

## SURFACES WITH 1-TYPE GAUSS MAP

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### 0. Introduction

Submanifolds of finite type were introduced by B.-Y. Chen about thirteen years ago [2]. Many works have been done in characterizing or classifying submanifolds in Euclidean space with this notion. On the other hand, several authors studied submanifolds with finite type Gauss map. B.-Y. Chen and P. Piccinni studied compact submanifolds with finite type Gauss map [3]. And C. Baikoussis, B.-Y. Chen and L. Verstraelen classified ruled surfaces and tubes with finite-type Gauss map [1]. Recently Y. H. Kim studied surfaces in 3-dimensional Euclidean space  $E^3$  with 1-type Gauss map and he proved that the only co-closed surfaces in  $E^3$  with 1-type Gauss map are spheres and circular cylinders [6]. In this paper we study surfaces in  $E^3$  with 1-type Gauss map without the assumption of co-closedness and obtain the following theorem.

**THEOREM.** *Let  $M$  be an orientable, connected surface in  $E^3$ . Then  $M$  has 1-type Gauss map if and only if  $M$  is an open part of a sphere or an open part of a circular cylinder.*

### 1. Preliminaries

Let  $M$  be an orientable, connected surface in  $E^3$ . We now choose  $e_1$  and  $e_2$  as principal normal vectors of  $M$  and let  $x$  and  $y$  the corresponding principal curvatures of the shape operator  $S$  associated with a unit normal vector  $e_3$ . Let  $\omega^1, \omega^2, \omega^3$  be the dual 1-forms to  $e_1, e_2$  and  $e_3$  and  $\omega_A^B$  the connection forms associated with  $\omega^1, \omega^2, \omega^3$  satisfying  $\omega_A^B + \omega_B^A = 0$  and

$$\nabla_{e_i} e_j = \sum_k \omega_j^k(e_i) e_k + h(e_i, e_j) e_3, \quad \nabla_{e_i} e_j = \sum_k \omega_j^k(e_i) e_k,$$

$$\nabla_{e_i} e_3 = \sum_k \omega_3^k(e_i) e_k = -S e_i,$$

$$x = \omega_1^3(e_1) = h(e_1, e_1), \quad y = \omega_2^3(e_2) = h(e_2, e_2), \quad h(e_1, e_2) = h(e_2, e_1) = 0,$$

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where  $\bar{\nabla}$  and  $\nabla$  are the Levi-Civita connections of  $E^3$  and  $M$  respectively and  $h$  the second fundamental form of  $M$ . The indices  $A, B$  run over the range  $\{1, 2, 3\}$  and  $i, j, k$  over  $\{1, 2\}$ . The covariant derivative of the second fundamental form  $h$  of  $M$  is given by

$$(\nabla_{e_k} h)(e_i, e_j) = e_k h(e_i, e_j) - h(\nabla_{e_k} e_i, e_j) - h(e_i, \nabla_{e_k} e_j).$$

We will use abbreviations  $h_{ij}, h_{ij,k}$  for  $h(e_i, e_j)$  and  $(\nabla_{e_k} h)(e_i, e_j)$  respectively. The Codazzi's equation  $h_{ij,k} = h_{ik,j}$  implies that

$$(1.1) \quad h_{11,1} = e_1 x, \quad h_{22,2} = e_2 y,$$

$$(1.2) \quad h_{12,1} = h_{21,1} = h_{11,2} = e_2 x = (y-x)\omega_2^1(e_1),$$

$$(1.3) \quad h_{12,2} = h_{21,2} = h_{22,1} = e_1 y = (x-y)\omega_1^2(e_2).$$

We now give the definition of co-closed surface introduced by Y.H. Kim for later use.

**DEFINITION [6].** *A surface of Euclidean 3-space is called co-closed if the connection form  $\omega_1^2$  is co-closed, that is, trace  $(\nabla \omega_1^2) = 0$ .*

For a smooth function  $f$  on  $M$ ,  $\nabla f$ , the gradient  $f$  and  $\Delta f$ , the Laplacian of  $f$  are given by

$$\begin{aligned} \nabla f &= \sum_i (e_i f) e_i, \\ \Delta f &= \sum_i \{e_i e_i f - (\nabla_{e_i} e_i f)\}. \end{aligned}$$

