

ON THE VALUE DISTRIBUTION OF $ff^{(k)}$

CHUNG CHUN YANG[†] AND PEI CHU HU

Abstract

Let f be a transcendental entire function. In this paper we will prove that if f is of finite order, then there exists at most one integer $k \geq 2$ such that $ff^{(k)}$ may have non-zero and finite Picard exceptional value. We also give a class of entire functions which have no non-zero finite Picard values. If f is a transcendental meromorphic function, we obtain that for non-negative integers n, n_1, \dots, n_k with $n \geq 1, n_1 + \dots + n_k \geq 1$, if $\delta(o, f) > 3/(3n + 3n_1 + \dots + 3n_k + 1)$, then $f^n(f')^{n_1} \dots (f^{(k)})^{n_k}$ has no finite non-zero Picard values.

I. Introduction and main results

Let f be a transcendental meromorphic function. In 1959, W.K. Hayman [4] proved that if n is an integer satisfying $n \geq 3$, then $f^n f'$ takes every non-zero complex value a infinitely often. He conjectured [5] that this remains valid for $n=1$ and $n=2$. The case $n=2$ was settled by E. Mues [9] on 1979. The case $n=1$ is still open.

J. Clunie [3] proved that Hayman's conjecture is true when f is entire and $n=1$. W. Hennekemper [7] extended Clunie's result and proved

$$(1) \quad T(r, f) \leq \left(4 + \frac{1}{k+1}\right) \left\{ \bar{N}(r, f) + \bar{N}\left(\frac{1}{(f^{k+1})^{(k)} - c}\right) \right\} + S(r, f)$$

for $k \in \mathbb{N}$, $c \in \mathbb{C} - \{0\}$, where the argument used here is based on the Nevanlinna theory, its associated standard symbols and notations, see, e.g. [6]. Particularly, $S(r, f)$ will be used to denote any quantity that satisfies $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$ and $r \notin E$ with E being a set of r values of finite linear measure. W. Bergweiler and A. Eremenko [2] proved this for functions of finite order. Recently, Q. Zhang [16] extended Hennekemper's result (1) for $k=1$ and c is replaced by any small function $a(z) (\not\equiv 0)$ of f , i.e. $a(z)$ satisfies $T(r, a) = S(r, f)$. W. Bergweiler [1] proved that if f is a transcendental meromorphic function of finite order and if a is a polynomial which does not vanish identically, then $ff' - a$ has infinitely many zeros.

[†] The research work is partially supported by a U.P.G.C. grant, Hong Kong.

AMS Subject Classification: 30D35.

Key words: Entire function, Meromorphic function, Picard value, Deficient condition.

Received January 6, 1995.

Recently a conjecture was raised in [14], it states that for any integer $k \geq 2$, and nonzero value c , the function $ff^{(k)} - c$ has infinitely many zeros. More generally is it true that for any non-negative integers n, n_1, \dots, n_k with $n=1, n_1 + \dots + n_k \geq 1$ and for any non-zero value c , the function $f^n(f')^{n_1} \dots (f^{(k)})^{n_k} - c$ always has infinitely many zeros? C.C. Yang, L. Yang and Y.F. Wang [15] proved that if f is a transcendental entire function and if n, k are non-negative integers with $n \geq 2$ and $k \geq 0$, then the only possible Picard value of $f(f^{(k)})^n$ is the value zero. In this note we shall provide some results in towards to a complete solution of the conjecture.

