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YANG-MILLS HOMOGENEOUS CONNECTIONS ON COMPACT SIMPLE LIE GROUPS

Dedicated to Tsunero Takahashi on his 60's birthday

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1. Introduction

Let M be a compact Riemannian manifold and P a principal G-bundle, where G is a compact Lie group. Fix a bi-invariant Riemannian metric on G. Let Ω_{Λ} denote the curvature form of a connection Λ on P. A critical point of the Yang-Mills functional

$$\Lambda \mapsto \frac{1}{2}\int_{M} \|\Omega_{\Lambda}\|^2$$

is called a Yang-Mills connection. A Yang-Mills connection Λ is said to be stable if the second variation of the Yang-Mills functional is non-negative. A flat connection is a stable Yang-Mills connection. H. T. Laquer [4] proved that (0)-connection on a compact Lie group is an unstable Yang-Mills connection. A compact Riemannian manifold M is called Yang-Mills unstable if, for every choice of G and every principal G-bundle P over M, stable Yang-Mills connection is always flat. S. Kobayashi, Y. Ohnita and M. Takeuchi [3] classified the compact simply connected irreducible symmetric spaces of type I which are Yang-Mills unstable. In their paper, they gave a following question:

Is every simply connected compact simple Lie group Yang-Mills unstable? In this paper, we consider an equivariant G-bundle P over a compact connected simple Lie group L. It is obtained by a Lie homomorphism $\rho: L \to G$. With respect to homogeneous connections on P, we get the following:

THEOREM 1. *Consider the following three conditions* (1), (2), *and* (3):

(1) *ρ is indecomposable* (*see* § 2 *for definition*),
(2) *Flat homogeneous connections are only* ($\pm$)-*connections,*
(3) (0)-*connection is a unique non-flat Yang-Mills homogeneous connection.*

Then (1) *and* (2) *are equivalent.* (3) *implies* (1).

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Moreover if $\rho(\mathfrak{l})$ contains a regular element of $\mathfrak{g}$, then (1) implies (3). In general, (1) does not imply (3) (see § 3).

THEOREM 2. *Assume $\rho(\mathfrak{l})$ contains a regular element of $\mathfrak{g}$. Then any non-flat Yang-Mills homogeneous connection is unstable.*

2. Proof of theorems

Let L be a compact connected simple Lie group with Lie algebra $\mathfrak{l}$. Take an $\mathrm{Ad}(L)$-invariant inner product $\langle\,,\rangle$ on $\mathfrak{l}$. Let G be an another compact connected Lie group with Lie algebra $\mathfrak{g}$. Take an $\mathrm{Ad}(G)$-invariant inner product $\langle\,,\rangle$ on $\mathfrak{g}$. Let $\rho: L \to G$ be a Lie homomorphism. We denote the differential Lie homomorphism of ρ by the same symbol ρ. Put

$$K = L \times L \supset H = \{(l, l);\ l \in L\} \cong L \quad ((l, l) \leftrightarrow l) \text{ and } M = K/H.$$

We define an inner product $\langle\,,\rangle$ on $\mathfrak{k}$ by

$$\langle (X, Y), (Z, W) \rangle = 2(\langle X, Z\rangle + \langle Y, W\rangle) \quad \text{for } X, Y, Z, W \in \mathfrak{l}.$$

We define an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{k}$ by

$$\mathfrak{m} = \{(X, -X);\ X \in \mathfrak{l}\}.$$

Then we have:

$$\mathfrak{k} = \mathfrak{h} + \mathfrak{m} \quad \text{(direct sum)}.$$

The induced $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$ naturally induces a K-invariant Riemannian metric on M. The mapping

$$(a, b)H \mapsto ab^{-1}$$

is an isometry from M onto L. The mapping

$$\mathfrak{m} \to \mathfrak{l};\ \left(\frac{1}{2}X, -\frac{1}{2}X\right) \mapsto X$$

is a linear isometry from $\mathfrak{m}$ onto $\mathfrak{l}$. In this correspondence, we have

$$(\mathrm{Ad}(H), \mathfrak{m}) \cong (\mathrm{Ad}(L), \mathfrak{l}).$$

We define a Lie homomorphism $\bar{\rho}$ from H into G by

$$\bar{\rho}: H \to G;\ (l, l) \mapsto \rho(l).$$

Every Lie homomorphism from H into G is obtained in this way. The space of homogeneous connections on the principal G-bundle $P = K \times_{\bar{\rho}} G$ over M is identified with

