

SOME RESULTS ON RIGIDITY OF HOLOMORPHIC MAPPINGS

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1. In this paper we study rigidity properties of holomorphic mappings. Let X and Y be complex normed spaces. Let D_1 be a balanced domain in X and D_2 be a bounded convex balanced domain in Y . We consider holomorphic mappings f from D_1 into D_2 . We prove two theorems. One of them is a generalization of the Schwarz lemma, which gives an upper bound for $\mu_{D_2}(f(x))$, $x \in D_1$. Here μ_{D_2} denotes the Minkowski functional of D_2 . We also discuss the extremal mappings related to the Schwarz lemma. We deduce as a corollary the following fact: if $f: X \rightarrow Y$ is a holomorphic mapping which satisfies $\|f(x)\| = \|x\|$ for all $x \in X$, then f is linear. Another theorem gives a lower bound for $\mu_{D_2}(f(x))$, $x \in D_1$. Finally we are concerned with the limits of sequences of automorphisms of bounded domains. It is known that if D is a bounded domain in \mathbb{C}^n and if a mapping $f: D \rightarrow D$ is a pointwise limit of a sequence of automorphisms of D , then f is also an automorphism of D . However, in the case that D is a bounded domain in a complex normed space X the limit $f: D \rightarrow D$ need not be an automorphism of D . We give a simple counterexample. Using the above two theorems we show that the limit f is one-to-one.

2. We summarize the main notation and terminology used in this paper. Let X be a complex normed space and let D be a domain in X . The Minkowski functional μ_D of D is defined by

$$\mu_D(x) = \inf \{t > 0 : t^{-1}x \in D\} \quad (x \in X).$$

We denote the open ball with center at a and radius r in X by $B(a, r)$. Then we have that $\mu_{B(0, r)}(x) = r^{-1}\|x\|$.

Let X and Y be complex normed spaces and let D be a domain in X . A mapping $f: D \rightarrow Y$ is said to be holomorphic in D if, corresponding to every $a \in D$, there exist a power series $\sum_{k=0}^{\infty} P_k$ and a positive number ρ such that f is expressed by

$$f(x) = \sum_{k=0}^{\infty} P_k(x-a) \quad (x \in B(a, \rho)).$$

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Here P_k is a continuous k -homogeneous polynomial from X to Y , and the convergence is uniform on $B(a, r)$ for every r with $0 < r < \rho$. We use the notation

$$\hat{d}^k f(a) = k! P_k \quad (k=0, 1, 2, \dots),$$

and we call the power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} \hat{d}^k f(a)(x-a)$$

the Taylor series of f at a . We refer to [1], [2] and [6] for further details.

3. There are some generalizations of the Schwarz lemma in complex normed spaces (see, for example, [2], [3], [4] and [5]). We also give a generalization of the Schwarz lemma. We adapt Rudin's proof [8, Theorem 8.1.2] in which only the finite dimensional case is considered.

Let X and Y be complex normed spaces and let D_1 and D_2 be balanced domains in X and Y , respectively. Suppose that $D_1 \neq X$. Then there exists a point $x \in X$ with $\mu_{D_1}(x) > 0$. For a holomorphic mapping $f: D_1 \rightarrow D_2$ we define

$$\lambda_f = \inf \left\{ \frac{\mu_{D_2}(\hat{d}f(0)(x))}{\mu_{D_1}(x)} : x \in X, \mu_{D_1}(x) > 0 \right\}$$

and

$$\Lambda_f = \sup \left\{ \frac{\mu_{D_2}(\hat{d}f(0)(x))}{\mu_{D_1}(x)} : x \in X, \mu_{D_1}(x) > 0 \right\}.$$

Here $\hat{d}f(0) = \hat{d}^1 f(0)$ is the linear part of the Taylor series of f at the origin 0 of X .

THEOREM 1. *Let X and Y be complex normed spaces. Suppose that*

- (i) D_1 is a balanced domain in X ,
- (ii) D_2 is a bounded convex balanced domain in Y ,
- (iii) $f: D_1 \rightarrow D_2$ is a holomorphic mapping with $f(0) = 0$.

Then

- (a) $\mu_{D_2}(f(x)) \leq \mu_{D_1}(x) \quad (x \in D_1)$,
- (b) $\mu_{D_2}(\hat{d}f(0)(x)) \leq \mu_{D_1}(x) \quad (x \in D_1)$,
- (c) *if $D_1 \neq X$, then $0 \leq \lambda_f \leq \Lambda_f \leq 1$.*

The equality $\lambda_f = 1$ holds if and only if the equality $\mu_{D_2}(f(x)) = \mu_{D_1}(x)$ holds for all $x \in D_1$.