THEOREM 1. *Let f be a transcendental entire function of finite order. Then there exists at most one integer $k \geq 2$, such that $ff^{(k)}$ has a non-zero and finite exceptional value.*

THEOREM 2. *Take non-negative integers n, n_1, \dots, n_k with $n \geq 1, n_1 + \dots + n_k \geq 1$ and define $d = n + n_1 + \dots + n_k$. Let f be a transcendental meromorphic function with $\delta(0, f) > 3/(3d+1)$. Then for any non-zero value c , the function $f^n(f')^{n_1} \dots (f^{(k)})^{n_k} - c$ always has infinitely many zeros.*

THEOREM 3. *Take $c_1, c_2, c_3 \in \mathbb{C}$, $k \in \mathbb{N}$ with $K_0 = 2k+1+2c_1 \neq 0$, and define*

$$(2) \quad P(f) = (f')^k + c_1 f(f')^{k-2} f'' + c_2 f^2(f')^{k-4} (f'')^2 + c_3 f^3(f')^{k-3} f'''$$

where $c_1 = c_2 = c_3 = 0$ if $k=1$, $c_2 = c_3 = 0$ if $k=2$ and $c_2 = 0$ if $k=3$. Write

$$(3) \quad K_1 = 3k+1+6c_1+6c_3, \quad K_3 = \frac{3}{K_0}(k^2+2(k-1)c_1+2c_2), \quad K_2 = K_0 - K_3$$

$$(4) \quad A = K_0^2 K_1 - K_0 K_1(k+1) + (K_1 - K_0 K_2)(k+1)^2$$

$$(5) \quad B = K_0^2 K_1 + K_0(3K_0 - 2K_1)(k+1)$$

Assume that (i) $c_1 + c_2 + c_3 \neq -1$ and $(1 + c_1 + c_2 + c_3)X^2 + (c_1 + 2c_2 + 3c_3)X + c_2 + 2c_3 = 0$ has no roots of positive integers; (ii) $A \neq 0$ and $B/A \notin \mathbb{N} - \{1\}$. If f is a transcendental meromorphic function, then, for $c \in \mathbb{C} - \{0\}$,

$$(6) \quad T(r, f) \leq \left(4 + \frac{1}{k+1}\right) \left\{ \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{fP(f)-c}\right) \right\} + S(r, f).$$

Remarks. If $k=1$, Theorem 3 yields Clunie's result. If $n \geq 2$, then the deficient condition in Theorem 2 can be omitted, see e.g. [11]. Furthermore, when $n=1, d \geq 1$, and $N_1(r, 1/f) = S(r, f)$, then Zhang-Li [17] obtained some results similar to that of Theorem 2. Also see [10].

2. Proof of Theorem 1

We first prove the following key point in the verification of Theorem 1.

LEMMA. Let $p_i(z) (\not\equiv 0, i=1, 2)$ and $q_i(z) (\not\equiv \text{constants}, i=1, 2)$ be some polynomials, with $\deg q_1 = \deg q_2$, and a_1 and a_2 are two constants. If

$$m\left(r, \frac{a_1 + p_1 e^{q_1}}{a_2 + p_2 e^{q_2}}\right) = o(1)T(r, e^{q_i}); \quad i=1, 2,$$

then q_1 and q_2 have the same leading coefficient.

Proof. By definition,

$$\begin{aligned} m\left(r, \frac{a_1 + p_1 e^{q_1}}{a_2 + p_2 e^{q_2}}\right) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{a_1 + p_1(r e^{i\theta}) e^{q_1(r e^{i\theta})}}{a_2 + p_2(r e^{i\theta}) e^{q_2(r e^{i\theta})}} \right| d\theta; \quad z = r e^{i\theta} \\ &= I. \end{aligned}$$

Let $q_j(z) = b_{j_0} z^{q_j} + b_{j_1} z^{q_j-1} + \dots + b_{j_{q_j-1}} z + b_{j_{q_j}}; q_j \geq 1, j=1, 2$ and $b_{j_0} \neq 0$.