$$\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g}) = \{\Lambda \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g});\ [\rho(X), \Lambda(Y)] = \Lambda([X, Y]) \text{ for } X, Y \in \mathfrak{l}\}.$$

by Wang's theorem ([2, pp. 106–107, Theorem 11.5]), where $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ is the space of linear mappings from the vector space $\mathfrak{l}$ to the vector space $\mathfrak{g}$. Remark that $\boldsymbol{R}\rho$ is contained in $\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$. The curvature from Ω of a homogeneous connection $\Lambda \in \mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ is an alternative linear mapping from $\mathfrak{l} \times \mathfrak{l}$ to $\mathfrak{g}$ which is given by

$$2\Omega(X, Y) = -\frac{1}{4}\rho([X, Y]) + [\Lambda(X), \Lambda(Y)].$$

In particular, the curvature form Ω_t of $t\rho \in \boldsymbol{R}\rho$ is

$$2\Omega_t(X, Y) = \left(t^2 - \frac{1}{4}\right)\rho([X, Y]).$$

Hence $\Lambda = (\pm 1/2)\rho$ are flat connections, which are called $(\pm)$-connection, respectively. A critical point of the Yang-Mills functional $\Lambda \mapsto \|\Omega\|^2$ is called a Yang-Mills connection. A homogeneous connection $\Lambda \in \mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ is Yang-Mills if and only if for each $X \in \mathfrak{l}$

$$\sum_{i=1}^{n} [\Lambda(E_i), \Omega(E_i, X)] = 0,$$

where $\{E_1, \cdots, E_n\}$ is an orthonormal basis of $\mathfrak{l}$. In particular, $\Lambda = 0$ is a Yang-Mills connection, which is called the (0)-connection.

DEFINITION 1. We say that ρ is indecomposable, if

$$\rho = \rho_1 + \rho_2,\ \rho_i : \mathfrak{l} \to \mathfrak{g} : \text{Lie homomorphism s.t. } [\mathrm{Im}\,\rho_1, \mathrm{Im}\,\rho_2] = 0 \quad (*)$$
$$\Rightarrow \rho_1 = 0,\ \rho_2 = \rho \text{ or } \rho_2 = 0,\ \rho_1 = \rho.$$

We say that $(*)$ is a decomposition of ρ.

Since the kernel of ρ is an ideal of $\mathfrak{l}$, ρ is injective or $\rho = 0$. If $\rho = 0$, then $\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g}) = \{0\}$ and (0)-connection is flat. Therefore we may assume that ρ is injective.

THEOREM 1. *Consider the following three conditions* (1), (2), *and* (3):

(1) *ρ is indecomposable,*
(2) *Flat homogeneous connections are only the $(\pm)$-connections,*
(3) *The* (0)*-connection is a unique non-flat Yang-Mills homogeneous connection.*

Then (1) *and* (2) *are equivalent. The condition* (3) *implies* (1). *Moreover if $\rho(\mathfrak{l})$ contains a regular element of $\mathfrak{g}$, then* (1) *implies* (3).

Remark 1. In general, (1) does not imply (3) (see § 3). ■

Proof of the first half of Theorem 1. If $\rho = \rho_1 + \rho_2$ is a non-trivial decomposition of ρ, then $1/2(\rho_1 - \rho_2)$ is a flat homogeneous connection except the $(\pm)$-connection and $(1/2)\rho_1$ is a non-flat Yang-Mills connection except the

(0)-connection. Hence (2) implies (1), and (3) implies (1). We show (1) implies (2). Let Λ be any flat homogeneous connection. Put

$$\rho_1=\frac{1}{2}\rho+\Lambda, \quad \rho_2=\frac{1}{2}\rho-\Lambda.$$

Then $\rho=\rho_1+\rho_2$ is a decomposition of ρ. Since ρ is indecomposable, $\rho_1=0$ or $\rho_1=\rho$. Hence $\Lambda=(\pm 1/2)\rho$. ∎

THEOREM 2. *Assume $\rho(\mathfrak{l})$ contains a regular element of $\mathfrak{g}$. Then any non-flat Yang-Mills homogeneous connection is unstable.*