Proof. We first note that since D_2 is bounded, convex and balanced, μ_{D_2} is a norm on Y and that since D_1 is balanced, the power series

$$\sum_{k=1}^{\infty} \frac{1}{k!} \hat{d}^k f(0)(x)$$

converges to f uniformly on every compact subset of D_1 . (See [1], Corollary 5.2).

Take a point $x_0 \in D_1$ with $\mu_{D_1}(x_0) > 0$. Put $y = \mu_{D_1}(x_0)^{-1}x_0$. Then $y \in \partial D_1$. Let φ be a continuous linear functional on Y of norm 1, i. e.,

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in Y, \mu_{D_2}(x) = 1\} = 1.$$

We define the function g by

$$g(\zeta) = \varphi(f(\zeta y)) \quad (\zeta \in \mathbb{C}, |\zeta| < 1).$$

Then g is holomorphic and $|g(\zeta)| \leq 1$ in $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and $g(0) = 0$. Now applying the classical Schwarz lemma we have

$$|g(\zeta)| \leq |\zeta| \quad (\zeta \in \Delta),$$

$$|g'(0)| \leq 1.$$

Since $g'(0) = \varphi(\hat{d}f(0)(y)) = \mu_{D_1}(x_0)^{-1}\varphi(\hat{d}f(0)(x_0))$, these inequalities imply

$$|\varphi(f(x_0))| \leq \mu_{D_1}(x_0),$$

$$|\varphi(\hat{d}f(0)(x_0))| \leq \mu_{D_1}(x_0).$$

The Hahn-Banach theorem assures the existence of continuous linear functionals φ_0 and φ_1 on Y of norm 1 such that $\varphi_0(f(x_0)) = \mu_{D_2}(f(x_0))$ and $\varphi_1(\hat{d}f(0)(x_0)) = \mu_{D_2}(\hat{d}f(0)(x_0))$. Hence we obtain the inequalities

$$\mu_{D_2}(f(x_0)) \leq \mu_{D_1}(x_0),$$

$$\mu_{D_2}(\hat{d}f(0)(x_0)) \leq \mu_{D_1}(x_0).$$

Suppose that $x_0 \in D_1$ and $\mu_{D_1}(x_0) = 0$. For every $\varepsilon > 0$ there is a positive number t such that $0 < t < \varepsilon$ and $t^{-1}x_0 \in D_1$. Considering the function

$$g(\zeta) = \varphi(f(\zeta t^{-1}x_0)) \quad (\zeta \in \Delta),$$

we have

$$\mu_{D_2}(f(x_0)) \leq t, \quad \mu_{D_2}(\hat{d}f(0)(x_0)) \leq t.$$

Hence $\mu_{D_2}(f(x_0)) = \mu_{D_2}(\hat{d}f(0)(x_0)) = 0$. Thus (a) and (b) are proved.

Now the inequality

$$0 \leq \tilde{\lambda}_f \leq \tilde{\lambda}_f \leq 1$$

is an immediate consequence of (b).

Suppose that $\tilde{\lambda}_f = 1$. Then $\mu_{D_2}(\hat{d}f(0)(x)) = \mu_{D_1}(x)$ for all $x \in X$. Let $x_0 \in D_1$. If $\mu_{D_1}(x_0) = 0$, then $\mu_{D_2}(f(x_0)) = 0$ as we have shown. If $\mu_{D_1}(x_0) > 0$, we consider the function

$$g_1(\zeta) = \varphi_1(f(\zeta y)), \quad y = \mu_{D_1}(x_0)^{-1}x_0.$$

Then

$$g'_1(0) = \mu_{D_1}(x_0)^{-1} \varphi_1(\hat{d}f(0)(x_0)) = \mu_{D_1}(x_0)^{-1} \mu_{D_2}(\hat{d}f(0)(x_0)) = 1$$

and so $g_1(\zeta) = \zeta$ for all $\zeta \in \Delta$. Hence

$$\mu_{D_1}(x_0) = g_1(\mu_{D_1}(x_0)) = \varphi_1(f(x_0)) \leq \|\varphi_1\| \mu_{D_2}(f(x_0)) = \mu_{D_2}(f(x_0)).$$