Then clearly on $|z|=r$, for $z \in J_j(\theta) = \{\theta | \operatorname{Re} b_{j_0} e^{i q_j \theta} > 0\} \quad j=1, 2$, the value $|a_j + p_j(r e^{i\theta}) e^{q_j(r e^{i\theta})}|, (j=1, 2)$ is dominated by the term $e^{b_{j_0} z^{q_j}}$. Thus if $b_{1_0} \neq b_{2_0}$ (either $\arg b_{1_0} \neq \arg b_{2_0}$ or $\arg b_{1_0} = \arg b_{2_0}$), the integrand in I , when $r > r_0$ will be $\geq \varepsilon r^q$ for some fixed $\varepsilon > 0$ along a piece of arc J_j^* on $|z|=r$ with $\operatorname{meas} J_j^* \geq c_0 > 0$, c_0 is a fixed positive number depends on q . Hence there exists a positive number ε_0 such that when $r > r_0$

$$m\left(r, \frac{a_1 + p_1 e^{q_1}}{a_2 + p_2 e^{q_2}}\right) \geq \varepsilon_0 r^q \neq o(1)T(r, e^{q_i}); \quad i=1, 2,$$

a contradiction. The lemma is thus proved.

Now we proceed to prove the theorem by contradiction.

Assume that there exist two distinct integers k_1 and $k_2; k_1 > k_2 \geq 2$ and constants a_1 and a_2 such that both $ff^{(k_1)} - a_1$ and $ff^{(k_2)} - a_2$ have only finitely many zeros. By this and the fact that f is of finite order, we have

$$(7) \quad ff^{(k_1)} - a_1 = p_1(z) e^{q_1(z)}$$

and

$$(8) \quad ff^{(k_2)} - a_2 = p_2(z) e^{q_2(z)},$$

where $p_i (\not\equiv 0)$ and $q_i(z) (\not\equiv \text{constant})$ are polynomials.

It follows that

$$(9) \quad ff^{(k_1)} = a_1 + p_1 e^{q_1}$$

and

$$(10) \quad ff^{(k_2)} = a_2 + p_2 e^{q_2}.$$

Hence

$$(11) \quad \frac{f^{(k_1)}}{f^{(k_2)}} = \frac{a_1 + p_1 e^{q_1}}{a_2 + p_2 e^{q_2}}.$$

We recall that $k_1 > k_2$. By applying the lemma of logarithmic derivative to (11), we have

$$(12) \quad m\left(r, \frac{f^{(k_1)}}{f^{(k_2)}}\right) = S(r, f^{(k_2)}) = S(r, f) \\ = o(1)T(r, f).$$

On the other hand, it is not difficult to verify that for any entire function f of finite order, $ff^{(k)}$ and f have the same order. Thus we have $\deg q_1 = \deg q_2$. Hence it follows from (12) and the lemma we conclude immediately that q_1 and q_2 have the same leading coefficient. Hence

$$e^{q_i(z)} = e^{bz^q} A_i(z); \quad i=1, 2;$$

where $b = b_{10} = b_{20}$, and $A_i(z)$ ($i=1, 2$) are functions of order no greater than $q-1$. In the following we shall treat two cases (i) $a_1 = a_2$ and (ii) $a_1 \neq a_2$ separately.

Case (i) $a_1 = a_2$. Then from equations (9) and (10), we obtain

$$(13) \quad f(f^{(k_1)} - f^{(k_2)}) = e^{bz^q} (p_1 A_1 - p_2 A_2).$$

If $f^{(k_1)} - f^{(k_2)} \equiv 0$ then this yields $f(z) = \sum_{i=1}^q c_i e^{\rho_i z}$, where ρ_i ($i=1, 2, \dots, q$) are roots of the unity and c_i ($i=1, 2, \dots, q$) are constants.

Substituting the form of f into (9) or (10) and then applying lemma on Borel type of identity [12], we can easily derive a contradiction.

Thus $f^{(k_1)} - f^{(k_2)} \not\equiv 0$ and it follows from (13) and the fact that the order of $(p_1 A_1 - p_2 A_2) \leq q-1$, we have

$$N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{p_1 A_1 - p_2 A_2}\right) \\ = O(1)r^{q-1}.$$

Hence f assume the form:

$$(14) \quad f(z) = h(z)e^{cz^q},$$

where $T(r, h) = O(1)r^{q-1}$ and c is a constant $\neq 0$.