Proof of the second half of Theorem 1 *and Theorem* 2. It is sufficient to prove that for each non-flat Yang-Mills connection $\Lambda\in\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$, there exists $\alpha(=\alpha_\Lambda)\in\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ such that

(A1) $\alpha=0$ implies $\Lambda=0$,
(A2) $\rho=\alpha+(\rho-\alpha)$ is a decomposition of ρ, and $\rho-\alpha\neq 0$,
(A3) $d^2/dt^2\|\Omega_t\|_{|t=0}<0$, where Ω_t is the curvature form of $\Lambda+t(\rho-\alpha)$.

Applying Whitehead's vanishing theorem of cohomology group ([6, p. 95, Theorem 13]) for the representation $(\mathrm{ad}\circ\rho, \mathfrak{g})$ of $\mathfrak{l}$, we have following:

If $\Lambda_1, \Lambda_2\in\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ satisfy

(B1) $[\Lambda_1(X), \Lambda_2(Y)]=-[\Lambda_1(Y), \Lambda_2(X)]$,
(B2) $\mathfrak{S}_{X,Y,Z}[\rho(X), [\Lambda_1(Y), \Lambda_2(Z)]]=0$, where $\mathfrak{S}_{X,Y,Z}$ is the sum over the cyclic permutations of X, Y, Z,

then there exists $\Lambda_3\in\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ such that

$$[\Lambda_1(X), \Lambda_2(Y)]=\Lambda_3([X, Y]).$$

Remark that under the condition (B1), the condition (B2) is equivalent to $\mathfrak{S}_{X,Y,Z}[\Lambda_1([X, Y]), \Lambda_2(Z)]=0$. Since $\rho(\mathfrak{l})$ contains a regular element of $\mathfrak{g}$, $[\Lambda_1, \Lambda_2]$ is skew-symmetric automatically. In fact, take Cartan subalgebras $\mathfrak{t}$ and $\mathfrak{h}$ of $\mathfrak{l}$ and $\mathfrak{g}$ respectively such that $\rho(\mathfrak{t})\subset\mathfrak{h}$. Then

$$[\rho(\mathfrak{t}), \Lambda_i(\mathfrak{t})]=\Lambda_i([\mathfrak{t}, \mathfrak{t}])=0.$$

This implies $\Lambda_i(\mathfrak{t})\subset\mathfrak{h}$ by assumption. In particular, $[\Lambda_1(\mathfrak{t}), \Lambda_2(\mathfrak{t})]=0$ and $[\Lambda_1(H), \Lambda_2(H)]=0$ for $H\in\mathfrak{t}$. Since $\mathfrak{l}=\mathrm{Ad}(L)\mathfrak{t}$ ([1, p. 248, Theorem 6.4]), we get $[\Lambda_1(X), \Lambda_2(X)]=0$.

Let $\Lambda\in\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ be any non-flat Yang-Mills homogeneous connection. First we prove $\mathfrak{S}_{X,Y,Z}[\rho(X), [\Lambda(Y), \Lambda(Z)]]=0$ using the classification of compact simple Lie algebras. The vector space

$$V=\mathfrak{l}\wedge\mathfrak{l}=\mathrm{span}\,\{X\wedge Y\,;\, X, Y\in\mathfrak{l}\}$$

is an $\mathfrak{l}$-module by the $\mathfrak{l}$-action:

$$(\operatorname{ad} Z)(X\wedge Y)=[Z, X]\wedge Y+X\wedge[Z, Y].$$

The space

$$W=\operatorname{span}\{[\Lambda(X), \Lambda(Y)] ; X, Y\in\mathfrak{l}\}$$

is an $\operatorname{ad}(\rho(\mathfrak{l}))$-invariant subspace of $\mathfrak{g}$. We consider the $\mathfrak{l}$-homomorphism Φ from V onto W which is defined by

$$\Phi : V=\mathfrak{l}\wedge\mathfrak{l}\rightarrow W ; X\wedge Y\mapsto[\Lambda(X), \Lambda(Y)].$$

