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# ON THE ORDER OF HOLOMORPHIC CURVES WITH MAXIMAL DEFICIENCY SUM

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## 1. Introduction

Let

$$f : \boldsymbol{C} \to P^n(\boldsymbol{C})$$

be a holomorphic curve from $\boldsymbol{C}$ into the $n$-dimensional complex projective space $P^n(\boldsymbol{C})$, where $n$ is a positive integer, and let

$$(f_1, \cdots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$$

be a reduced representation of $f$. We then write $f=[f_1, \cdots, f_{n+1}]$.

For a vector $\boldsymbol{a}=(a_1, \cdots, a_{n+1})$ in $\boldsymbol{C}^{n+1}$, we write

$$(\boldsymbol{a}, f)=\sum_{j=1}^{n+1} a_j f_j \quad \text{and} \quad \|\boldsymbol{a}\|=\left\{\sum_{j=1}^{n+1} |a_j|^2\right\}^{1/2},$$

and put

$$\|f(z)\|=\left\{\sum_{j=1}^{n+1} |f_j(z)|^2\right\}^{1/2}.$$

Then we define as usual the characteristic function of $f$ as follows.

$$T(r, f)=\frac{1}{2\pi}\int_0^{2\pi} \log\|f(re^{i\theta})\|d\theta - \log\|f(0)\|.$$

In addition, put

$$U(z)=\max_{1 \leq j \leq n+1} |f_j(z)|,$$

then

$$U(z) \leqq \|f(z)\| \leqq (n+1)^{1/2} U(z)$$

and we have

$$(1) \qquad T(r, f)=\frac{1}{2\pi}\int_0^{2\pi} \log U(re^{i\theta})d\theta + O(1) \qquad \text{(see [1]).}$$

We suppose that $f$ is transcendental; that is to say,

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$$\lim_{r\to\infty}\frac{T(r,\ f)}{\log r}=+\infty\,.$$

We denote the order of $f$ by $\rho(f)$ and the lower order of $f$ by $\mu(f)$, respectively:

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,\ f)}{\log r}\quad\text{and}\quad \mu(f)=\liminf_{r\to\infty}\frac{\log T(r,\ f)}{\log r}\,.$$

It is said that $f$ is of regular growth if $\rho(f)=\mu(f)$.

We write for $\boldsymbol{a}=(a_1,\ \cdots,\ a_{n+1})$ in $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ such that $(\boldsymbol{a},\ f)\not\equiv 0$

$$m(r,\ \boldsymbol{a},\ f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\|f\|}{|(\boldsymbol{a},\ f)|}d\theta\quad\text{and}\quad N(r,\ \boldsymbol{a},\ f)=N\left(r,\ \frac{1}{(\boldsymbol{a},\ f)}\right).$$

Then we have

$$T(r,\ f)=N(r,\ \boldsymbol{a},\ f)+m(r,\ \boldsymbol{a},\ f)+O(1) \tag{2}$$

(the first fundamental theorem (see [13], p. 76)).

We call the quantity

$$\delta(\boldsymbol{a},\ f)=1-\limsup_{r\to\infty}\frac{N(r,\ \boldsymbol{a},\ f)}{T(r,\ f)}$$
$$=\liminf_{r\to\infty}\frac{m(r,\ \boldsymbol{a},\ f)}{T(r,\ f)}$$

the deficiency of $\boldsymbol{a}$ with respect to $f$. It is easy to see that

$$0\leqq\delta(\boldsymbol{a},\ f)\leqq 1$$

by (2) since $m(r,\ \boldsymbol{a},\ f)\geqq 0$. Put

$$\lambda=\dim\{(c_1,\ \cdots,\ c_{n+1})\in\boldsymbol{C}^{n+1}:\ c_1f_1+\ \cdots\ c_{n+1}f_{n+1}=0\},$$

then it is easy to see that $0\leqq\lambda\leqq n-1$. We say that $f$ is (linearly) non-degenerate if $\lambda=0$ and that $f$ is (linearly) degenerate if $\lambda>0$.

It is well-known that $f$ is non-degenerate if and only if the Wronskian $W(f_1,\ \cdots,\ f_{n+1})$ of $f_1,\ \cdots,\ f_{n+1}$ is not identically equal to 0.

Let $X$ be a subset of $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ in general position; that is to say, any $n+1$ vectors of $X$ are linearly independent. Then it is well-known that the following defect relation is easily obtained from the fundamental inequality of H. Cartan ([1]):

The defect relation. If $f$ is non-degenerate,

$$\sum_{\boldsymbol{a}\in X}\delta(\boldsymbol{a},\ f)\leqq n+1\,. \tag{3}$$

As a generalization of the case of meromorphic functions to holomorphic curves, it is natural to ask the following problem:

PROBLEM. What properties does $f$ possess if the equality holds in (3)?

Our main purpose of this paper is to generalize the following well-known result to holomorphic curves, which gives an answer to a special case of this problem.

THEOREM A. *Let $f(z)$ be a transcendental meromorphic function of order finite in the complex plane. If*

$$\delta(\infty, f)=1 \quad \text{and} \quad \sum_{a\neq\infty}\delta(a, f)=1$$

*then $f$ is of regular growth and the order of $f(z)$ is a positive integer* ([2], p. 299).

To prove Theorem A, the following result is essential.

THEOREM B. *Let $f(z)$ be as in Theorem A. Then for any $a_1, \cdots, a_q\in \boldsymbol{C}$ $(q<\infty)$,*

$$\sum_{j=1}^{q} m(r, a_j, f)\leqq m(r, 1/f')+O(\log r)$$

(see [3], p. 89).

Our method to obtain a generalization of Theorem A is parallel to the case of meromorphic functions. We shall first generalize Theorem B by using the derived holomorphic curve introduced in [12] as an extension of the derivative of meromorphic functions to holomorphic curves and then we shall give a generalization of Theorem A.

The first attempt to extend Theorem A to holomorphic curves is the following result due to Mori ([4]).

THEOREM C. *Suppose that $f$ is non-degenerate and $\rho(f)<+\infty$. If there exist $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+1\leqq q\leqq+\infty)$ in $X$ such that*

*(i) the order of $N(r, \boldsymbol{a}_j, f)$ is smaller than $\rho(f)$ for $j=1, \cdots, n$,*

*(ii) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)=n+1$,*

*then $\rho(f)$ is a positive integer.*

*Remark* 1. If (i) and (ii) of this theorem hold, then $\delta(\boldsymbol{a}_j, f)=1$ $(j=1, \cdots, n)$ (see [4], Remark 2).

We prepare several lemmas in Section 2 and give a generalization of Theorem A for non-degenerate holomorphic curves in Section 3, which contains Theorem C. In Section 4, we extend a result obtained in Section 3 to moving targets. In Section 5, we treat the degenerate case.

We use the standard notation of the Nevanlinna theory of meromorphic functions ([3], [6]).

## 2. Lemma

We shall give some lemmas in this section for later use. Let $f$ and $X$ be as in Section 1.

LEMMA 1. (a) $T(r, f_k/f_j)<T(r, f)+O(1)$ $(k\neq j)$ ([1]).
(b) *For any* $\boldsymbol{a}, \boldsymbol{b}$ *in* $X$ *such that* $(\boldsymbol{a}, f)\not\equiv 0$ *add* $(\boldsymbol{b}, f)\not\equiv 0$,

$$T(r, (\boldsymbol{a}, f)/(\boldsymbol{b}, f))<T(r, f)+O(1) \qquad ([1]).$$

LEMMA 2. *If there are* $n+1$ *elements* $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{n+1}$ *in* $X$ *such that*

$$\delta(\boldsymbol{a}_j, f)=1 \quad (j=1, \cdots, n+1),$$

*then* $f$ *is of regular growth and* $\rho(f)$ *is equal to either a positive integer or infinity* ([11], Théorème 3).

Put for any $\boldsymbol{a}_j\in X$ $(j=1, \cdots, n+1)$

$$K(f)=\limsup_{r\to\infty}\frac{\sum_{j=1}^{n+1} N(r, \boldsymbol{a}_j, f)}{T(r, f)}$$

(see [10], Definition 3). Then we have the followings.