Consequently we obtain that

$$\mu_{D_2}(f(x)) = \mu_{D_1}(x) \quad (x \in D_1).$$

Conversely, suppose that $\mu_{D_2}(f(x_0)) = \mu_{D_1}(x_0)$ for some $x_0 \in D_1$ with $\mu_{D_1}(x_0) > 0$. We consider the function

$$g_0(\zeta) = \varphi_0(f(\zeta y)), \quad y = \mu_{D_1}(x_0)^{-1} x_0.$$

Since $g_0(\mu_{D_1}(x_0)) = \mu_{D_1}(x_0)$, we have $g_0(\zeta) = \zeta$ for all $\zeta \in \Delta$ and so $g'_0(0) = 1$. Hence we obtain

$$\mu_{D_1}(x_0) = \varphi_0(\hat{d}f(0)(x_0)) \leq \mu_{D_2}(\hat{d}f(0)(x_0)).$$

Thus if $\mu_{D_2}(f(x)) = \mu_{D_1}(x)$ for all $x \in D_1$, then $\tilde{\lambda}_f \geq 1$ and hence $\tilde{\lambda}_f = 1$.

Remarks. If $\mu_{D_2}(f(x_0)) = \mu_{D_1}(x_0)$ for some $x_0 \in D_1$ with $\mu_{D_1}(x_0) > 0$, then $\tilde{\lambda}_f = 1$. However, $\tilde{\lambda}_f = 1$ does not imply that there exists a point $x_0 \in D_1$ with $\mu_{D_1}(x_0) > 0$ such that $\mu_{D_2}(f(x_0)) = \mu_{D_1}(x_0)$. Indeed, let $X = Y = c_{00}$ and $D_1 = D_2 = \{x \in c_{00} : \|x\| < 1\}$. Here c_{00} is the vector space of all sequences $x = (x_1, x_2, \dots, x_n, \dots)$ of complex numbers having only a finite number of non-vanishing terms, with norm

$$\|x\| = \max_n |x_n|.$$

The mapping $f : X \rightarrow Y$ defined by

$$f : (x_1, x_2, \dots, x_n, \dots) \mapsto \left(\frac{1}{2}x_1, \frac{2}{3}x_2, \dots, \frac{n}{n+1}x_n, \dots \right)$$

maps D_1 into D_2 and satisfies $f(0) = 0$ and $\tilde{\lambda}_f = 1$. But $\mu_{D_2}(f(x)) = \|f(x)\| \neq \|x\| = \mu_{D_1}(x)$ for every $x \in D_1$ with $\mu_{D_1}(x) = \|x\| > 0$.

Moreover, we consider the mapping $g : X \rightarrow Y$ defined by

$$g : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (x_1^2, x_1, x_2, \dots, x_{n-1}, \dots).$$

Then g maps D_1 into D_2 and satisfies $g(0) = 0$ and $\tilde{\lambda}_g = 1$. But $g \neq \hat{d}g(0)$. Thus $\tilde{\lambda}_g = 1$ does not imply that $g = \hat{d}g(0)$.

COROLLARY. *Let X and Y be complex normed spaces. If $f : X \rightarrow Y$ is a holomorphic mapping which satisfies $\|f(x)\| = \|x\|$ for all $x \in X$, then f is linear.*

Proof. Let M be a positive number. By the assumptions it follows that f

maps $B(0, M)$ into $B(0, M)$ and satisfies $f(0)=0$ and

$$\mu_{B(0, M)}(f(x)) = \mu_{B(0, M)}(x) \quad (x \in B(0, M)).$$

Hence by Theorem 1 we have $\hat{\lambda}_f=1$, and so

$$\mu_{B(0, M)}(\hat{d}f(0)(x)) = \mu_{B(0, M)}(x) \quad (x \in X).$$

Therefore it follows that the equality

$$\|f(x)\| = \|\hat{d}f(0)(x)\|$$

holds for all $x \in X$. Replacing x by ζx we have

$$\left\| \sum_{k=1}^{\infty} \frac{1}{k!} \zeta^{k-1} \hat{d}^k f(0)(x) \right\| = \|\hat{d}f(0)(x)\| \quad (x \in X, \zeta \in \mathbb{C}).$$

Let φ be a continuous linear functional on Y . Then the function

$$h(\zeta) = \varphi\left(\sum_{k=1}^{\infty} \frac{1}{k!} \zeta^{k-1} \hat{d}^k f(0)(x)\right)$$

is holomorphic and bounded in \mathbb{C} . Hence the Liouville theorem says that h is constant. Since the dual space Y^* of Y separates points on Y , it now follows that

$$\sum_{k=2}^{\infty} \frac{1}{k!} \zeta^{k-1} \hat{d}^k f(0)(x) = 0 \quad (\zeta \in \mathbb{C}).$$

Therefore we conclude that $f = \hat{d}f(0)$.

