From (3.9), (3.10), (3.20) and (3.21) we find

$$(3.22) \quad T(r, \Delta_2) = S(r, f) + S(r, g).$$

Let z'_∞ be a pole for f of order $p \geq 2$. Then from (3.20) we find

$$(3.23) \quad \Delta_2(z'_\infty) = 0.$$

If $\Delta_2 \equiv 0$, then from (3.20) we find

$$(3.24) \quad \frac{f-1}{f} = c \frac{g-1}{g},$$

where c is a nonzero constant. From this it is easy to see f and g share the value 1 CM. And hence from (3.24) and Lemma 3 we find f and g satisfy one of the identities in (1.1). We now suppose $\Delta_2 \neq 0$. From (3.22) and (3.23) we find

$$(3.25) \quad N_{(2)}(r, f) \leq 2N\left(r, \frac{1}{\Delta_2}\right) \leq 2T(r, \Delta_2) + O(1) \leq S(r, f) + S(r, g).$$

From (3.19) and (3.25) we find

$$(3.26) \quad N(r, f) = S(r, f) + S(r, g).$$

Set

$$(3.27) \quad \Delta_3 = \frac{f''}{f'} - 2\frac{f'}{f} - \frac{g''}{g'} + 2\frac{g'}{g}.$$

Similar to the above, from (3.5), (3.6) and (3.27) it is easy to see that

$$(3.28) \quad T(r, \Delta_3) = S(r, f) + S(r, g).$$

Let z_0 be a simple zero of f . Then from (3.27) we find

$$(3.29) \quad \Delta_3(z_0) = 0.$$

If $\Delta_3 \equiv 0$, then from (3.27) we easily arrive at that f and g satisfy one of identities in (1.1). We now suppose $\Delta_3 \neq 0$. Then from (3.28) and (3.29) we find

$$(3.30) \quad N_{(1)}\left(r, \frac{1}{f}\right) = S(r, f) + S(r, g).$$

Set

$$(3.31) \quad \Delta_4 = \frac{f'}{f-1} - \frac{g'}{g-1}.$$

Again by a similar way as the above, we find from (3.9), (3.10) and (3.31) that

$$(3.32) \quad T(r, \Delta_4) = S(r, f) + S(r, g).$$

Let z'_0 be a zero for f of order $p \geq 2$. Then from (3.31) we find

$$(3.33) \quad \Delta_4(z'_0) = 0.$$

If $\Delta_4 \equiv 0$, then from (3.31) we easily arrive at that f and g satisfy one of the identities in (1.1). We now suppose $\Delta_4 \neq 0$. Then from (3.32) and (3.33) we find

$$(3.34) \quad N_{(2)}\left(r, \frac{1}{f}\right) = S(r, f) + S(r, g).$$

And hence from (3.30) and (3.34) we find

$$(3.35) \quad N\left(r, \frac{1}{f}\right) = S(r, f) + S(r, g).$$

Set

$$(3.36) \quad \Delta_5 = \frac{f''}{f'} - 2 \frac{f'}{f-1} - \frac{g''}{g'} + 2 \frac{g'}{g-1}.$$

Similar to the above we find from (3.5), (3.6), (3.9), (3.10) and (3.26) that

$$(3.37) \quad T(r, \Delta_5) = S(r, f) + S(r, g).$$

Let z_1 be a common simple 1-point of f and g . Then from (3.36) we find

$$(3.38) \quad \Delta_5(z_1) = 0.$$

If $\Delta_5 \equiv 0$, then from (3.36) we easily arrive at that f and g satisfy one of the identities in (1.1). We now suppose $\Delta_5 \not\equiv 0$. Then from (3.37) and (3.38), we find

$$\begin{aligned} N_{11}(r, 1) &\leq N\left(r, \frac{1}{\Delta_5}\right) \leq T(r, \Delta_5) + O(1) \\ &\leq S(r, f) + S(r, g). \end{aligned}$$

From this and (3.9), we have

$$(3.39) \quad \bar{N}\left(r, \frac{1}{f-1}\right) = S(r, f) + S(r, g).$$

Similarly, we get

$$(3.40) \quad \bar{N}\left(r, \frac{1}{g-1}\right) = S(r, f) + S(r, g).$$

Thus from (3.26), (3.35), (3.39), (3.40) and the second fundamental theorem for f and g we find

$$\begin{aligned} T(r, f) + T(r, g) &\leq 2\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}\left(r, \frac{1}{g-1}\right) \\ &\quad + 2\bar{N}(r, f) + S(r, f) + S(r, g) \\ &\leq S(r, f) + S(r, g), \end{aligned}$$

this is impossible. And so the proof of Theorem 2 is finished. \square

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