LEMMA 3. (I) *If* $\rho=\rho(f)$ *is finite and non-integer,*

$$K(f)\geqq\frac{|\sin\pi\rho|}{2\cdot 2\rho+|\sin\pi\rho|/2} \qquad ([11], \text{Théorème } 1).$$

(II) *If* $\mu(f)<\rho(f)$, *for any* $\tau\neq\infty$ *such that* $\mu(f)\leqq\tau\leqq\rho(f)$

$$K(f)\geqq\frac{n+1}{n}\cdot\frac{|\sin\pi\tau|}{4\cdot 4e(\tau+1)+|\sin\pi\tau|} \qquad ([11], \text{Théorème } 4).$$

Note that $f$ is not always non-degenerate in these two lemmas.

Suppose now that $f$ is non-degenerate. Let $d(z)$ be an entire function such that the functions

$$f_j^{n+1}/d \quad (j=1, \cdots, n) \quad \text{and} \quad W(f_1, \cdots, f_{n+1})/d$$

are entire functions without common zeros.

DEFINITION ([12]). We call the holomorphic curves induced by the mapping

$$(f_1^{n+1}, \cdots, f_n^{n+1}, W(f_1, \cdots, f_{n+1})): \boldsymbol{C}\to\boldsymbol{C}^{n+1}$$

the derived holomorphic curve of $f$ and we write it by $f^*$:

$$f^*=[f_1^{n+1}/d, \cdots, f_n^{n+1}/d, W(f_1, \cdots, f_{n+1})/d].$$

*Remark* 2. When $n=1$, $f^*$ corresponds exactly to the derivative of the meromorphic function $f_2/f_1$.

*Remark* 3. The definition of $f^*$ does not depend on the choice of a reduced representation of $f$ (Proposition 1 ([12])).

LEMMA 4. *When* $\rho(f)<\infty$,

$$T(r, f^*)\leqq(n+1)T(r, f)-N(r, 1/d)+O(\log r)$$

(Lemma 3 ([12])).

In addition, $f^*$ has the following properties:

PROPOSITION 1 ([12]). (a) *$f^*$ is transcendental.* (b) $\rho(f^*)=\rho(f)$. (c) *$f^*$ is not always non-degenerate.*

## 3. Non-degenerate case

Let $f=[f_1, \cdots, f_{n+1}]$ and $X$ be as in Section 1. We shall give a generalization of Theorem A when $f$ is non-degenerate in this section. We need another lemma.

LEMMA 5. *Suppose that $f$ is non-degenerate and $\rho(f)<\infty$. For any $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+1\leqq q<\infty)$ of $X$, we have*

$$(q-n-1)T(r, f)<\sum_{j=1}^{q} N(r, \boldsymbol{a}_j, f)-N(r, 1/W(f_1, \cdots, f_{n+1}))+O(\log r)$$

(see [1]).

*Proof.* We have only to change slightly the proof of the fundamental inequality of Cartan ([1], p. 12-p. 15). We make use of the formula

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{F_1\cdots F_q}{W(f_1, \cdots, f_{n+1})}\right|d\theta$$
$$=\sum_{j=1}^{q} N\left(r, \frac{1}{F_j}\right)-N\left(r, \frac{1}{W(f_1, \cdots, f_{n+1})}\right)+O(1)$$

instead of the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{F_1\cdots F_q}{W(f_1, \cdots, f_{n+1})}\right|\leqq\sum_{j=1}^{q} N_n(r, F_j)+O(1)$$

used in [1], where $F_j=(\boldsymbol{a}_j, f)$.

Since the error term $S(r)$ used in [1] is equal to a finite sum of integrals of the form

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{W(h_1', \cdots, h_n')}{h_1\cdots h_n}\right|d\theta+O(1)\leqq\sum_{j=1}^{n}\sum_{k=1}^{n-1}m\left(r, \frac{h_j^{(k)}}{h_j}\right)+O(1),$$

where $h_j$ is a ratio of the form $F_{j_1}/F_{j_2}$ $(j_1\neq j_2)$, it is easy to see that

$$S(r)=O(\log r)\quad(r\to\infty)$$

since $h_j$ is of order finite by Lemma 1 (b) and

$$m(r, h_j^{(k)}/h_j)=O(\log r)\quad(k=1, \cdots, n-1)\qquad\text{(see [6]).}$$

COROLLARY 1. *Under the same condition as in Lemma* 5, *if the equality holds in* (3), *then*

$$\lim_{r\to\infty}\frac{N(r, 1/W(f_1, \cdots, f_{n+1}))}{T(r, f)}=0.$$

Let $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}\}$ be the standard basis of $\boldsymbol{C}^{n+1}$ and put

$$X_0=\{\boldsymbol{a}=(a_1, \cdots, a_{n+1})\in X: a_{n+1}=0\}.$$

Since $X$ is in general position, $\#X_0\leqq n$.

We shall generalize Theorem B first.

THEOREM 1. *Suppose that $f$ is non-degenerate and $\rho(f)<\infty$. For any $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(1\leqq q<\infty)$ in $X-X_0$, we have the following inequality:*

$$\sum_{j=1}^{q}m(r, \boldsymbol{a}_j, f)\leqq m(r, \boldsymbol{e}_{n+1}, f^*)+O(\log r).$$

*Proof.* We put

$$(\boldsymbol{a}_j, f)=F_j\quad(j=1, \cdots, q)\quad\text{and}\quad u(z)=\max_{1\leq j\leq n}|f_j(z)|$$

and for any $z$ $(\neq 0)$ arbitrarily fixed, let

$$|F_{j_1}(z)|\leqq|F_{j_2}(z)|\leqq\cdots\leqq|F_{j_q}(z)|\quad(1\leqq j_1, \cdots, j_q\leqq q).$$

Then there is a positive constant $K$ such that

$$U(z)\leqq K|F_{j_k}(z)|\quad(k=n+1, \cdots, q)$$

(Lemma in [1], p. 11),

$$|F_{j_k}(z)|\leqq KU(z)\quad(k=1, \cdots, q)$$

and since the $n+1$-th elements of $\boldsymbol{a}_j$ are different from zero,

$$|f_{n+1}(z)|\leqq K\{u(z)+|F_{j_k}(z)|\}\quad(k=1, \cdots, q).$$

(From now on we denote by $K$ a positive number, which may be different from each other in each case where it appears.)

(I) The case when $u(z) \leqq |F_{j_1}(z)|$.

Since $\|f\| \leqq K|F_{j_1}(z)|$ in this case, we have

$$\prod_{j=1}^{q} \frac{\|a_j\|\|f\|}{|F_j|} \leqq K. \tag{4}$$

(II) The case when $|F_{j_1}(z)| < u(z)$.

Since

$$\|f\| \leqq K\{|f_1|^2 + \cdots + |f_n|^2 + |F_{j_1}|^2\}^{1/2} \leqq K(n+1)^{1/2} u(z)$$

in this case, we have

$$\begin{aligned}\prod_{j=1}^{q} \frac{\|a_j\|\|f\|}{|F_j|} &\leqq K \prod_{k=1}^{n+1} \frac{u(z)}{|F_{j_k}(z)|} = K \frac{u(z)^{n+1}}{|W(f_1, \cdots, f_{n+1})|} \cdot \frac{|W(f_1, \cdots, f_{n+1})|}{|F_{j_1} \cdots F_{j_{n+1}}|} \\ &= K \frac{u(z)^{n+1}}{|W(f_1, \cdots, f_{n+1})|} \cdot \frac{|W(F_{j_1}, \cdots, F_{j_{n+1}})|}{|F_{j_1} \cdots F_{j_{n+1}}|}\end{aligned} \tag{5}$$

since $W(F_{j_1}, \cdots, F_{j_{n+1}}) = cW(f_1, \cdots, f_{n+1})$ ($c \neq 0$, constant).

From (4) and (5) we obtain the inequality

$$\begin{aligned}&\sum_{j=1}^{q} m(r, a_j, f) \\ &\leqq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{u(z)^{n+1}}{|W(f_1, \cdots, f_{n+1})|} d\theta + \sum_{(j_1, \cdots, j_{n+1})} m\left(r, \frac{W(F_{j_1}, \cdots, F_{j_{n+1}})}{F_{j_1} \cdots F_{j_{n+1}}}\right) + O(1) \\ &\leqq m(r, e_{n+1}, f^*) + S(r, f),\end{aligned}$$

where $\sum_{(j_1, \cdots, j_{n+1})}$ is the summation taken over all combinations $(j_1, \cdots, j_{n+1})$ chosen from $\{1, \cdots, q\}$ and

$$\begin{aligned}S(r, f) &= \sum_{(j_1, \cdots, j_{n+1})} m\left(r, \frac{W(F_{j_1}, \cdots, F_{j_{n+1}})}{F_{j_1} \cdots F_{j_{n+1}}}\right) + O(1) \\ &= O(\log r)\end{aligned}$$

as in the case of Lemma 5. Thus, our proof is complete.