Next we prove a theorem which gives a lower bound for $\mu_{D_2}(f(x))$, $x \in D_1$. In our proof the following fact plays an important role.

PROPOSITION. *Let X be a complex normed space and D be a convex balanced domain in X . Let k_D denote the Kobayashi pseudodistance of D . Then*

$$k_D(0, x) = \frac{1}{2} \log \frac{1 + \mu_D(x)}{1 - \mu_D(x)}$$

and

$$\{x \in D : k_D(0, x) < \alpha\} = rD, \quad r = \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1}.$$

(See [2], Theorem IV.1.8).

THEOREM 2. *Let X and Y be complex normed spaces. Suppose that*

- (i) D_1 is a balanced domain in X ,
- (ii) D_2 is a bounded convex balanced domain in Y ,
- (iii) $f: D_1 \rightarrow D_2$ is a holomorphic mapping with $f(0)=0$.

Then

$$\mu_{D_2}(f(x)) \geq \frac{\mu_{D_1}(x)(\tilde{\lambda}_f - \mu_{D_1}(x))}{1 - \tilde{\lambda}_f \mu_{D_1}(x)} \quad (x \in D_1).$$

Proof. Take a point $x_0 \in D_1$ with $\mu_{D_1}(x_0) > 0$. Put $y = \mu_{D_1}(x_0)^{-1}x_0$ and define the mapping g from Δ into D_2 by

$$g(\zeta) = f(\zeta y) \quad (\zeta \in \Delta).$$

Since D_1 is balanced, the Taylor series of f at 0

$$\sum_{k=1}^{\infty} \frac{1}{k!} \hat{d}^k f(0)(x)$$

converges to f uniformly on every compact subset of D_1 . Hence we have

$$g(\zeta) = \sum_{k=1}^{\infty} \frac{1}{k!} \zeta^k \hat{d}^k f(0)(y) \quad (\zeta \in \Delta),$$

and hence we can write

$$g(\zeta) = \zeta h(\zeta) \quad (\zeta \in \Delta),$$

where h is a holomorphic mapping from Δ into Y with the Taylor series

$$h(\zeta) = \sum_{k=1}^{\infty} \frac{1}{k!} \zeta^{k-1} \hat{d}^k f(0)(y)$$

at 0. Let $0 < t < 1$. By Proposition and the definition of k_{D_2} we have that if $\zeta h(\zeta) \neq 0$, then

$$k_{D_2}(0, t\zeta h(\zeta)) < k_{D_2}(0, \zeta h(\zeta)) = k_{D_2}(g(0), g(\zeta)) \leq \frac{1}{2} \log \frac{1+|\zeta|}{1-|\zeta|},$$

and so

$$t\zeta h(\zeta) \in |\zeta| D_2.$$

Since D_2 is balanced, this implies that

$$th(\zeta) \in D_2 \quad (\zeta \in \Delta, \zeta \neq 0).$$

By Theorem 1, $h(0) = \hat{d}f(0)(y) \in \bar{D}_2$ and so $th(0) \in D_2$. Thus th is also a holomorphic mapping from Δ into D_2 . Therefore by Proposition and the definition of k_{D_2} we have the following two inequalities:

$$\begin{aligned} k_{D_2}(0, th(0)) &= k_{D_2}(0, t\hat{d}f(0)(y)) = \frac{1}{2} \log \frac{1+t\mu_{D_2}(\hat{d}f(0)(y))}{1-t\mu_{D_2}(\hat{d}f(0)(y))} \\ &\geq \frac{1}{2} \log \frac{1+t\tilde{\lambda}_f}{1-t\tilde{\lambda}_f}, \end{aligned}$$

and

$$k_{D_2}(th(0), th(\zeta)) \leq \frac{1}{2} \log \frac{1+|\zeta|}{1-|\zeta|}.$$

Moreover, by the triangle inequality we have

$$k_{D_2}(0, th(\zeta)) \geq k_{D_2}(0, th(0)) - k_{D_2}(th(0), th(\zeta)).$$

Combining these inequalities we obtain

$$k_{D_2}(0, th(\zeta)) \geq \frac{1}{2} \log \frac{(1+t\tilde{\lambda}_f)(1-|\zeta|)}{(1-t\tilde{\lambda}_f)(1+|\zeta|)}.$$