Since Φ is surjective, $V/V_0\cong W$ as $\mathfrak{l}$-modules, where $V_0=\operatorname{Ker}\Phi$. On the other hand, we consider the $\mathfrak{l}$-homomorphism Ψ from V into $\mathfrak{l}$ which is defined by

$$\Psi : V=\mathfrak{l}\wedge\mathfrak{l}\rightarrow\mathfrak{l} ; X\wedge Y\mapsto[X, Y].$$

Since $[\mathfrak{l}, \mathfrak{l}]=\mathfrak{l}$, Ψ is surjective. We show that the irreducibility of $V_1=\operatorname{Ker}\Psi$. We denote by $\mathfrak{l}^C$, $\mathfrak{t}^C$ and ρ^C the complexifications of $\mathfrak{l}$, $\mathfrak{t}$ and ρ respectively. The complex Lie algebra $\mathfrak{l}^C$ is simple. We denote by Δ the set of nonzero roots of $\mathfrak{l}^C$ with respect to $\mathfrak{t}^C$. For $\alpha\in\Delta$, there exists a non-zero vector $E_\alpha\in\mathfrak{l}^C$ such that

$$[H, E_\alpha]=\alpha(H)E_\alpha \quad \text{for all } H\in\mathfrak{t}^C.$$

We have a direct-sum decomposition:

$$\mathfrak{l}^C=\mathfrak{t}^C+\sum_{\alpha\in\Delta}\boldsymbol{C}E_\alpha .$$

Fix a lexicographic ordering on $\mathfrak{t}^C$. We denote by δ_0 the highest root of Δ and by $\{\alpha_1, \cdots, \alpha_r\}$ the set of simple roots of Δ. The set

$$\{\delta_0-\alpha_i\in\Delta\}\neq\emptyset$$

is a single point set $\{\delta_1\}$ or two points set $\{\delta_1, \delta_2\}$, and the set consists two points if and only if $\mathfrak{l}=\mathfrak{su}(m)$.

In the case where $\{\delta_0-\alpha_i\in\Delta\}=\{\delta_1\}$, we define an $\mathfrak{l}$-invariant subspace $V_1(\delta_0+\delta_1)$ of V_1^C by

$$V_1(\delta_0+\delta_1)=\operatorname{ad}(U(\mathfrak{l}^C))(E_{\delta_0}\wedge E_{\delta_1}),$$

where $U(\mathfrak{l}^C)$ is the universal enveloping algebra of $\mathfrak{l}^C$. The highest weight of $V_1(\delta_0+\delta_1)$ is $\delta_0+\delta_1$ and the multiplicity of $\delta_0+\delta_1$ is equal to 1. Hence $V_1(\delta_0+\delta_1)$ is irreducible. By virtue of Weyl's dimensionality formula ([6, p. 257]), we get

$$\dim V_1(\delta_0+\delta_1)=\frac{\dim\mathfrak{l}(\dim\mathfrak{l}-3)}{2}=\dim V_1 .$$

Hence $V_1^C=V_1(\delta_0+\delta_1)$. In particular, V_1^C is irreducible so V_1 is.

In the case where $\{\delta_0-\alpha_i\in\Delta\}=\{\delta_1, \delta_2\}$, we define $\mathfrak{l}$-invariant subspaces $V_1(\delta_0+\delta_1)$ and $V_1(\delta_0+\delta_2)$ of V_1^C by

$$V_1(\delta_0+\delta_1)=\mathrm{ad}(U(\mathfrak{l}^C))(E_{\delta_0}\wedge E_{\delta_1}),$$
$$V_1(\delta_0+\delta_2)=\mathrm{ad}(U(\mathfrak{l}^C))(E_{\delta_0}\wedge E_{\delta_2}).$$

For $i=1, 2$, the highest weight of $V_1(\delta_0+\delta_i)$ is $\delta_0+\delta_i$ and the multiplicity of $\delta_0+\delta_i$ is equal to 1 Hence $V_1(\delta_0+\delta_i)$ $(i=1, 2)$ is irreducible. By virtue of Weyl's dimensionality formula, we get

$$\dim V_1(\delta_0+\delta_1)=\dim V_1(\delta_0+\delta_2)=\frac{1}{2}\dim V_1.$$

Hence we have

$$V_1^C=V_1(\delta_0+\delta_1)+V_1(\delta_0+\delta_2) \quad \text{(direct sum)}.$$