COROLLARY 2. *Let $f$ be as in Theorem* 1. *Then we have*

$$\frac{1}{n+1} \sum_{a \in X - X_0} \delta(a, f) \leqq \delta(e_{n+1}, f^*), \tag{6}$$

$$\sum_{a \in X - X_0} \delta(a, f) \leqq \liminf_{r \to \infty} \frac{T(r, f^*)}{T(r, f)} \leqq \limsup_{r \to \infty} \frac{T(r, f^*)}{T(r, f)} \leqq n+1. \tag{7}$$

We can easily prove this corollary by Lemma 4 and Theorem 1.

Now, we can prove a generalization of Theorem A, which contains Theorem C.

THEOREM 2. *Suppose that $f$ is non-degenerate, $\rho(f)<\infty$ and*

(i) $\delta(\boldsymbol{e}_j, f)=1$ $(j=1, \cdots, n)$.

*If there exist $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+1\leqq q\leqq\infty)$ in $X$ such that*

(ii) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)=n+1$,

*then $f$ is of regular growth and $\rho(f)$ is equal to a positive integer.*

*Proof.* Suppose that $X_0$ consists of $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_l$. Then, $0\leqq l\leqq n$. By Corollary 1, we have from (ii)

$$\lim_{r\to\infty}\frac{N(r, 1/W(f_1, \cdots, f_{n+1}))}{T(r, f)}=0. \tag{8}$$

By (7) and (ii), we have

$$1\leqq n+1-l\leqq\sum_{j=l+1}^{q}\delta(\boldsymbol{a}_j, f)\leqq\liminf_{r\to\infty}\frac{T(r, f^*)}{T(r, f)}\leqq\limsup_{r\to\infty}\frac{T(r, f^*)}{T(r, f)}\leqq n+1. \tag{9}$$

This relation (9) implies that $f^*$ is transcendental,

$$\rho(f^*)=\rho(f) \quad\text{and}\quad \mu(f^*)=\mu(f). \tag{10}$$

From (8) and (9), we have

$$\delta(\boldsymbol{e}_{n+1}, f^*)=1 \tag{11}$$

and from (i) and (9)

$$\delta(\boldsymbol{e}_j, f^*)=1 \quad (j=1, \cdots, n). \tag{12}$$

By Lemma 2, (10), (11) and (12) imply that $f$ is of regular growth and $\rho(f)$ is a positive integer since the set $\{e_j\}_{j=1}^{n+1}$ is in general position.

COROLLARY 3. *Suppose that $f$ is non-degenerate and $\rho(f)<\infty$. If there are $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ in $X$ $(n+1\leqq q\leqq\infty)$ such that*

(i) $\delta(\boldsymbol{a}_j, f)=1$ $(j=1, \cdots, n)$;

(ii) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)=n+1$,

*then $f$ is of regular growth and $\rho(f)$ is equal to a positive integer.*

*Proof.* Put

$$(\boldsymbol{a}_j, f)=F_j \quad (j=1, \cdots, q)$$

and let $M$ be the $(n+1)\times(n+1)$ matrix whose $j$-th row is $\boldsymbol{a}_j$ $(j=1, \cdots, n+1)$.

Then $F_1, \cdots, F_{n+1}$ are linearly independent and have no common zeros, $M$ is a regular matrix and

$${}^t(F_1, \cdots, F_{n+1})=M\,{}^t(f_1, \cdots, f_{n+1}).$$

Let $F$ be the holomorphic curve induced by $(F_1, \cdots, F_{n+1})$; that is to say, $F=$

$[F_1, \cdots, F_{n+1}]$. Then

(13) $$T(r, F)=T(r, f)+O(1) \qquad ([1], \text{ p. } 9),$$

and so $F$ is transcendental, $\rho(F)=\rho(f)$ and $\mu(F)=\mu(f)$.

Put

$$Y=\{\boldsymbol{b}=\boldsymbol{a}M^{-1} : \boldsymbol{a}\in X\}.$$

Then, $Y$ is in general position, $(\boldsymbol{a}, f)=(\boldsymbol{b}, F)$ and by (13)

(14) $$\delta(\boldsymbol{a}, f)=\delta(\boldsymbol{b}, F),$$

where $\boldsymbol{b}=\boldsymbol{a}M^{-1}$ $(\boldsymbol{a}\in X)$. Let $Y_0=\{\boldsymbol{b}\in Y :$ the $n+1$-th element of $\boldsymbol{b}=0\}$ and put

$$\boldsymbol{b}_j=\boldsymbol{a}_j M^{-1} \quad (j=1, \cdots, q).$$

Then, $\boldsymbol{b}_j=\boldsymbol{e}_j$ $(j=1, \cdots, n+1)$, $Y_0=\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$ and by (14)

(i)′ $\delta(\boldsymbol{b}_j, F)=1$ $(j=1, \cdots, n)$;

(ii)′ $\sum_{j=1}^{q} \delta(\boldsymbol{b}_j, F)=n+1$.

Let $P(z)$ be an entire function such that the functions

$$F_1^{n+1}/P, \cdots, F_n^{n+1}/P, W(F_1, \cdots, F_{n+1})/P$$

are entire functions without common zeros. Then,

$$F^*=[F_1^{n+1}/P, \cdots, F_n^{n+1}/P, W(F_1, \cdots, F_{n+1})/P].$$

Applying Theorem 2 to $F$ and $Y$, we have this corollary.

## 4. Extension

Let $f=[f_1, \cdots, f_{n+1}]$ be a transcendental holomorphic curve from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$. We use the same notation as in Section 1. Let $S_0(r, f)$ be any quantity satisfying

$$S_0(r, f)=o(T(r, f)) \quad (r\to\infty)$$

and $\Gamma$ the field consisting of meromorphic functions $a$ in $|z|<\infty$ such that $T(r, a)=S_0(r, f)$.

Throughout the section we suppose that $f$ is non-degenerate over $\Gamma$. Let

$$\boldsymbol{S}_0(f)=\left\{\boldsymbol{A}=[a_1, \cdots, a_{n+1}] : \begin{matrix}\text{holomorphic curve from } \boldsymbol{C} \text{ into } P^n(\boldsymbol{C})\\ \text{such that } T(r, \boldsymbol{A})=S_0(r, f)\end{matrix}\right\}$$

and let $H$ be a subset of $\boldsymbol{S}_0(f)$ in general position. It is clear that $\boldsymbol{S}_0(f)\supset P^n(\boldsymbol{C})$. For $\boldsymbol{A}=[a_1, \cdots, a_{n+1}]\in\boldsymbol{S}_0(f)$ we set

$$(\boldsymbol{A}, f)=a_1f_1+\cdots+a_{n+1}f_{n+1}.$$

Then, we have the following

PROPOSITION 2. (a) $a_k/a_j \in \Gamma$ *if* $a_j \not\equiv 0$. (b) $(A, f) \not\equiv 0$.

*Proof.* (a) Applying Lemma 1 (a) to $A$, we have

$$T(r, a_k/a_j) < T(r, A) + O(1) = S_0(r, f).$$

(b) Since there is at least one $a_j \not\equiv 0$ $(1 \leqq j \leqq n+1)$,

$$\frac{(A, f)}{a_j} = \frac{a_1}{a_j} f_1 + \cdots + \frac{a_{n+1}}{a_j} f_{n+1}$$

is a linear combination of $f_1, \cdots, f_{n+1}$ with $\Gamma$-coefficients. As $f$ is non-degenerate over $\Gamma$, $(A, f)/a_j \not\equiv 0$. That is, $(A, f) \not\equiv 0$.