Now this inequality and Proposition show that

$$\begin{aligned} \mu_{D_2}(tg(\zeta)) &= |\zeta| \mu_{D_2}(th(\zeta)) = |\zeta| \Phi(k_{D_2}(0, th(\zeta))) \\ &\geq |\zeta| \Phi\left(\frac{1}{2} \log \frac{(1+t\tilde{\lambda}_f)(1-|\zeta|)}{(1-t\tilde{\lambda}_f)(1+|\zeta|)}\right) = \frac{|\zeta|(t\tilde{\lambda}_f - |\zeta|)}{1-t\tilde{\lambda}_f|\zeta|}, \end{aligned}$$

where $\Phi(s) = (e^{2s} - 1)/(e^{2s} + 1)$. Letting $t \rightarrow 1$ and putting $\zeta = \mu_{D_1}(x_0)$ we obtain the desired inequality

$$\mu_{D_2}(f(x_0)) \geq \frac{\mu_{D_1}(x_0)(\tilde{\lambda}_f - \mu_{D_1}(x_0))}{1 - \tilde{\lambda}_f \mu_{D_1}(x_0)}.$$

4. Finally we study the limits of sequences of automorphisms of bounded domains. If D is a bounded domain in \mathbb{C}^n and if $f: D \rightarrow D$ is a pointwise limit of a sequence $\{F_n\}$ of automorphisms of D , then f is also an automorphism of D . This follows from the fact that $\{F_n\}$ has a subsequence $\{F_{n_k}\}$ which converges to f uniformly on every compact subset of D . (See [7], pp. 78-82). However, in the case that D is a bounded domain in a complex normed space X the limit $f: D \rightarrow D$ of a sequence of automorphisms of D need not be an automorphism of D . In this section using Theorems 1 and 2 we prove that f is one-to-one.

Let X be a complex normed space and D be a domain in X . The automorphisms of D are the biholomorphic mappings from D onto D . We denote by $\text{Aut}(D)$ the group of all automorphisms of D . We begin with a simple example.

Example. Let $X = c_{00}$ and $D = \{x \in X : \|x\| < 1\}$. Define the mappings F_n , $n=1, 2, \dots$, and f by

$$F_n : (x_1, x_2, x_3, \dots, x_n, x_{n+1}, \dots) \longmapsto (x_n, x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots)$$

and

$$f : (x_1, x_2, x_3, \dots, x_n, \dots) \longmapsto (0, x_1, x_2, \dots, x_{n-1}, \dots).$$

Then D is a bounded domain in X and $F_n \in \text{Aut}(D)$, $n=1, 2, \dots$. Moreover,

$$f(x) = \lim_{n \rightarrow \infty} F_n(x) \quad (x \in D).$$

However, $f \notin \text{Aut}(D)$.

We next prove two lemmas. For a holomorphic mapping $f: D \rightarrow X$ we define

$$\lambda_f(a) = \inf \left\{ \frac{\|\hat{d}f(a)(x)\|}{\|x\|} : x \in X, x \neq 0 \right\} \quad (a \in D),$$

and

$$A_f(a) = \sup \left\{ \frac{\|\hat{d}f(a)(x)\|}{\|x\|} : x \in X, x \neq 0 \right\} \quad (a \in D).$$

Here we note that if f maps $B(0, r_1)$ into $B(0, r_2)$, then

$$\lambda_f(0) = \frac{r_2}{r_1} \tilde{\lambda}_f, \quad A_f(0) = \frac{r_2}{r_1} \tilde{A}_f.$$

LEMMA 1. Let D be a bounded domain in X . Let $F \in \text{Aut}(D)$ and $a \in D$. If M, r and ρ are positive numbers such that $D \subset B(0, M)$, $B(a, r) \subset D$ and $B(F(a), \rho) \subset D$, then

$$\frac{\rho}{2M} \leq \lambda_F(a) \leq A_F(a) \leq \frac{2M}{r}.$$

Proof. Since $F^{-1} \circ F = F \circ F^{-1} = \text{id.}$, we have $\hat{d}F^{-1}(F(a)) \circ \hat{d}F(a) = \hat{d}F(a) \circ \hat{d}F^{-1}(F(a)) = \text{id.}$ Hence it follows that $\hat{d}F(a) \in \text{Aut}(X)$ and $\lambda_F(a) A_{F^{-1}}(F(a)) = 1$.