We denote by $W(L)$ the Weyl group of L. Clearly, there exist $\sigma_1, \sigma_2\in W(L)$ such that

$$\sigma_1(\delta_0+\delta_1)=-(\delta_0+\delta_2), \quad \sigma_2(\delta_0+\delta_2)=-(\delta_0+\delta_1).$$

Hence V_1 is real irreducible, whether $\{\delta_0-\alpha_i\in\Delta\}$ is a single point set or two points set. So we get

$$V_1=\mathrm{ad}(U(\mathfrak{l}))(\mathfrak{t}\wedge\mathfrak{t})\subset V_0.$$

Hence Φ naturally induces $\mathfrak{l}$-homomorphism φ from V/V_1 onto W defined by

$$\varphi: V/V_1\to W;\ \overline{X\wedge Y}\mapsto[\Lambda(X), \Lambda(Y)],$$

where $\overline{X\wedge Y}$ is the equivalence class of $X\wedge Y$. From Jacobi's identity, we have

$$\begin{aligned}\mathfrak{S}_{X,Y,Z}\,\mathrm{ad}(Z)\overline{X\wedge Y}&=\mathfrak{S}_{X,Y,Z}(\overline{[Z, X]\wedge Y+X\wedge[Z, Y]})\\&=2\mathfrak{S}_{X,Y,Z}\overline{[Z, X]\wedge Y}\\&=0.\end{aligned}$$

Hence we have

$$0=\varphi(\mathfrak{S}_{X,Y,Z}\,\mathrm{ad}(Z)\overline{X\wedge Y})=\mathfrak{S}_{X,Y,Z}[\rho(Z), [\Lambda(X), \Lambda(Y)]].$$

By Whitehead's vanishing theorem of cohomology group, there exists $\alpha\in \mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ such that

$$\alpha([X, Y])=4[\Lambda(X), \Lambda(Y)].$$

By Jacobi's identity, we have

$$\mathfrak{S}_{X,Y,Z}[\alpha([X, Y]), \Lambda(Z)]=\frac{1}{4}\mathfrak{S}_{X,Y,Z}[[\Lambda(X), \Lambda(Y)], \Lambda(Z)]=0.$$

By Whitehead's vanishing theorem of cohomology group, there exists $\Gamma\in \mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ such that

$$[\alpha(X), \Lambda(Y)]=\Gamma([X, Y]).$$

Since Λ is Yang-Mills, we have

$$
\begin{aligned}
-\frac{c}{4}\Gamma(X) &= \frac{1}{4}\sum_{i=1}^{n}[\Lambda(E_i), \alpha([E_i, X])] \\
&= \sum_{i=1}^{n}[\Lambda(E_i), [\Lambda(E_i), \Lambda(X)]] \\
&= -\frac{c}{4}\Lambda(X),
\end{aligned}
$$

where c is the eigenvalue of the negative of the Casimir operator of (ad, $\mathfrak{l}$). Hence $\Gamma=\Lambda$, that is,

$$[\alpha(X), \Lambda(Y)]=\Lambda([X, Y]).$$

Hence we get (A1). We show α is a Lie homomorphism. From Jacobi's identity, we have

$$
\begin{aligned}
\frac{1}{4}[\alpha(X), \alpha([Z, W])] &= [\alpha(X), [\Lambda(Z), \Lambda(W)]] \\
&= [[\alpha(X), \Lambda(Z)], \Lambda(W)]+[\Lambda(Z), [\alpha(X), \Lambda(W)]] \\
&= [\Lambda([X, Z]), \Lambda(W)]+[\Lambda(Z), [\Lambda([X, W])]] \\
&= \frac{1}{4}\alpha([[X, Z], W]+[Z, [X, W]]) \\
&= \frac{1}{4}\alpha([X, [Z, W]]).
\end{aligned}
$$

Hence $\alpha\in\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ is a Lie homomorphism. So, if we put $\delta=\rho-\alpha$, then $\rho=\alpha+\delta$ is a decomposition of ρ. The curvature form Ω of Λ is given by $\Omega(X, Y)=(-1/4)\delta([X, Y])$. Since Λ is not flat, we have $\delta\neq 0$. Hence we have (A2). Since $[\delta(X), \Lambda(Y)]=0$, the curvature form Ω_t of $\Lambda+t\delta$ is given by

$$\Omega_t(X, Y)=\frac{4t^2-1}{4}\delta([X, Y]).$$

Hence we have (A3).