We put

$$m(r, A, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|A\| \|f\|}{|(A, f)|} d\theta,$$

which is non-negative as in Section 1 and independent of the choice of reduced representations of $f$ and $A$, and

$$N(r, A, f) = N(r, 1/(A, f)).$$

Then we have the first fundamental theorem:

$$T(r, f) = m(r, A, f) + N(r, A, f) + S_0(r, f).$$

The defect of $A$ with respect to $f$ is defined as follows:

$$\delta(A, f) = \liminf_{r \to \infty} \frac{m(r, A, f)}{T(r, f)}$$

which is equal to

$$1 - \limsup_{r \to \infty} \frac{N(r, A, f)}{T(r, f)}$$

by the first fundamental theorem. Then, $0 \leqq \delta(A, f) \leqq 1$.

The defect relation ([9], see also [7]):

$$\sum_{A \in H} \delta(A, f) \leqq n+1.$$

Similar to Problem in Section 1, we would like to know what properties $f$ has when the equality of the defect relation holds.

Concerning this, Mori ([5]) has recently proved the following

THEOREM D. *Suppose that* $\rho(f)$ *is finite and that*

$$N(r, 1/f_1) = S_0(r, f) \quad \textit{and} \quad T(r, f_j/f_1) = S_0(r, f) \quad (j = 2, \cdots, n).$$

*If there exist* $A_1, \cdots, A_q$ $(n+1 \leqq q < \infty)$ *in* $H$ *such that*

$$\sum_{j=1}^{q} \delta(A_j, f) = n+1,$$

*then* $f$ *is of regular growth and* $\rho(f)$ *is a positive integer.*

The purpose of this section is to improve this theorem by applying the idea used in the proofs of Theorems 1 and 2 to the case of moving targets in the usual way (see, for example, [7], [9], [5]). We need the following lemma.

LEMMA 6. *For any* $A=[a_1, \cdots, a_{n+1}]$ *and* $B=[b_1, \cdots, b_{n+1}]$ *of* $S_0(f)$ *such that* $a_j \not\equiv 0$, $b_k \not\equiv 0$, *put* $(A, f)=F$ *and* $(B, f)=G$. *Then,*

$$T\left(r, \frac{F/a_j}{G/b_k}\right) \leqq 2nT(r, f) + S_0(r, f).$$

*Proof.* Since

$$\begin{aligned}\frac{F/a_j}{G/b_k} &= \left\{\sum_{\nu=1}^{n+1} (a_\nu/a_j) f_\nu\right\} \Big/ \left\{\sum_{\nu=1}^{n+1} (b_\nu/b_k) f_\nu\right\} \\ &= \left\{\sum_{\nu=1}^{n+1} (a_\nu/a_j)(f_\nu/f_1)\right\} \Big/ \left\{\sum_{\nu=1}^{n+1} (b_\nu/b_k)(f_\nu/f_1)\right\},\end{aligned}$$

$$\begin{aligned}T\left(r, \frac{F/a_j}{G/b_k}\right) &\leqq \sum_{\nu=1}^{n+1} \left\{2T\left(r, \frac{f_\nu}{f_1}\right) + T\left(r, \frac{a_\nu}{a_j}\right) + T\left(r, \frac{b_\nu}{b_k}\right)\right\} + O(1) \\ &\leqq 2nT(r, f) + S_0(r, f)\end{aligned}$$

by Lemma 1 (a) and Proposition 2 (a).

For $A=[a_1, \cdots, a_{n+1}]$ of $H$, let $a_{j_0}$ be the first element not identically equal to zero. Then we put

$$\tilde{A} = \left(\frac{a_1}{a_{j_0}}, \cdots, \frac{a_{n+1}}{a_{j_0}}\right) = (g_1, \cdots, g_{n+1}), \quad \|\tilde{A}\| = \|A\|/|a_{j_0}|, \ \tilde{H} = \{\tilde{A} : A \in H\}$$

and for $(A, f)=F$

$$\tilde{F} = F/a_{j_0} = (\tilde{A}, f) = \sum_{j=1}^{n+1} g_j f_j.$$

Then, it is clear that $\tilde{H}$ is in general position and $g_j = a_j/a_{j_0} \in \Gamma$ by Proposition 2 (a).

Put

$$H_0 = \{A = [a_1, \cdots, a_{n+1}] \in H : a_{n+1} = 0\}.$$

Then we have

THEOREM 3. *Suppose that* $\rho(f) < \infty$ *and that*

(i) $\delta(e_j, f) = 1$ $(j=1, \cdots, n)$.

*If there exist* $A_1, \cdots, A_q$ $(n+1 \leqq q \leqq \infty)$ *in* $H$ *such that*

(ii) $\sum_{j=1}^{q} \delta(A_j, f)=n+1$,

*then,* $f$ *is of regular growth and* $\rho(f)$ *is equal to a positive integer.*

*Proof.* We may suppose without loss of generality that $q \geqq 2n+1$. If the number of the set $Q=\{A \in H: \delta(A, f)>0\}$ is not greater than $2n$, we have only to add a finite number of $A \in H$ such that $\delta(A, f)=0$ to $Q$ so that $q \geqq 2n+1$. This does not affect our result.

Let $\varepsilon$ be any positive number smaller than 1/4. Then, there exists a finite number $\nu$ $(\geqq 2n+1)$ such that

$$\sum_{j=1}^{\nu} \delta(A_j, f)>n+1-\varepsilon . \tag{15}$$

Put for $j=1, \cdots, \nu$

$$A_j=[a_{j_1}, \cdots, a_{j_{n+1}}] \quad \text{and} \quad \tilde{A}_j=(g_{j_1}, \cdots, g_{j_{n+1}}).$$

For any integer $p$, let $V(p)$ be the vector space generated by

$$\left\{\prod_{k=1}^{n+1} \prod_{j=1}^{\nu} g_{jk}^{p(j, k)}: \sum_{k=1}^{n+1} \sum_{j=1}^{\nu} p(j, k) \leqq p,\ p(j, k) \geqq 0 \text{ and integer}\right\}$$

over $\boldsymbol{C}$ and

$$d(p)=\dim V(p).$$

Then, $V(p)$ is a subspace of $V(p+1)$ and

$$\liminf_{p \to \infty} d(p+1)/d(p)=1$$

since $d(p) \leqq \binom{(n+1)\nu+p}{p}$ (see [8], see also [9]).

Note that any element of $V(p)$ belongs to $\Gamma$ since $g_{jk} \in \Gamma$.

Let $p$ be so large that the following inequality holds:

$$d(p+1)/d(p)<1+\varepsilon/(n+1). \tag{16}$$

Let

$$b_1, \cdots, b_{d(p)}, b_{d(p)+1}, \cdots, b_{d(p+1)}$$

be a basis of $V(p+1)$ such that

$$b_1, \cdots, b_{d(p)}$$

form a basis of $V(p)$. Then, it is clear that the functions

$$\{b_t f_k: t=1, \cdots, d(p+1);\ k=1, \cdots, n+1\}$$

are linearly independent over $\boldsymbol{C}$. We put for convenience

$$W=W(b_1 f_1,\ b_2 f_1,\ \cdots,\ b_{d(p+1)} f_{n+1}).$$

Then, we can prove the following inequality as in [7]:

$$(17)\qquad \begin{aligned} &N(r,\ 1/W)+d(p)(\nu-n-1)T(r,\ f)\\ &\leqq d(p)\sum_{j=1}^{\nu} N(r,\ \boldsymbol{A}_j,\ f)+(n+1)\{d(p+1)-d(p)\}T(r,\ f)+S_0(r,\ f)\end{aligned}$$

since $\rho(f)<\infty$ and $\log r=S_0(r,\ f)$.

We put

$$(18)\qquad (\tilde{\boldsymbol{A}}_j,\ f)=\tilde{F}_j \quad (j=1,\ \cdots,\ \nu).$$

Suppose without loss of generality that $H_0$ consists of $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l$. Then, $0\leqq l\leqq n$. Let $z$ be a point of $\boldsymbol{C}-\{0\}$. We rearrange $\{\tilde{F}_j\}_{j=l+1}^{\nu}$ as follows.

$$|\tilde{F}_{j_1}(z)|\leqq|\tilde{F}_{j_2}(z)|\leqq\cdots\leqq|\tilde{F}_{j_n}(z)|\leqq\cdots\leqq|\tilde{F}_{j_{\nu-l}}(z)|,$$

where $l+1\leqq j_1,\ \cdots,\ j_{\nu-l}\leqq\nu$.

From now on we use $S(z,\ f)$ as a non-negative function defined on $\boldsymbol{C}$ such that

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+ S(re^{i\theta},\ f)d\theta=S_0(r,\ f),$$

which may be different from each other in each case when it appears.