Put $G(x) = F(x+a) - F(a)$. Then G maps $B(0, r)$ into $B(0, 2M)$ and $G(0) = 0$. Hence, by Theorem 1, we obtain

$$A_F(a) = A_G(0) = \frac{2M}{r} \tilde{A}_G \leq \frac{2M}{r}.$$

On the other hand, since F^{-1} maps $B(F(a), \rho)$ into $B(0, M)$, we have

$$\lambda_F(a) = \frac{1}{A_{F^{-1}}(F(a))} \geq \frac{\rho}{2M}.$$

LEMMA 2. Let D be a bounded domain in X . Let $F \in \text{Aut}(D)$, $a \in D$ and $\lambda_F(a) \geq \lambda_0 > 0$. Let M and r be positive numbers such that $B(a, r) \subset D \subset B(0, M)$. If $0 < t < (r^2 \lambda_0 / 4M)$, then

$$F(B(a, t)) \supset B\left(F(a), \frac{\lambda_0}{2} t\right).$$

Proof. Put $G(x) = F(x+a) - F(a)$. Then G maps $B(0, r)$ into $B(0, 2M)$ and $G(0) = 0$. Hence applying Theorem 2 to G we have

$$\|G(x)\| \geq \frac{2M}{r^2} \frac{\|x\| (r^2 \lambda_G(0) - 2M \|x\|)}{2M - \lambda_G(0) \|x\|} \quad (x \in B(0, r)).$$

Since $\lambda_G(0) = \lambda_F(a) \geq \lambda_0$, we have

$$\|F(x) - F(a)\| \geq \frac{2M}{r^2} \frac{\|x - a\| (r^2 \lambda_0 - 2M \|x - a\|)}{2M - \lambda_0 \|x - a\|} \quad (x \in B(a, r)).$$

Hence using the inequality $\lambda_0 \leq \lambda_F(a) \leq A_F(a) \leq (2M/r)$, we obtain that if $\|x-a\| \leq (r^2\lambda_0/4M)$, then

$$\|F(x)-F(a)\| \geq \frac{4M^2\lambda_0}{8M^2-r^2\lambda_0^2} \|x-a\| \geq \frac{\lambda_0}{2} \|x-a\|.$$

Since $F \in \text{Aut}(D)$, this inequality shows that if $0 < t < (r^2\lambda_0/4M)$, then

$$F(B(a, t)) \supset B\left(F(a), \frac{\lambda_0}{2}t\right).$$

Now we can prove the following theorem.

THEOREM 3. *Let D be a bounded domain in X . Suppose that $f: D \rightarrow D$ is a pointwise limit of a sequence $\{F_n\}$ of automorphisms of D . Then f is one-to-one in D .*

Proof. Assume that there exist two distinct points a_1 and a_2 in D such that $f(a_1) = f(a_2) = b$. Since $b \in D$, there is a positive number ρ with $B(b, 2\rho) \subset D$. Hence we can choose a positive integer n_0 such that if $n > n_0$, then $B(F_n(a_i), \rho) \subset D$, $i=1, 2$. Take positive numbers M, r_1 and r_2 such that $B(0, M) \supset D$, $B(a_1, r_1) \subset D$ and $B(a_2, r_2) \subset D$. Then, by Lemma 1, we have, if $n > n_0$, then

$$\lambda_{F_n}(a_i) \geq \frac{\rho}{2M} \quad (i=1, 2),$$

and hence, by Lemma 2, if $n > n_0$ and $0 < t < (r_i^2\rho/8M^2)$, then

$$F_n(B(a_i, t)) \supset B\left(F_n(a_i), \frac{\rho}{4M}t\right) \quad (i=1, 2).$$

On the other hand, we can choose a positive number t and a positive integer n satisfying conditions:

$$(i) \quad 0 < t < \min\left\{\frac{r_1^2\rho}{8M^2}, \frac{r_2^2\rho}{8M^2}\right\}, \text{ and } B(a_1, t) \cap B(a_2, t) = \emptyset,$$

$$(ii) \quad n > n_0 \text{ and } B\left(F_n(a_1), \frac{\rho}{4M}t\right) \cap B\left(F_n(a_2), \frac{\rho}{4M}t\right) \neq \emptyset.$$

These facts contradict that $F_n \in \text{Aut}(D)$. Therefore it follows that f is one-to-one in D .

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