The Laplacian  $\Delta$  can be extended in a natural way to  $E^3$ -valued smooth maps on  $M$ . In fact, if  $\nu$  is an  $E^3$ -valued smooth map on  $M$ . Then

$$(1.4) \quad \Delta \nu = \sum_i \{\bar{\nabla}_{e_i} \bar{\nabla}_{e_i} \nu - (\bar{\nabla}_{\nabla_{e_i} e_i} \nu)\}.$$

Applying (1.4) to the unit normal vector  $e_3$ , we find

$$(1.5) \quad \Delta e_3 = -\nabla H - \text{tr } S^2 e_3,$$

where  $H$  and  $\text{tr } S^2$  denote the mean curvature function of  $M$  in  $E^3$  and the square length of the second fundamental form  $h$  respectively. A smooth map  $\nu$  is said to be of  $k$ -type if  $\nu$  can be written as

$$\nu = \nu_0 + \nu_1 + \cdots + \nu_k,$$

where  $\nu_0$  is a constant vector,  $\nu_1, \nu_2, \dots, \nu_k$  are non-constant maps satisfying  $\Delta \nu_i = \lambda_i \nu_i$ ,  $i=1, 2, \dots, k$  and all eigen values  $\{\lambda_1, \dots, \lambda_k\}$  are mutually different. Suppose that the Gauss map  $e_3: M \rightarrow S_0^2(1) \subseteq E^3$  of  $M$  is of 1-type. Then there exist a constant  $a$  and a constant vector  $c$  such that

$$(1.6) \quad \Delta e_3 = a(e_3 - c).$$

Then, (1.5) and (1.6) imply that

$$(1.7) \quad a(e_3 - c) = -\nabla H - \text{tr } S^2 e_3.$$

So we have

$$(1.8) \quad \langle ac, ac \rangle = \langle \nabla H, \nabla H \rangle + (\text{tr } S^2 + a)^2,$$

where  $\langle, \rangle$  means the Euclidean metric of  $E^3$ . Comparing the tangential and normal components in (1.7), we obtain the followings

$$(1.9) \quad \nabla H = ac^T,$$

$$(1.10) \quad \text{tr } S^2 = a \langle c, e_3 \rangle - a,$$

where  $()^T$  means the projection to the tangent space of  $M$ . By (1.9) and (1.10), differentiating the mean curvature  $H$  in the principal normal vectors  $e_1, e_2$  on  $M$ , we obtain

$$(1.11) \quad \begin{aligned} e_j H &= a \langle e_j, c \rangle, \\ e_i e_j H - (\nabla_{e_i} e_j) H &= h_{ij} (\text{tr } S^2 + a). \end{aligned}$$

So we have

$$(1.12) \quad e_1 e_2 (x + y) + e_1 (x + y) \omega_1^2(e_1) = 0,$$

$$(1.13) \quad e_2 e_1 (x + y) + e_2 (x + y) \omega_2^1(e_2) = 0,$$

$$(1.14) \quad e_1 e_1 (x + y) + e_2 (x + y) \omega_2^1(e_1) - x(x^2 + y^2 + a) = 0,$$

$$(1.15) \quad e_2 e_2 (x + y) + e_1 (x + y) \omega_1^2(e_2) - y(x^2 + y^2 + a) = 0,$$

since  $H = x + y$  and  $\text{tr } S^2 = x^2 + y^2$ . From (1.10), we get

$$\nabla \text{tr } S^2 = -S(\nabla H).$$

This imply

$$(1.16) \quad 3x e_1 x + (x + 2y) e_1 y = 0,$$

$$(1.17) \quad (2x + y) e_2 x + 3y e_2 y = 0.$$

From (1.5) and (1.10), we find

$$\Delta \text{tr } S^2 = -\langle \nabla H, \nabla H \rangle - \text{tr } S^2 (\text{tr } S^2 + a).$$

This and (1.18) imply

$$(1.18) \quad \Delta \text{tr } S^2 = a(\text{tr } S^2 + a) - \alpha,$$

where  $\alpha = \langle ac, ac \rangle$ . We need to mention a well known identity.