It is easy to see by a simple calculation that for $k\geqq n+1$

$$(19)\qquad U(z)\leqq S(z,\ f)|\tilde{F}_{j_k}(z)|$$

and that for $k=1,\ \cdots,\ \nu-l$

$$(20)\qquad |\tilde{F}_{j_k}(z)|\leqq S(z,\ f)U(z),$$

where $U(z)=\max_{1\leqq j\leqq n+1}|f_j(z)|$. We then have the following:

$$(21)\qquad \begin{aligned}\Big(\prod_{j=l+1}^{\nu}\frac{\|\boldsymbol{A}_j\|\|f\|}{|(\boldsymbol{A}_j,\ f)|}\Big)^{d(p)} &=\Big(\prod_{j=l+1}^{\nu}\frac{\|\tilde{\boldsymbol{A}}_j\|\|f\|}{|\tilde{F}_j|}\Big)^{d(p)}\\ &=\Big(\prod_{j=l+1}^{\nu}\|\tilde{\boldsymbol{A}}_j\|\Big)^{d(p)}\Big(\prod_{k=1}^{n}\frac{\|f\|}{|\tilde{F}_{j_k}|}\Big)^{d(p)}\Big(\prod_{k=n+1}^{\nu-l}\frac{\|f\|}{|\tilde{F}_{j_k}|}\Big)^{d(p)}\\ &\leqq S(z,\ f)\Big(\prod_{k=1}^{n}\frac{\|f\|}{|\tilde{F}_{j_k}|}\Big)^{d(p)}\end{aligned}$$

from (19) since $\boldsymbol{A}_j\in H\subset \boldsymbol{S}_0(f)$ and $U\leqq\|f\|\leqq(n+1)^{1/2}U$. We put

$$u(z)=\max_{1\leqq j\leqq n}|f_j(z)|.$$

It then holds that

$$(22)\qquad |f_{n+1}(z)|\leqq S(z,\ f)\{|\tilde{F}_{j_k}(z)|+u(z)\}\quad (k=1,\ \cdots,\ \nu-l)$$

since $a_{j_{n+1}} \not\equiv 0$ for any $A_j \in H - H_0$.

(I) The case when $u(z) \leqq |\tilde{F}_{j_1}(z)|$. In this case, from (22)

$$\|f\| \leqq S(z, f)|\tilde{F}_{j_k}(z)| \quad (k=1, \cdots, n)$$

and we have

$$\left(\prod_{k=1}^{n} \frac{\|f\|}{|\tilde{F}_{j_k}(z)|}\right)^{d(p)} \leqq S(z, f). \tag{23}$$

(II) The case when $|\tilde{F}_{j_1}(z)| < u(z)$. In this case, from (22) for $k=1$

$$\|f\| \leqq S(z, f)u(z)$$

and we have

$$\left(\prod_{k=1}^{n} \frac{\|f\|}{|\tilde{F}_{j_k}(z)|}\right)^{d(p)} \leqq S(z, f) \frac{u(z)^{nd(p)}}{\left(\prod_{k=1}^{n} |\tilde{F}_{j_k}(z)|\right)^{d(p)}}. \tag{24}$$

Now, $\tilde{F}_{j_1}, \cdots, \tilde{F}_{j_{n+1}}$ are linearly independent over $\Gamma$ and it is easy to see that

$$\{b_1\tilde{F}_{j_1}, b_2\tilde{F}_{j_1}, \cdots, b_{d(p)}\tilde{F}_{j_{n+1}}\}$$

are linearly independent over $\boldsymbol{C}$. From (18), these functions can be represented as linear combinations of

$$\{b_t f_k : 1 \leqq t \leqq d(p+1),\ 1 \leqq k \leqq n+1\}$$

with constant coefficients:

$$(b_1\tilde{F}_{j_1}, b_2\tilde{F}_{j_1}, \cdots, b_{d(p)}\tilde{F}_{j_{n+1}}) = (b_1 f_1, b_2 f_1, \cdots, b_{d(p+1)} f_{n+1})D_1$$

where $D_1$ is a $(n+1)d(p+1) \times (n+1)d(p)$ matrix whose elements are constants. The rank of $D_1$ is equal to $(n+1)d(p)$. Let $D_2$ be a

$$(n+1)d(p+1) \times (n+1)\{d(p+1) - d(p)\}$$

matrix consisting of constant elements such that the matrix

$$D = [D_1 D_2]$$

is regular. Put

$$(G_1, \cdots, G_L) = (b_1 f_1, b_2 f_1, \cdots, b_{d(p+1)} f_{n+1})D_2,$$

where $L = (n+1)\{d(p+1) - d(p)\}$, then

$$(b_1\tilde{F}_{j_1}, \cdots, b_{d(p)}\tilde{F}_{j_{n+1}}, G_1, \cdots, G_L) = (b_1 f_1, \cdots, b_{d(p+1)} f_{n+1})D$$

from which we obtain

$$W(j_1, \cdots, j_{n+1}) \equiv W(b_1\tilde{F}_{j_1}, \cdots, G_L) = (\det D)W \tag{25}$$

where $W = W(b_1 f_1, \cdots, b_{d(p+1)} f_{n+1})$.

We then have from (25)

$$\frac{1}{\left(\prod_{k=1}^{n}|\tilde{F}_{j_k}|\right)^{d(p)}}=\frac{|W(j_1, \cdots, j_{n+1})|}{|W||\det D|}\cdot\frac{1}{\left(\prod_{k=1}^{n}|\tilde{F}_{j_k}|\right)^{d(p)}} \tag{26}$$

$$\leqq S(z, f)\frac{(u(z))^{L+d(p)}}{|W|}\cdot\frac{|W(j_1, \cdots, j_{n+1})|}{|b_1\tilde{F}_{j_1}\cdot b_2\tilde{F}_{j_1}\cdots G_L|}$$

since $|G_j(z)|\leqq S(z, f)U(z)$ $(j=1, \cdots, L)$, $|\tilde{F}_{n+1}(z)|\leqq S(z, f)U(z)$ and $U(z)\leqq S(z, f)u(z)$ in this case. Note that $b_j\in\Gamma$ and $\det D\neq 0$.

Further, by using the inequalities for $j=1, \cdots, L$

$$T(r, G_j/b_1\tilde{F}_{j_1})\leqq 2nT(r, f)+S_0(r, f),$$

which we can prove as in Lemma 6 since $b_t\in\Gamma$ $(1\leqq t\leqq d(p+1))$, and by Lemma 6, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|W(j_1, \cdots, j_{n+1})|}{|b_1\tilde{F}_{j_1}\cdots G_L|}d\theta=O(\log r) \tag{27}$$

as usual (see [1]) since $\rho(f)<\infty$.

From (21), (23), (24), (26) and (27), we have

$$d(p)\sum_{j=l+1}^{\nu}m(r, A_j, f)\leqq\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{\{u(re^{i\theta})\}^{(n+1)d(p+1)}}{|W|}d\theta+S_0(r, f). \tag{28}$$

Let $g(z)$ be a meromorphic function such that the functions

$$\frac{1}{g(z)}\{f_j(z)\}^{(n+1)d(p+1)}\quad (j=1, \cdots, n)\quad\text{and}\quad\frac{1}{g(z)}W$$

are entire functions without common zeros.