Substituting this into either (9) or (10) and applying lemma on Borel type of identity again, we will derive a contradiction.

Now case (ii): $a_1 \neq a_2$. Let $a_1/a_2 = d$ for some constant d ($\neq 0$). Then equation (10) is equivalent to

$$df f^{(k_2)} = a_2 d + d p_2 e^{q_2} = a_1 + d p_2 e^{q_2}.$$

Combining this and (9), and by following arguments similar to case (i)'s proof, we will arrive at the same contradiction.

This also completes the proof of the theorem.

3. Proof of Theorem 2

We first prove the following key point in the verification of Theorem 2.

THEOREM 4. *Take non-negative integers n, n_1, \dots, n_k with $n_1 + \dots + n_k \geq 1$ and define*

$$(15) \quad d = n + n_1 + \dots + n_k, \quad w = n + 2n_1 + 3n_2 + \dots + (k+1)n_k.$$

Let f be a transcendental meromorphic function and set

$$(16) \quad \phi = a f^n (f')^{n_1} \dots (f^{(k)})^{n_k} \quad (a \in \mathbb{C} - \{0\}).$$

Then either

$$(17) \quad (3d-2)T(r, f) < (3d+1)N\left(r, \frac{1}{f}\right) + 4\bar{N}\left(r, \frac{1}{\phi-1}\right) - 2N\left(r, \frac{\phi-1}{\phi'}\right) + S(r, f)$$

or ϕ satisfies the identity

$$(18) \quad (w-n)\frac{\phi''}{\phi'} - \left(\frac{n}{w} + w + 1 - n\right)\frac{\phi'}{\phi-1} + 2n\frac{f'}{f} = 0.$$

Proof. If $N_1(r, f)$ counts simple poles of f and if $\bar{N}_2(r, f)$ counts multiple poles of f , each counted only once irrespective of multiplicity, then

$$(19) \quad \bar{N}(r, f) = N_1(r, f) + \bar{N}_2(r, f),$$

$$(20) \quad N_1(r, f) + 2\bar{N}_2(r, f) \leq N(r, f) \leq T(r, f).$$

Note that (see Hu [8])

$$(21) \quad dT(r, f) \leq \bar{N}(r, f) + dN\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{\phi-1}\right) - N\left(r, \frac{\phi-1}{\phi'}\right) + S(r, f).$$

Then (20) and (21) imply

$$(22) \quad (d-1)N_1(r, f) + (2d-1)\bar{N}_2(r, f) \leq dN\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{\phi-1}\right) - N\left(r, \frac{\phi-1}{\phi'}\right) + S(r, f).$$

Suppose that z_0 is a simple pole of f . Then we may write

$$(23) \quad f(z) = \frac{b}{z-z_0} + \sum_{i=0}^{\infty} b_i (z-z_0)^i \quad (b \neq 0)$$

near $z=z_0$. Consequently

$$(24) \quad \phi(z) = \frac{\lambda}{(z-z_0)^w} \left\{ 1 + \frac{nb_0}{b}(z-z_0) + \dots \right\}$$

where

$$(25) \quad \lambda = ab^d(-1)^{w-d} \prod_{i=1}^k (i!)^{n_i}.$$

Then we have

$$(26) \quad \frac{\psi'(z)}{\psi(z)-1} = -\frac{w}{z-z_0} + \frac{nb_0}{b} + \dots$$

$$(27) \quad \frac{\psi''(z)}{\psi'(z)} = -\frac{w+1}{z-z_0} + \frac{n(w-1)b_0}{wb} + \dots$$

near $z=z_0$. Combining (26) and (27) we obtain, near $z=z_0$,

$$(28) \quad w \frac{\psi''(z)}{\psi'(z)} - (w+1) \frac{\psi'(z)}{\psi(z)-1} = -2 \frac{nb_0}{b} + O(z-z_0).$$