LEMMA 1 (Simons' identity) [5]. *Let  $M$  is a surface in  $E^3$  with the induced metric. Let  $H$ ,  $S$  and  $h$  be the mean curvature function of  $M$  in  $E^3$ , the shape operator of  $M$  and the second fundamental form of  $M$ , respectively. Then, for given orthonormal frame  $e_1, e_2$ , the value  $\Delta \operatorname{tr} S^2$  is calculated as follows.*

$$(1.19) \quad \Delta \operatorname{tr} S^2 = 2 \sum_{i,j} h_{ij}(e_i e_j H - (\nabla_{e_i} e_j H)) + 2|\nabla S|^2 + 2H \operatorname{tr} S^3 - 2(\operatorname{tr} S^2)^2,$$

where  $|\nabla S|^2$  means  $\sum_{i,j,k} (h_{ij,k})^2$ .

From (1.11), (1.18) and (1.19), we get

$$(1.20) \quad 2|\nabla S|^2 = -a \operatorname{tr} S^2 - 2H \operatorname{tr} S^3 + a^2 - \alpha.$$

## 2. Proof of Theorem

At first we will prove that the mean curvature function  $H$  of  $M$  is constant. We need the following lemmas.

LEMMA 2. *If  $(e_1 x, e_1 y) = (0, 0)$  or  $(e_2 x, e_2 y) = (0, 0)$  in an open subset  $U$  of  $M$ , then  $H$  is constant in  $U$ .*

*Proof.* Suppose that  $(e_1 x, e_1 y) = (0, 0)$  in  $U$ . Then from (1.13), we find  $e_2 H \omega_1^2(e_2) = 0$ . So we may assume that  $\omega_1^2(e_2) = 0$ . Differentiating (1.14) in the direction  $e_1$  and using (1.12), we obtain  $e_1(\omega_1^2(e_1)) = 0$ . So we have that

$$\operatorname{tr}(\nabla \omega_1^2) = e_1(\omega_1^2(e_1)) - \omega_1^2(e_2) \omega_1^2(e_1) + e_2(\omega_1^2(e_2)) - \omega_1^2(e_1) \omega_1^2(e_2) = 0.$$

This imply that  $U$  is a co-closed surface with 1-type Gauss map. Hence, due to the result of Kim [6], we see that  $H$  is constant in  $U$ . In case that  $(e_2 x, e_2 y) = (0, 0)$  we can get the same conclusion by similar computation.

LEMMA 3. *Let  $f(u, v)$  be a nonconstant real polynomial in two variables  $u, v$ . If the principal curvatures  $x, y$  satisfy  $f(u, v)$ , that is  $f(x, y) = 0$ , in a open subset  $U$  of  $M$ , then  $H$  is constant in  $U$ .*

*Proof.* Suppose that  $H$  is nonconstant in  $U$ . Then

$$V = \{p \in U \mid \nabla H(p) \neq 0\}$$

is a nonempty open subset. Since the real polynomial ring  $R[u, v]$  is a UFD, the polynomial  $f(u, v)$  can be factored as  $f = f_1 f_2 \cdots f_k$ , where  $f_i$  are irreducible polynomials in  $R[u, v]$ . From the condition  $f(x, y) = 0$  on  $U$ , we can guarantee the existence of a nonempty open subset  $W$  of  $U$ , where  $x, y$  satisfy a non-constant irreducible polynomial  $f_i$ . We may assume  $f_i = f_1$  without loss of generality. Differentiating  $f_1(x, y) = 0$  in the direction  $e_i$ , we have

$$(2.1) \quad (f_1)_u(x, y) e_i x + (f_1)_v(x, y) e_i y = 0$$

where  $(f_1)_u$  and  $(f_1)_v$  mean partial derivatives of  $f_1$  with respect to  $u$  and  $v$ . If  $(e_1x, e_1y)=(0, 0)$  or  $(e_2x, e_2y)=(0, 0)$  holds in  $W$ , then  $H$  is constant in  $W$  by Lemma 2. So we may assume that  $(e_1x, e_1y) \neq (0, 0)$  and  $(e_2x, e_2y) \neq (0, 0)$  in  $W$ . From (1.16), (1.17) and (2.1) we get

$$\begin{aligned} 3x(f_1)_v(x, y) - (x+2y)(f_1)_u(x, y) &= 0, \\ (2x+y)(f_1)_v(x, y) - 3y(f_1)_u(x, y) &= 0. \end{aligned}$$