We put

$$h^*=\left[\frac{1}{g}(f_1)^{(n+1)d(p+1)}, \cdots, \frac{1}{g}(f_n)^{(n+1)d(p+1)}, \frac{1}{g}W\right].$$

Then, we have the inequality

$$T(r, h^*)\leqq(n+1)d(p+1)T(r, f)+S_0(r, f) \tag{29}$$

(cf. Lemma 4) by using the inequality

$$N(r, g)\leqq(n+1)d(p+1)\sum_{t=1}^{d(p+1)}N(r, b_t)=S_0(r, f).$$

From (28) and (29), we have the following as in the case of (7):

$$d(p)\left\{\sum_{j=l+1}^{\nu}\delta(A_j, f)\right\}\leqq\liminf_{r\to\infty}\frac{T(r, h^*)}{T(r, f)} \tag{30}$$

$$\leqq\limsup_{r\to\infty}\frac{T(r, h^*)}{T(r, f)}\leqq(n+1)d(p+1)$$

and from (15), (16) and (17) we have

$$\limsup_{r\to\infty}\frac{N(r, 1/W)}{T(r, f)}<2\varepsilon d(p). \tag{31}$$

From (30) with (15) and $0\leqq l\leqq n$ we have as in the case of (10)

$$\rho(f)=\rho(h^*) \quad \text{and} \quad \mu(f)=\mu(h^*).$$

Suppose further that $\rho(f)$ is not an integer. Let $\varepsilon$ satisfy

$$0<4\varepsilon<\min\{1, |\sin\pi\rho|/(2.2\rho+|\sin\pi\rho|/2)\},$$

where $\rho=\rho(f)$. By the hypothesis (i), (15), (30) and (31)

$$K(h^*)=\limsup_{r\to\infty}\frac{\sum_{j=1}^{n+1}N(r, e_j, h^*)}{T(r, h^*)}\leqq\frac{2\varepsilon}{1-2\varepsilon}<4\varepsilon \tag{32}$$

since $\varepsilon<1/4$. This inequality contradicts with Lemma 3 (I). This shows that $\rho(f)$ must be an integer. Due to Corollaire 1 in [11], $\mu(h^*)$ is positive, since

$$\delta(e_j, h^*)>0 \quad (j=1, \cdots, n+1)$$

by the hypothesis (i), (15), (30) and (31). This implies that $\rho(f)$ is a positive integer.

Suppose next that $f$ is not of regular growth. Let $\varepsilon$ satisfy

$$0<4\varepsilon<\min\left\{1, \max_{\mu\leqq\tau\leqq\rho}\frac{n+1}{n}\cdot\frac{|\sin\pi\tau|}{4.4e(\tau+1)+|\sin\pi\tau|}\right\}$$

where $\mu=\mu(f)$ and $\rho=\rho(f)$. Then, as in the case of (32), we have

$$K(h^*)<4\varepsilon,$$

which contradicts with Lemma 3 (II) since $\rho(f)=\rho(h^*)$, $\mu(f)=\mu(h^*)$. This shows that $f$ must be of regular growth.

Our proof is complete.

## 5. Degenerate case

Let $f$, $X$ and $\lambda$ be as in Section 1. Throughout the section we suppose that $\lambda>0$.

LEMMA 7. *Let $a_1, \cdots, a_{n+1}$ be any $n+1$ elements of $X$ and put*

$$(\boldsymbol{a}_j, f)=F_j \quad (j=1, \cdots, n+1).$$

*Then, the holomorphic curve $F$ induced by $(F_1, \cdots, F_{n+1})$ is transcendental. Further, if we put*

$$V'=\{(d_1, \cdots, d_{n+1})\in \boldsymbol{C}^{n+1}: d_1F_1+\cdots d_{n+1}F_{n+1}=0\},$$

*then* $\dim V'=\lambda$.

*Proof.* Since it is known ([1]) that

$$T(r, F)=T(r, f)+O(1),$$

it is trivial that $F$ is transcendental since so is $f$.

Let $M$ be the $(n+1)\times(n+1)$ matrix whose $j$-th row is $\boldsymbol{a}_j$. Then, $M$ is regular and

$${}^t(F_1, \cdots, F_{n+1})=M\,{}^t(f_1, \cdots, f_{n+1}).$$

It is clear that for $V=\{\boldsymbol{a}\in\boldsymbol{C}^{n+1}: (\boldsymbol{a}, f)=0\}$

$$\boldsymbol{a}\in V \text{ if and only if } \boldsymbol{a}M^{-1}\in V'$$

and $\lambda=\dim V=\dim V'$.

By the definition of $\lambda$, there are $n+1-\lambda$ functions in $\{f_1, \cdots, f_{n+1}\}$ which are linearly independent over $\boldsymbol{C}$. We suppose without loss of generality that $f_1, \cdots, f_{n+1-\lambda}$ are linearly independent over $\boldsymbol{C}$. Then $f_{n+2-\lambda}, \cdots, f_{n+1}$ can be represented as linear combinations of $f_1, \cdots, f_{n+1-\lambda}$ with constant coefficients. Put

$$U_1(z)=\max_{1\leq j\leq n+1-\lambda}|f_j(z)|.$$

We then have the following.

PROPOSITION 3. $T(r, f)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\log U_1(re^{i\theta})d\theta+O(1).$

*Proof.* It is trivial that

$$U_1(z)\leqq U(z). \tag{33}$$

On the other hand, since $f_{n+2-\lambda}, \cdots, f_{n+1}$ are linear combinations of $f_1, \cdots, f_{n+1-\lambda}$ with constant coefficients, we have

$$U(z)\leqq KU_1(z), \tag{34}$$

where $K$ is a positive constant. From (1), (33) and (34) we have our result.

From now on we put

$$n-\lambda=l$$

for simplicity.

For any $\boldsymbol{a}=(a_1, \cdots, a_{n+1})$ of $\boldsymbol{C}^{n+1}$ such that $(\boldsymbol{a}, f)\neq 0$, there exists only one vector $\boldsymbol{a}'=(a_1', \cdots, a_{l+1}', 0, \cdots, 0)$ of $\boldsymbol{C}^{n+1}$ such that

$$(\boldsymbol{a}, f)=(\boldsymbol{a}', f)$$

since $f_{l+2}, \cdots, f_{n+1}$ can be uniquely represented as linear combinations of $f_1, \cdots, f_{l+1}$ with constant coefficients. We map $\boldsymbol{a}$ to $\boldsymbol{a}'$. In this mapping, we put

$$X_0'=\{\boldsymbol{a}\in X : a_{l+1}'=0\}.$$

LEMMA 8. (I) *The number of vectors of $X_0'$ is at most $n$.*
(II) *For any vectors $\boldsymbol{a}_{j_1}, \cdots, \boldsymbol{a}_{j_m}$ $(1\leqq m\leqq l)$ of $X-X_0'$ such that $\boldsymbol{a}_{j_1}', \cdots, \boldsymbol{a}_{j_m}'$ are linearly independent over $\boldsymbol{C}$, we can choose $\boldsymbol{e}_{i_1}, \cdots, \boldsymbol{e}_{i_{l+1-m}}$ from $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_l\}$ such that*

$$\boldsymbol{e}_{i_1}', \cdots, \boldsymbol{e}_{i_{l+1-m}}', \boldsymbol{a}_{j_1}', \cdots, \boldsymbol{a}_{j_m}'$$

*are linearly independent over $\boldsymbol{C}$.*
(III) *There is a subset $X_0''$ of $X_0'$ such that $\#X_0''\leqq\lambda$ and such that* (*) *from any $n+1$ vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{n+1}$ of $X-X_0''$, we can find $l+1$ vectors $\boldsymbol{a}_{j_1}, \cdots, \boldsymbol{a}_{j_{l+1}}$ for which*

$$(\boldsymbol{a}_{j_1}, f), \cdots, (\boldsymbol{a}_{j_{l+1}}, f)$$

*are linearly independent over $\boldsymbol{C}$ and $\boldsymbol{a}_{j_{l+2}}, \cdots, \boldsymbol{a}_{j_{n+1}}$ do not belong to $X_0'$.*

*Proof.* (I) Suppose that $X_0'$ contains $n+1$ vectors $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_{n+1}$. Put

$$(\boldsymbol{b}_j, f)=G_j \quad (j=1, \cdots, n+1).$$

By Lemma 7, there are $l+1$ functions (say, $G_1, \cdots, G_{l+1}$) in $\{G_1, \cdots, G_{n+1}\}$ and linearly independent over $\boldsymbol{C}$. There is a regular matrix $B$ such that

$${}^t(G_1, \cdots, G_{l+1})=B\,{}^t(f_1, \cdots, f_{l+1}).$$

On the other hand,

$$G_j=(\boldsymbol{b}_j', f) \quad (j=1, \cdots, n+1),$$

where

$$\boldsymbol{b}_j'=(b_{j_1}', \cdots, b_{j_l}', 0, \cdots, 0) \quad (j=1, \cdots, n+1).$$

This means that the $l+1$-th column of $B$ is 0 and $B$ is not regular.

This is a contradiction. $X_0'$ contains at most $n$ vectors.

(II) This is because the rank of $m\times(l+1)$ matrix whose $k$-th row is $\boldsymbol{a}_{j_k}'$ is equal to $m$.