Noting that, near $z=z_0$,

$$(29) \quad \frac{f'(z)}{f(z)} = -\frac{1}{z-z_0} + \frac{b_0}{b} + \dots.$$

Then

$$(30) \quad w \frac{f'(z)}{f(z)} - \frac{\psi'(z)}{\psi(z)-1} = (w-n) \frac{b_0}{b} + O(z-z_0).$$

By (28), (30), it follows that, near $z=z_0$,

$$(31) \quad h(z) := w(w-n) \frac{\psi''(z)}{\psi'(z)} - \{(w-n)(w+1)+2n\} \frac{\psi'(z)}{\psi(z)-1} + 2nw \frac{f'(z)}{f(z)} \\ = O(z-z_0).$$

Now suppose that (18) is not true, i.e., $h \not\equiv 0$. Then $h(z_0)=0$, and thus we have

$$(32) \quad N_1(r, f) \leq N\left(r, \frac{1}{h}\right).$$

Applying Jensen's formula to h , (32) yields

$$(33) \quad N_1(r, f) \leq m(r, h) + N(r, h) + O(1) = N(r, h) + S(r, f).$$

Note that $h(z)$ can only have simple poles at zeros and poles of ψ' , $\psi-1$ and f which are not simple poles of f . Thus

$$(34) \quad N(r, h) \leq \bar{N}_2(r, f) + \varepsilon \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{\psi-1}\right) + \bar{N}\left(r, \frac{\psi-1}{\psi'}\right)$$

where $\varepsilon=1$ if $n \neq 0$, and $\varepsilon=0$ if $n=0$. Therefore by (22), (33) and (34), we have

$$(35) \quad (3d-2)N_1(r, f) \leq dN\left(r, \frac{1}{f}\right) + \varepsilon(2d-1)\bar{N}\left(r, \frac{1}{f}\right) + 2d\bar{N}\left(r, \frac{1}{\phi-1}\right) \\ + 2(d-1)N\left(r, \frac{\phi-1}{\phi'}\right) + S(r, f).$$

Multiply (35) by $d/(3d-2)$ and add to (22) to obtain

$$(36) \quad (3d-2)\bar{N}(r, f) \leq 2dN\left(r, \frac{1}{f}\right) + \varepsilon d\bar{N}\left(r, \frac{1}{f}\right) + (d+2)\bar{N}\left(r, \frac{1}{\phi-1}\right) \\ + (d-2)N\left(r, \frac{\phi-1}{\phi'}\right) + S(r, f).$$

Hence (21) and (36) yield

$$(37) \quad (3d-2)T(r, f) \leq 3dN\left(r, \frac{1}{f}\right) + \varepsilon\bar{N}\left(r, \frac{1}{f}\right) + 4\bar{N}\left(r, \frac{1}{\phi-1}\right) \\ - 2N\left(r, \frac{\phi-1}{\phi'}\right) + S(r, f).$$

Now (17) follows from (37).

Q. E. D.

Now we proceed to prove Theorem 2 by contradiction.

Assume that there exists $c \in C - \{0\}$ such that $f^n(f')^{n_1} \dots (f^{(k)})^{n_k} - c$ has only finitely many zeros. Set $a = 1/c$ and define ψ by (16). Then

$$(38) \quad N\left(r, \frac{1}{\psi-1}\right) = O(\log r) = S(r, f).$$

If (17) is true, then

$$(3d-2)T(r, f) < (3d+1)N\left(r, \frac{1}{f}\right) + S(r, f).$$

It follows that

$$(3d-2) \leq (3d+1)(1 - \delta(0, f))$$

i.e., $\delta(0, f) \leq 3/(3d+1)$, which is a contradiction.