If  $(3x)(-3y) + (x+2y)(2x+y) = 2(x-y)^2 = 0$  in  $W$ , then  $W$  is totally umbilical and hence  $H$  is constant in  $W$ , which contradicts to the assumption. So we get  $(f_1)_u(x, y) = (f_1)_v(x, y) = 0$  in  $W$ . Since  $f_1(u, v)$  is a nonconstant polynomial, both of  $(f_1)_u$  and  $(f_1)_v$  are not zero polynomials. Assume that  $(f_1)_u$  is not a zero polynomial. Since  $f_1$  and  $(f_1)_u$  are relatively prime, the system

$$\begin{aligned} f_1(u, v) &= 0 \\ (f_1)_u(u, v) &= 0 \end{aligned}$$

has only finitely many zeros [4, page 18]. But  $x, y$  satisfy this system in  $W$ . Hence  $x$  and  $y$  must be constant, which contradicts to the assumption. So we can conclude that  $H$  is constant in  $U$ .

Suppose that the mean curvature function  $H$  of  $M$  is nonconstant. Then there exists an open subset  $U$  of  $M$  where  $\nabla H$  never vanishes. By Lemma 3, we also assume that  $y \neq 0$ ,  $x+2y \neq 0$ ,  $x-y \neq 0$  and  $x+y \neq 0$  in  $U$ . We will work in  $U$ . By (1.1), (1.2) and (1.3) we see that  $|\nabla S|^2 = (e_1x)^2 + 3(e_2x)^2 + 3(e_1y)^2 + (e_2y)^2$ . So from (1.20) we have

$$2\{(e_1x)^2 + 3(e_2x)^2 + 3(e_1y)^2 + (e_2y)^2\} = -a(x^2 + y^2) - 2(x+y)(x^3 + y^3) + a^2 - \alpha.$$

From (1.16), (1.17) and this we find

$$\begin{aligned} & 2\left\{(e_1x)^2 + 3(e_2x)^2 + 3\left(\frac{3x}{x+2y}\right)^2(e_1x)^2 + \left(\frac{2x+y}{3y}\right)^2(e_2x)^2\right\} \\ &= -a(x^2 + y^2) - 2(x+y)(x^3 + y^3) + a^2 - \alpha. \end{aligned}$$

After some calculation we get

$$\begin{aligned} (2.2) \quad & 8(3y^2)(7x^2 + xy + y^2)(e_1x)^2 + 8(x+2y)^2(7y^2 + xy + x^2)(e_2x)^2 \\ &= \{-a(x^2 + y^2) - 2(x+y)(x^3 + y^3) + a^2 - \alpha\}(x+2y)^2(3y)^2. \end{aligned}$$

From (1.8) we obtain

$$\alpha - (\text{tr } S^2 + a)^2 = (e_1x)^2 + (e_2x)^2 + 2(e_1x)(e_1y) + 2(e_2x)(e_2y) + (e_1y)^2 + (e_2y)^2.$$

Using (1.16) and (1.17) and after some computations, we get

$$(2.3) \quad \begin{aligned} & 4(3y)^2(x-y)^2(e_1x)^2 + 4(x+2y)^2(x-y)^2(e_2x)^2 \\ &= \{\alpha - (x^2 + y^2 + a)^2\}(x+2y)^2(3y)^2. \end{aligned}$$

From (2.2) and (2.3) we obtain

$$(2.4) \quad (e_1x)^2 = \frac{1}{48(x^2 - y^2)(x-y)^2} F(x, y),$$

where

$$\begin{aligned} F(x, y) &= (x+2y)^2 [(x-y)^2 \{a(a-x^2-y^2) - 2(x+y)(x^2+y^2) - \alpha\} \\ &\quad - 2(7y^2 + xy + x^2) \{\alpha - (x^2 + y^2 + a)^2\}] \\ &= (4y)x^7 + \sum_{j < 7} f_j(y)x^j, \end{aligned}$$

where  $f_j(y)$  are real polynomials in one variable  $y$ . From (2.4) we see that

$$e_1x = \pm \sqrt{\frac{1}{48(x^2 - y^2)(x-y)^2} F(x, y)}.$$

We will denote  $e_1x$  by  $G(x, y)$ . Substituting

$$\omega_1^2(e_2) = -\frac{e_1y}{x-y}, \quad e_1y = -\frac{3x}{x+2y}e_1x \quad \text{and} \quad e_2y = -\frac{2x+y}{3y}e_2x$$

into (1.13), and after some computations we have

$$\left[ \frac{3y^2 + 3xy + 3x^2}{x+2y} G + \frac{(x-y)\{3yG_x - (2x+y)G_y\}}{3} \right] e_2x = 0,$$