(III) $X_0''=\emptyset$ when $X_0'=\emptyset$. Otherwise, let $X_0'=\{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_p\}$ $(1\leqq p\leqq n)$, $B'$ the $p\times(l+1)$ matrix whose $j$-th row is $\boldsymbol{b}_j'$ and $s=\operatorname{rank} B'$. Then $1\leqq s\leqq\min(p, l+1)$. We may suppose without loss of generality that $\boldsymbol{b}_{k+1}', \cdots, \boldsymbol{b}_p'$ are linearly independent over $\boldsymbol{C}$. Then, $k\leqq\lambda$. In fact, for any $\boldsymbol{b}_{p+1}, \cdots, \boldsymbol{b}_{n+1}\in X-X_0$, there are $l+1$ linearly independent vectors in $\{\boldsymbol{b}_1', \cdots, \boldsymbol{b}_{n+1}'\}$ and so it must be $n+1-k\geqq l+1$. That is, $k\leqq\lambda$. Put

$$X_0''=\{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_k\}.$$

Then, it is easy to see that $X_0''$ has the desired property (*).

LEMMA 9. *Suppose that $f_1, \cdots, f_{l+1}$ $(l=n-\lambda)$ are linearly independent over $\boldsymbol{C}$ and $\rho(f)<\infty$. Then for any $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+1\leqq q<\infty)$ of $X-X_0''$, we have*

$$\sum_{j=1}^{q} m(r, \boldsymbol{a}_j, f)\leqq(n+1)T(r, f)+\lambda\sum_{j=1}^{l} N(r, \boldsymbol{e}_j, f)-(\lambda+1)N(r, 1/W)+O(\log r),$$

*where $W=W(f_1, \cdots, f_{l+1})$.*

*Proof.* Put

$$(\boldsymbol{a}_j, f)=F_j \quad (j=1, \cdots, q).$$

For any $z$ $(\neq 0)$, let

$$|F_{j_1}(z)|, \cdots, |F_{j_{n+1}}(z)|$$

be the least $n+1$ values of $\{|F_j(z)|\}_{j=1}^{q}$ and let $|F_{j_{n+2}}(z)|, \cdots, |F_{j_q}(z)|$ be others. For a positive constant $K$, it holds that

$$\|f(z)\|\leqq K\max_{1\leq i\leq n+1}|F_{j_i}(z)|$$

and

$$|F_j(z)|\leqq K\|f(z)\| \quad (j=1, \cdots, q)$$

as in Proof of Theorem 1, since $U(z)\leqq\|f(z)\|\leqq(n+1)^{1/2}U(z)$. At the point $z$

$$\prod_{j=1}^{q}\frac{\|a_j\|\|f\|}{|F_j|}=K\prod_{i=1}^{q}\frac{\|f\|}{|F_{j_i}|}\leqq K\prod_{i=1}^{n+1}\frac{\|f\|}{|F_{j_i}|}$$
$$=K\frac{\|f\|^{n+1}}{|W|^{\lambda+1}}\cdot\frac{|W(F_{j_1}, \cdots, F_{j_{l+1}})|}{\prod_{i=1}^{l+1}|F_{j_i}|}\cdot\prod_{i=l+2}^{n+1}\frac{|W(f_1, \cdots, f_l, F_{j_i}|}{|F_{j_i}|},$$

where we suppose without loss of generality that $F_{j_1}, \cdots, F_{j_{l+1}}$ are linearly independent over $\boldsymbol{C}$ and $F_{j_i}$ $(i=l+2, \cdots, n+1)$ do not belong to $X_0'$ by Lemma 8 (III). Integrating both sides of this inequality from zero to $2\pi$ with respect to $\theta$ $(z=re^{i\theta})$, we have this lemma as in Lemma 5, since for $i=l+2, \cdots, n+1$

$$\frac{W(f_1, \cdots, f_l, F_{j_i})}{F_{j_i}}=f_1\cdots f_l\frac{W(f_1, \cdots, f_l, F_{j_i})}{f_1\cdots f_l\cdot F_{j_i}}.$$

THEOREM 4. *Suppose that $f_1, \cdots, f_{l+1}$ are linearly independent over $\boldsymbol{C}$ and $\rho(f)<\infty$. Let $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+\lambda+1\leqq q<\infty)$ be any elements of $X$ such that $X_0''\cap\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}=\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k\}$. Then we have*

$$\sum_{j=1}^{q} m(r, \boldsymbol{a}_j, f) \tag{35}$$
$$\leqq(n+\lambda+1)T(r, f)+\lambda\sum_{j=1}^{l} N(r, \boldsymbol{e}_j, f)-(\lambda+1)N(r, 1/W)+O(\log r),$$

*where $W=W(f_1, \cdots, f_{l+1})$ and $X_0''$ is the set obtained in Lemma 8 (III).*

*Further if* $\delta(\boldsymbol{e}_j, f)=1$ $(j=1, \cdots, l)$, *then*

$$\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)\leqq n+1+\sum_{j=1}^{k}\delta(\boldsymbol{a}_j, f)\leqq n+\lambda+1. \tag{36}$$

*Proof.* We first note that $0\leqq k\leqq\lambda$ by Lemma 8 (III). Applying Lemma 9 to $\{\boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_q\}$, we have

$$\begin{aligned}&\sum_{j=k+1}^{q} m(r, \boldsymbol{a}_j, f)\\ &\leqq(n+1)T(r, f)+\lambda\sum_{j=1}^{l}N(r, \boldsymbol{e}_j, f)-(\lambda+1)N\left(r, \frac{1}{W}\right)+O(\log r).\end{aligned} \tag{37}$$

Adding $\sum_{j=1}^{k} m(r, \boldsymbol{a}_j, f)$ to both sides of (37), using

$$m(r, \boldsymbol{a}_j, f)\leqq T(r, f)+O(1)$$

and noting $k\leqq\lambda$, we have (35).

If $\delta(\boldsymbol{e}_j, f)=1$ $(j=1, \cdots, l)$, then from (37) we have

$$\sum_{j=k+1}^{q}\delta(\boldsymbol{a}_j, f)\leqq n+1.$$

Adding $\sum_{j=1}^{k}\delta(\boldsymbol{a}_j, f)$ to both sides of this inequality, we obtain (36).

COROLLARY 4. *Suppose that* $f_1, \cdots, f_{l+1}$ *are linearly independent over* $\boldsymbol{C}$, $\rho(f)<\infty$ *and that*

(i) $\delta(\boldsymbol{e}_j, f)=1$ $(j=1, \cdots, l)$.

*If there exist* $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+\lambda+1\leqq q\leqq\infty)$ *in* $X$ *such that*

(ii) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)=n+\lambda+1$

*and such that*

$$X_0''\cap\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}=\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k\},$$

*then*

(a) $k=\lambda$ *and* $\delta(\boldsymbol{a}_j, f)=1$ $(j=1, \cdots, \lambda)$;

(b) $\lim_{r\to\infty}\dfrac{N(r, 1/W)}{T(r, f)}=0$.

*Proof.* (a) From the hypothesis (ii) and (36), we have

$$n+\lambda+1=\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)\leqq n+1+\sum_{j=1}^{k}\delta(\boldsymbol{a}_j, f)\leqq n+\lambda+1,$$

so that we have

$$k=\lambda \quad\text{and}\quad \delta(\boldsymbol{a}_j, f)=1 \ (j=1, \cdots, \lambda).$$

(b) From (35) of Theorem 4 and the hypothesis (i), we have

$$\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)+(\lambda+1)\limsup_{r\to\infty}\frac{N(r, 1/W)}{T(r, f)}\leqq n+\lambda+1,$$

so that by the hypothesis (ii) we obtain

$$\lim_{r\to\infty}\frac{N(r,\,1/W)}{T(r,\,f)}=0\,.$$

Suppose that $f_1, \cdots, f_{l+1}$ are linearly independent over $\boldsymbol{C}$. Let $f^*$ be the holomorphic curve induced by the mapping

$$(f_1^{l+1},\,\cdots,\,f_l^{l+1},\,W)\colon \boldsymbol{C}\to\boldsymbol{C}^{l+1},$$

where $W=W(f_1, \cdots, f_{l+1})$ is the Wronskian of $f_1, \cdots, f_{l+1}$.

Note that there is an entire function $d(z)$ such that the functions $f_j^{l+1}/d$ $(j=1, \cdots, l)$ and $W/d$ have no common zeros.