Now suppose that (18) is true. Integrating (18) to obtain

$$(39) \quad (\psi')^m / (\psi-1)^l = c_1 / f^{2n}$$

where $m = w - n$, $l = (n/w) + w + 1 - n$, $c_1 \in C - \{0\}$. Solving (39) to obtain

$$(40) \quad \frac{1}{(\psi-1)^{w+n}} = \left\{ c_3 + c_2 \int^z \frac{ds}{f^{2n(w-n)}(s)} \right\}^{w(w-n)}.$$

Then (38) and (40) (or (39)) yield

$$(41) \quad N\left(r, \frac{1}{f}\right) = O\left\{N\left(r, \frac{1}{\psi-1}\right)\right\} = S(r, f).$$

Hence (21), (38) and (41) imply

$$(d-1)T(r, f) \leq S(r, f)$$

which is impossible.

Thus we complete the proof of Theorem 2.

4. Proof of Theorem 3

Set $b=1/c$ and consider the following differential polynomial

$$(42) \quad \phi = bfP(f)$$

where $P(f)$ is defined by (2). Define

$$(43) \quad h = -\frac{\phi'}{\phi-1}, \quad B_0 = -\frac{1}{f} \left\{ K_0 f'' + \left(h - \frac{h'}{h} \right) f' \right\}$$

and

$$(44) \quad \begin{aligned} F = & (K_1 - K_0 K_2) h^4 + (K_0 K_3 + K_1 - K_0 K_1) (h')^2 \\ & + (2K_0^2 - K_0 K_1 + K_0 K_2 - K_0 K_3 - 2K_1) h^2 h' \\ & + (K_0 K_1 - K_0^2) h h'' - K_0 K_1 h^2 B_0. \end{aligned}$$

According to W. Hennekemper [7], we prove the following properties:

- (iii) If z_0 is a p -fold zero of f , then z_0 is at least a p -fold zero of F ;
- (iv) F don't vanish identically;
- (v) $T(r, F) \leq 4\bar{N}(r, f) + 4\bar{N}(r, 1/(\phi-1)) + S(r, f)$.

Firstly, we prove (iii). Trivially one see

$$(45) \quad \phi' + h\phi - h = 0.$$

Differentiating (45) successively we obtain

$$(46) \quad \phi'' + h\phi' + h'\phi - h' = 0,$$

$$(47) \quad \phi''' + h\phi'' + 2h'\phi' + h''\phi - h'' = 0.$$

If $p=1$, then by (42) and (2), we obtain

$$(48) \quad \phi(z_0) = 0,$$

$$(49) \quad \phi'(z_0) = b(f'(z_0))^{k+1},$$

$$(50) \quad \phi''(z_0) = bK_0(f'(z_0))^k f''(z_0),$$

$$(51) \quad \phi'''(z_0) = bK_1(f'(z_0))^k f'''(z_0) + bK_0 K_3(f'(z_0))^{k-1} (f''(z_0))^2.$$

By (45), (48) and (49), we have

$$(52) \quad h(z_0) = b(f'(z_0))^{k+1}.$$

By (46), (48)-(50) and (52), then

$$(53) \quad h'(z_0) = (h(z_0))^2 + K_0 h(z_0) f''(z_0) / f'(z_0)$$

and (47)-(53) yield

$$(54) \quad h''(z_0) = K_1 h(z_0) \frac{f'''(z_0)}{f'(z_0)} + (K_2 h(z_0)^2 + K_3 h'(z_0)) \frac{f''(z_0)}{f'(z_0)} + 2h(z_0) h'(z_0).$$

Note that B_0 is regular at $z=z_0$ according to (52) and (53), and that

$$(55) \quad B_0(z_0) = -K_0 \frac{f'''(z_0)}{f'(z_0)} - \left(h(z_0) - \frac{h'(z_0)}{h(z_0)} \right) \frac{f''(z_0)}{f'(z_0)} - h'(z_0) + \frac{h''(z_0)}{h(z_0)} - \left(\frac{h'(z_0)}{h(z_0)} \right)^2.$$

Substituting (52)-(55) into (44), by simple computation we see $F(z_0)=0$.

If $p \geq 2$, then z_0 is at least $(k+1)(p-1)$ -fold zero of h . Hence z_0 is at least zero point of F of order $2(k+1)(p-1)-2(\geq p)$. Thus we obtain (iii).