3. An example

When $\rho(\mathfrak{l})$ does not contain any regular element of $\mathfrak{g}$, the (0)-connection is not necessarily a unique non-flat Yang-Mills homogeneous connection, even if ρ is indecomposable. We show such an example. Put $L=SU(m)$ for $m\geqq 3$. We define an $\mathrm{Ad}(L)$-invariant inner product $\langle\,,\rangle$ on $\mathfrak{l}$ by

$$\langle X, Y\rangle=-\mathrm{tr}(XY) \quad \text{for } X, Y\in\mathfrak{l}.$$

The inner product $\langle\,,\rangle$ naturally induces a Hermitian inner product $\langle\,,\rangle$ on $\mathfrak{l}^C$. Put $G=SU(\mathfrak{l}^C)$ and $\rho=\mathrm{Ad}: L\to G$. In this case, $\rho(\mathfrak{l})$ does not contain any regular element of $\mathfrak{g}$. We define an $\mathrm{Ad}(G)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by

$$\langle A, B\rangle=\sum_{i=1}^{m^2-1}\langle AE_i, BE_i\rangle \quad \text{for } A, B\in\mathfrak{g},$$

where $\{E_i\}_{1\leq i\leq m^2-1}$ is an orthonormal basis of $\mathfrak{l}$. We define a homogeneous connection $\Lambda\in\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$ by

$$(\Lambda(X))(Y)=\frac{-m}{2\sqrt{m^2+4}}\left\{(XY+YX)-\frac{2}{m}\mathrm{tr}(XY)1_m\right\},$$

where 1_m is the identity matrix (cf. [5]).

Remark 2. If $m=2$, then $\Lambda=0$. ■

PROPOSITION 1. (1) $\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})=\boldsymbol{R}\rho+\boldsymbol{R}\Lambda$ (orthogonal direct sum),

(2) *ρ is indecomposable,*

(3) *Λ $(\neq 0)$ is a non-flat Yang-Mills homogeneous connection, which is a local minimum on the space of homogeneous connections* $\mathrm{Hom}_L(\mathfrak{l}, \mathfrak{g})$.

Proof. (1) is obtained by simple calculation. (2) is obtained by (1) and Theorem 1.

(3) The equations

$$\sum_{i=1}^{m^2-1}[E_i, [E_i, X]]=-2mX, \quad \sum_{i=1}^{m^2-1}E_i^2=-\frac{m^2-1}{m}1_m$$

and

$$\begin{aligned}&[\Lambda(X), [\Lambda(Y), \Lambda(Z)]](W)\\ &=\frac{m^2}{4(m^2+4)}\Lambda([X, [Y, Z]])(W)\\ &\quad+\frac{m}{m^2+4}\{\mathrm{tr}(YW)\Lambda(X)Z-\mathrm{tr}(ZW)\Lambda(X)Y\\ &\qquad-\mathrm{tr}(Y\Lambda(X)W)Z+\mathrm{tr}(Z\Lambda(X)W)Y\}\end{aligned}$$

imply that Λ is a non-flat Yang-Mills homogeneous connection.

Put $\Lambda(x, y)=(x/2)\rho+y\Lambda$ and $f(x, y)=4\|\Omega(x, y)\|^2$, where $\Omega(x, y)$ is the curvature form of $\Lambda(x, y)$. The equations

$$\sum_{i,j}\|\rho([E_i, E_j])\|^2=4m^2(m^2-1),$$

$$\sum_{i,j}\|\Lambda([E_i, E_j])\|^2=\frac{m^2(m^2-1)(m^2-4)}{m^2+4},$$

$$\sum_{i,j}\|[\Lambda(E_i), \Lambda(E_j)]\|^2=\frac{m^2(m^2-1)(m^2-4)}{4(m^2+4)}$$

imply that

$$f(x, y)=m^2(m^2-1)\left\{\frac{1}{4}(x^2-1)^2+\frac{m^2-4}{4(m^2+4)}y^4 + \frac{m^2-4}{m^2+4}x^2y^2+\frac{m^2-4}{2(m^2+4)}(x^2-1)y^2\right\}.$$

Hence f is a local minimum at $(0, 1)$. ■

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