Let $\{\tilde{\boldsymbol{e}}_1, \cdots, \tilde{\boldsymbol{e}}_{l+1}\}$ be the standard basis of $\boldsymbol{C}^{l+1}$. Then, we have

THEOREM 5. *Suppose that* $\rho(f)<\infty$. *For any* $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+1\leqq q<\infty)$ *in* $X-X_0'$, *we have*

$$\sum_{j=1}^{q} m(r,\,\boldsymbol{a}_j,\,f)\leqq(\lambda+1)m(r,\,\tilde{\boldsymbol{e}}_{l+1},\,f^*)+O(\log r)\,.$$

*Proof.* Put

$$(\boldsymbol{a}_j,\,f)=F_j\quad (j=1,\,\cdots,\,q)\quad\text{and}\quad u(z)=\max_{1\leq j\leq l}|f_j(z)|\,.$$

For any $z$ $(\neq 0)$ arbitrarily fixed, let

$$|F_{j_1}(z)|\leqq|F_{j_2}(z)|\leqq\cdots\leqq|F_{j_q}(z)|\quad(1\leqq j_1,\,\cdots,\,j_q\leqq q).$$

Then

$$\|z(f)\|\leqq K|F_{j_k}(z)|\quad(k=n+1,\,\cdots,\,q)$$

(see Lemme in [1], p. 11),

$$|F_{j_k}(z)|\leqq K\|f(z)\|\quad(k=1,\,\cdots,\,q)$$

and since the $l+1$-th elements of vectors $\boldsymbol{a}_j'$ are different from zero,

$$|f_{l+1}(z)|\leqq K\{u(z)+|F_{j_k}(z)|\}\quad(k=1,\,\cdots,\,q).$$

(I) The case when $u(z)\leqq|F_{j_1}(z)|$.
Since $\|f(z)\|\leqq K|F_{j_1}(z)|$ in this case, we have

$$\prod_{j=1}^{q}\frac{\|\boldsymbol{a}_j\|\|f\|}{|F_j|}\leqq K. \tag{38}$$

(II) The case when $|F_{j_1}(z)|<u(z)$.
We can find linearly independent $l$ functions from $\{F_{j_1}, \cdots, F_{j_n}\}$ including $F_{j_1}$. Let $H_1$ $(=F_{j_1}), \cdots, H_l$ $(|H_1(z)|\leqq|H_2(z)|\leqq\cdots\leqq|H_l(z)|)$ be those functions and

$$\{F_{j_1},\,\cdots,\,F_{j_n}\}-\{H_1,\,\cdots,\,H_l\}=\{H_{l+1},\,\cdots,\,H_n\}\,.$$

Then, since $H_1 \in X - X_0'$, we have

$$\|f\| \leqq K\{|f_1|^2 + |f_2|^2 + \cdots + |f_l|^2 + |H_1|^2\}^{1/2} \leqq Ku(z).$$

Let $e_{i_0}$ be a vector in $\{e_1, \cdots, e_l\}$ such that

$$e_{i_0}', b_{j_1}', \cdots, b_{j_l}'$$

are linearly independent over $C$ (see Lemma 8 (II)), where

$$(b_{j_k}, f) = H_k \quad (k=1, \cdots, l).$$

Then, for a non-zero constant $c$

$$W(f_{i_0}, H_1, \cdots, H_l) = cW(f_1, \cdots, f_{l+1}).$$

We put $W = W(f_1, \cdots, f_{l+1})$. Then,

$$\prod_{k=1}^{l} \frac{\|f\|}{|H_k|} \leqq K \frac{u(z)^l}{|W|} \cdot \frac{|W|}{|H_1 \cdots H_l|} \tag{39}$$
$$\leqq K \frac{u(z)^{l+1}}{|W|} \cdot \frac{|W(f_{i_0}, H_1, \cdots, H_l)|}{|f_{i_0} \cdot H_1 \cdots H_l|}$$

and for $k = l+1, \cdots, n$

$$\frac{\|f\|}{|H_k|} \leqq K \quad \text{if } u(z) \leqq |H_k(z)| \tag{40}$$

since $\|f\| \leqq K\{|f_1|^2 + \cdots + |f_l|^2 + |H_k|^2\}^{1/2} \leqq K|H_k(z)|$ and if $|H_k(z)| < u(z)$

$$\frac{\|f\|}{|H_k|} = \frac{\|f\|}{|W|} \cdot \frac{|W|}{|H_k|} \leqq K \frac{u(z)^{l+1}}{|W|} \cdot \frac{|W(f_1, \cdots, f_l, H_k)|}{|f_1 \cdots f_l \cdot H_k|}. \tag{41}$$

By using (38), (39), (40), (41) and the following inequality

$$\sum_{j=1}^{q} \frac{\|a_j\| \|f\|}{|F_j|} \leqq K \prod_{k=1}^{l} \frac{\|f\|}{|H_k|} \cdot \prod_{k=l+1}^{n} \frac{\|f\|}{|H_k|}$$
$$\leqq K \left\{ \max\left( \frac{u(z)^{l+1}}{|W|}, 1 \right) \right\}^{\lambda+1} \frac{|W(f_{i_0}, \cdots, H_l)|}{|f_{i_0} \cdots H_l|} \prod_{k=l+1}^{n} \frac{|W(f_1, \cdots, H_k)|}{|f_1 \cdots H_k|},$$

we obtain the inequality

$$\sum_{j=1}^{q} m(r, a_j, f) \leqq (\lambda+1) m(r, \tilde{e}_{l+1}, f^*) + O(\log r)$$

since $\rho(f) < \infty$.

COROLLARY 5. *Under the same assumption as in Theorem* 5, *we have*

$$\frac{1}{(\lambda+1)(l+1)} \sum_{a \in X - X_0'} \delta(a, f) \leqq \delta(\tilde{e}_{l+1}, f^*), \tag{42}$$

$$(43)\qquad \frac{1}{(\lambda+1)}\sum_{a\in X-X_0'}\delta(\boldsymbol{a}, f)\leqq\liminf_{r\to\infty}\frac{T(r, f^*)}{T(r, f)}\leqq\limsup_{r\to\infty}\frac{T(r, f^*)}{T(r, f)}\leqq l+1.$$

We can prove this corollary by Theorem 5 and Lemma 4 as in the case of Corollary 2 in Section 3.

THEOREM 6. *Suppose that* $f_1, \cdots, f_{l+1}$ *are linearly independent over* $\boldsymbol{C}$, $\rho(f)<\infty$ *and that*
(i) $\delta(\boldsymbol{e}_j, f)=1$ $(j=1, \cdots, l)$.
*If there exist* $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+\lambda+1\leqq q\leqq\infty)$ *in* $X$ *such that*
(ii) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)=n+\lambda+1,$
*then* $f$ *is of regular growth and* $\rho(f)$ *is equal to a positive integer.*

*Proof*. By Lemma 8 (I), $X_0'$ contains at most $n$ vectors. We may suppose without loss of generality that

$$X_0'=\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p\}\quad (0\leqq p\leqq n).$$

Then from the hypothesis (ii), we have

$$(44)\qquad \lambda+1\leqq n+\lambda+1-p\leqq\sum_{j=p+1}^{q}\delta(\boldsymbol{a}_j, f).$$

(43) and (44) imply that

$$(45)\qquad \rho(f)=\rho(f^*)\quad\text{and}\quad \mu(f)=\mu(f^*).$$

The hypothesis (i), (43) and (44) imply that

$$(46)\qquad \delta(\tilde{\boldsymbol{e}}_j, f^*)=1\quad (j=1, \cdots, l).$$

Further, Corollary 4 (b), (43) and (44) imply that

$$(47)\qquad \delta(\tilde{\boldsymbol{e}}_{l+1}, f^*)=1.$$

By Lemma 2, (45), (46) and (47) imply that $f$ is of regular growth and $\rho(f)$ is equal to a positive integer.

As in Corollary 3, we have the following

COROLLARY 6. *Suppose that* $\rho(f)<\infty$. *If there exist* $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ $(n+\lambda+1\leqq q\leqq\infty)$ *in* $X$ *such that*
(i) $\delta(\boldsymbol{a}_j, f)=1$ $(j=1, \cdots, n)$,
(ii) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)=n+\lambda+1,$
*then* $f$ *is of regular growth and* $\rho(f)$ *is equal to a positive integer.*

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