where  $G_x$  and  $G_y$  are partial derivatives of  $G(x, y)$  with respect to  $x$  and  $y$ . So the following holds

$$[9(y^2 + xy + x^2)G + (x+2y)(x-y)\{3yG_x - (2x+y)G_y\}]e_2x = 0.$$

Suppose  $e_2x = 0$  locally. Then it follows that  $e_2y = 0$  from (1.17). This is a contradiction to the assumption by Lemma 2. Thus the following holds in  $U$ ,

$$(2.5) \quad 9(y^2 + xy + x^2)G + (x+2y)(x-y)\{3yG_x - (2x+y)G_y\} = 0.$$

From (2.4), we have

$$(2.6) \quad 48(x^2 - y^2)(x-y)^2G^2 = 4yx^7 + \sum_{j < 7} f_j(y)x^j.$$

Differentiating this with respect to  $x$ , we get

$$\begin{aligned} & 96x(x-y)^2G^2 + 96(x^2 - y^2)(x-y)G^2 + 96(x^2 - y^2)(x-y)^2GG_x \\ &= 28yx^6 + \sum \text{lower degree terms with respect to } x. \end{aligned}$$

Multiplying  $(x^2 - y^2)(x-y)$  at both sides of this, we get

$$(2.7) \quad \begin{aligned} & 96(x^2-y^2)(x-y)^3(2x+y)G^2+96(x^2-y^2)^2(x-y)^3GG_x \\ & =28yx^9+\sum \text{ lower degree terms with respect to } x. \end{aligned}$$

Similarly we get

$$(2.8) \quad \begin{aligned} & -96(x^2-y^2)(x-y)^3(x+2y)G^2+96(x^2-y^2)^2(x-y)^3GG_y \\ & =4x^{10}+\sum \text{ lower degree terms with respect to } x. \end{aligned}$$

From (2.6), (2.7) and (2.8) we find

$$(2.9) \quad 96(x^2-y^2)^2(x-y)^3GG_x=12yx^9+\sum \text{ lower degree terms to } x,$$

$$(2.10) \quad 96(x^2-y^2)^2(x-y)^3GG_y=4x^{10}+\sum \text{ lower degree terms to } x.$$

Multiplying  $96(x^2-y^2)^2(x-y)^3G$  at both sides of (2.5), we get

$$(2.11) \quad \begin{aligned} & 9(x^2+xy+y^2)\{96(x^2-y^2)^2(x-y)^3G^2\} \\ & \quad + (x+2y)(3y)(x-y)\{96(x^2-y^2)^2(x-y)^3GG_x\} \\ & \quad - (x+2y)(2x+y)(x-y)\{96(x^2-y^2)^2(x-y)^3GG_y\} = 0. \end{aligned}$$

Substituting (2.6), (2.9) and (2.10) into (2.11), we can see that the highest degree term with respect to  $x$  in (2.11) is  $-8x^{13}$ . So  $x$  and  $y$  satisfy a non-constant polynomial (2.11). Hence, by Lemma 3,  $H$  is constant in  $U$ , which contradicts to our assumption. Consequently  $H$  is constant in  $M$ . And from (1.14) and (1.15) we can see that  $x$  and  $y$  are constant. So  $M$  is an open part of a plane or a sphere or a circular cylinder. But the Gauss map of a plane is constant. Hence a plane has not 1-type Gauss map. The converse is an easy computation.

## REFERENCES

- [1] C. BAIKOUSSIS, B.-Y. CHEN AND L. VERSTRAELEN, Ruled surfaces and tubes with finite-type Gauss map, *Tokyo J. Math.*, **16** (1993), 341-349.
- [2] B.-Y. CHEN, Total Mean Curvature and Submanifolds Finite-type, World Scientific Publishing, Singapore, 1984.
- [3] B.-Y. CHEN AND P. PICCINNI, Submanifolds with finite-type Gauss map, *Bull. Austral. Math. Soc.*, **35** (1987), 161-186.
- [4] W. FULTON, Algebraic Curves, the Benjamin/Cummings Company, Inc., 1978.
- [5] G. HUISKEN, Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.*, **20** (1984), 237-266.
- [6] Y.H. KIM, Co-closed surfaces of 1-type Gauss map, *Bull. Korean Math. Soc.*, **31** (1994), 125-132.

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