Next we prove (iv). If z_0 is a pole of f of order p , writing

$$f(z) = \frac{a}{(z-z_0)^p} + \dots \quad (a \in C - \{0\}),$$

near $z=z_0$. We have, near $z=z_0$,

$$\phi = a \tilde{a} / (z-z_0)^{k(p+1)+p} + \dots$$

where

$$(56) \quad \tilde{a} = b(-a)^k p^{k-2} \{ (1+c_1+c_2+c_3)p^2 + (c_1+2c_2+3c_3)p + c_2+2c_3 \}.$$

By the condition (i), we see $\tilde{a} \neq 0$. Hence, near $z=z_0$, we have Laurent expansions

$$h = (k(p+1)+p)/(z-z_0) + \dots$$

$$B_0 = \{ p(k(p+1)+p+1) - K_0 p(p+1) \} / (z-z_0)^2 + \dots$$

$$F = a' / (z-z_0)^4 + \dots$$

where

$$(57) \quad a' = (k(p+1)+p)^2(p+1) \{ A(p+1) - B \}.$$

The condition (ii) implies $a' \neq 0$, thus $F \neq 0$.

Now we proceed to prove (iv) for entire functions f by contradiction. Assume that there exists an entire function f such that $F \equiv 0$. Then we have

$$(58) \quad \begin{aligned} & K_0^2 K_1 \phi(\phi')^2 f'' - K_0 K_1 \phi \phi' \phi'' f' + (K_0 K_3 - K_0 K_1 + K_1) \phi(\phi'')^2 f \\ & + (K_0^2 - K_0 K_3 - K_0 K_2) (\phi')^2 \phi'' f + (K_0 K_1 - K_0^2) \phi \phi' \phi''' f \\ & - K_0^2 K_1 (\phi')^2 f'' + K_0 K_1 \phi' \phi'' f' - (K_0 K_3 + K_1 - K_0 K_1) (\phi'')^2 f \\ & - (K_0 K_1 - K_0^2) \phi' \phi''' f \equiv 0. \end{aligned}$$

Note that the sum of the coefficients of ϕ is

$$a_0 = b(1 + c_1 + c_2 + c_3) \neq 0.$$

Obviously, the sum of the coefficients of $\phi^{(t)}$ is $(k+1)^t a_0$. Hence the sum of the coefficients of former five summands in (58) is

$$\begin{aligned} \sigma_1 &= a_0^3 \{K_0^2 K_1 (k+1)^2 - K_0 K_1 (k+1)^3 + (K_0 K_3 - K_0 K_1 + K_1) (k+1)^4 \\ &\quad + (K_1^2 - K_0 K_3 - K_0 K_2) (k+1)^4 + (K_0 K_1 - K_0^2) (k+1)^4\} \\ &= a_0^3 (k+1)^2 A \neq 0. \end{aligned}$$

Let σ_2 denote the sum of the coefficients of later four summands in (58). By using the theory of Wiman-Valiron (see [13]), there exists complex sequences $\{\xi_n\}$, $\{\delta_n\}$ and $\{\Delta_n\}$ ($n=1, 2, \dots$) such that

$$\lim_{n \rightarrow \infty} |\xi_n| = \infty, \quad \lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} \Delta_n = 0,$$

and for all $n \in N$,

$$(60) \quad 0 < |f(\xi_n)| = M(|\xi_n|, f), \quad \nu(|\xi_n|) \geq 1,$$

$$(61) \quad \left(\frac{\nu(|\xi_n|)}{\xi_n} \right)^{3k+4} f^{3k+4}(\xi_n) \{\sigma_1 + \delta_n\} + \left(\frac{\nu(|\xi_n|)}{\xi_n} \right)^{2k+4} f^{2k+3}(\xi_n) \{\sigma_2 + \Delta_n\} = 0$$

where we apply the formula (8) of Chapter I from [13], into (58) to obtain (61). Thus for all $n \in N$, we have

$$\sigma_1 + \delta_n = -(\sigma_2 + \Delta_n) \left(\frac{\xi_n}{\nu(|\xi_n|)} \right)^k / f^{k+1}(\xi_n).$$

which implies $\sigma_1 = 0$ since f is transcendental with

$$\lim_{n \rightarrow \infty} \frac{\xi_n^k}{f^{k+1}(\xi_n)} = 0.$$

We get a contradiction. This completes the proof of (iv).

By the definition of F , we see

$$(62) \quad m(r, F) = S(r, f).$$

Note that poles of F only come from poles of f and zeros of $\phi - 1$; its multiplicities are 4 at most. Hence

$$(63) \quad N(r, F) \leq 4\bar{N}(r, f) + 4\bar{N}\left(r, \frac{1}{\phi-1}\right).$$

Thus (v) follows.

Finally, we prove Theorem 3. By (iii), (iv) and (v), we have

$$(64) \quad N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{F}\right) \leq T(r, F) + O(1) \leq 4\bar{N}\left(r, \frac{1}{\phi-1}\right) + 4\bar{N}(r, f) + S(r, f)$$

Theorem 1 of Hu [8] yields

$$(65) \quad T(r, f) \leq \frac{1}{k+1} \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + \frac{1}{k+1} \bar{N}\left(r, \frac{1}{\phi-1}\right) + S(r, f).$$

Then (6) follows from (64) and (65).

Q. E. D.

References

- [1] BERGWEILER, W., On the product of a meromorphic function and its derivative, to appear.
- [2] BERGWEILER, W. AND EREMENKO, E., On the singularities of the inverse of a meromorphic function of finite order, *Rev. Mat. Iberoamericana*, **11** (1995), 355–373.
- [3] CLUNIE, J., On a result of Hayman, *J. London Math. Soc.*, **42** (1967), 389–392.
- [4] HAYMAN, W.K., Picard values of meromorphic functions and their derivatives, *Ann. of Math.*, **70** (1959), 9–42.
- [5] HAYMAN, W.K., *Research Problems in Function Theory*, The Athlone Press, London, 1967.
- [6] HAYMAN, W.K., *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
- [7] HENNEKEMPER, W., Über die Wertverteilung von $(f^{k+1})^{(k)}$, *Math. Z.*, **177** (1981), 375–380.
- [8] HU, P.C., An extension of Hayman's inequality, *J. Math. (Wuhan)*, **10** (1990), 405–412.
- [9] MUES, E., Über ein Problem von Hayman, *Math. Z.*, **164** (1979), 239–259.
- [10] SONS, L.R., Deficiencies of monomials, *Math. Z.*, **111** (1969), 53–68.
- [11] STEINMETZ, N., Über die Nullstellen von differentiapolynomen, *Math. Z.*, **176** (1981), 255–264.
- [12] WITTICH, H., Über ein Borelsche Identität, *Math. Z.*, **84** (1964), 233–243.
- [13] WITTICH, H., *Neuere Untersuchungen Über eindeutige analytische Funktionen*, Berlin-Göttingen-Heidelberg, Springer, 1955.
- [14] YANG, L. AND YANG, C.C., Angular distribution of values of ff' , *Sci. China Ser. A*, **37** (1994), 284–294.
- [15] YANG, C.C., YANG, L. AND WANG, Y.F., On the zeros of $(f^k)^n f - a$, *Chinese Sci. Bull.*, **38** (1993), 2125–2128.
- [16] ZHANG, Q., On the value distribution of $\varphi(z)f(z)f'(z)$, *Acta Math. Sinica*, **37** (1994), 36–98.
- [17] ZHANG, Z.L. AND LI, W., On monomials and Hayman's problem, *Kodai Math. J.*, **16** (1993), 44–51.

DEPARTMENT OF MATHEMATICS
THE HONG KONG UNIVERSITY OF SCIENCE & TECHNOLOGY
CLEAR WATER BAY, KOWLOON, HONG KONG