

ON ALGEBRAICITY OF VECTOR VALUED SIEGEL MODULAR FORMS

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0. Introduction.

Let n be a positive integer and let $k, l \geq 0$ be integers. Let Sym^l be the natural representation of $GL(n, \mathbb{C})$ on $Sym^l(\mathbb{C}^n)$, the l -th symmetric tensor product of the vector space \mathbb{C}^n .

A $Sym^l(\mathbb{C}^n)$ -valued holomorphic function f on the Siegel upper half space of degree n is called a Siegel modular form of degree n and type $\det^k \otimes Sym^l$ when f satisfies certain automorphic condition with respect to the action of the integral symplectic group of size $2n$ through the representation $\det^k \otimes Sym^l$.

Let $M_{k,l}^n$ be the \mathbb{C} -vector space of Siegel modular forms of degree n and type $\det^k \otimes Sym^l$. Let $S_{k,l}^n$ be the subspace of $M_{k,l}^n$ consisting of cuspforms. Precise definitions of them are in §1 below.

The purpose of this paper is to prove several algebraic results on Fourier coefficients of $f \in M_{k,l}^n$ described as follows:

RESULTS. (*Precise statements are in §2.*)

Suppose that k, l are even and $k \geq 2n+2$.

(1) *$S_{k,l}^n$ has a basis consisting of forms whose Fourier coefficients lie in $Sym^l(\mathbb{Q}^n)$.*

(2) *Let $f \in S_{k,l}^n$ be an eigenforms (i.e. a non-zero eigenfunction of the Hecke algebra) and let $\mathbb{Q}(f)$ be the extension field of \mathbb{Q} generated by the eigenvalues on f of the Hecke algebra over \mathbb{Q} . Then $\mathbb{Q}(f)$ is a totally real number field and the degree of extension does not exceed $S_{k,l}^n$.*

(3) *$S_{k,l}^n$ has an orthogonal basis consisting of eigenforms such that the Fourier coefficients of each element f lie in $Sym^l(\mathbb{Q}(f)^n)$.*

(4) *Let m be a integer with $m \geq n$ and $k > m+n+1$, let $[\]_n^m: S_{k,l}^n \rightarrow M_{k,l}^m$ be the Eisenstein lifting. Let $f \in S_{k,l}^n$ be an eigenform whose Fourier coefficients lie in $Sym^l(\mathbb{Q}(f)^n)$. Then, $[f]_n^m$ has Fourier coefficients in $Sym^l(\mathbb{Q}(f)^m)$.*

For the case $l=0$, i.e. $Sym^0(\mathbb{C}^n)=\mathbb{C}$ -valued case, above results are proved by several authors. The assertion (2) is due to Kurokawa [6]. In [5], Garrett

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showed the "Pullback Formula" which reduces problems on Siegel modular forms to smaller degree ones, and using this formula he showed (a similitude of) (4). Böcherer [4] showed (3), by effective use of the pullback formula. In [7], Mizumoto gave a way to prove (3), (4) as well as (1) simulatenously, also using the pullback formula.

In the paper [2] of Böcherer-Satoh-Yamazaki, they have obtained the pullback formula for the case $l \in 2\mathbb{Z} > 0$, which enables us to apply the above proofs for the case $l=0$ to the case $l \in 2\mathbb{Z} > 0$ without essential change. A brief description of the pullback formula is given in §3, where we shall also a connection between Fourier coefficients of eigen cuspforms and the partial Fourier expansion of pullback of Siegel's Eisenstein series.

The stated results shall be proved in §4.

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1. Notations and definitions.

Let n be a positive integer and k, l be positive even integers. Let $\mathbf{x} := (x_1, \dots, x_n)$ be a row vector with x_1, \dots, x_n being indeterminates. We define a \mathbf{C} -vector space

$$V := \mathbf{C}x_1 \oplus \dots \oplus \mathbf{C}x_n$$

and a Hermitian inner product on V by

$$(1.1) \quad \left(\sum_{i=1}^n a_i x_i, \sum_{i=1}^n b_i x_i \right) := \sum_{i=1}^n a_i \bar{b}_i$$

where $a_i, b_i \in \mathbf{C} (1 \leq i \leq n)$ and \bar{b}_i denotes the complex conjugate of b_i . Put $V^{(l)} := \text{Sym}^l(V)$, the l -th symmetric tensor product of V , which is identified with $\mathbf{C}[x_1, \dots, x_n]_{(l)}$, the \mathbf{C} -vector space of homogeneous polynomials in x_1, \dots, x_n of degree l . The inner product (1.1) induces an inner product on $V^{(l)}$ by

$$(1.2) \quad (\alpha_1 \cdots \alpha_l, \beta_1 \cdots \beta_l) := \frac{1}{l!} \sum_{\sigma \in \mathfrak{S}_l} \prod_{j=1}^l (\alpha_{\sigma(j)}, \beta_j)$$

where $\alpha_j, \beta_j \in V$, \cdot denotes the symmetric tensor product and \mathfrak{S}_l denotes the symmetric group of degree l .

Let $\rho = \rho_{k,l}^n$ be the representation

$$\det^k \otimes \text{Sym}^l: GL(n, \mathbf{C}) \longrightarrow GL(V^{(l)}).$$

Let \mathfrak{H}_n be Siegel upper half space of degree n , and $\Gamma_n := Sp(n, \mathbf{Z})$ be the group of integral symplectic group of size $2n$.

For a function $f: \mathfrak{H}_n \rightarrow V^{(l)}$ and $M \in Sp(n, \mathbf{R})$, put

$$f|_{k,l}^n M(Z) := \rho(CZ + D)^{-1} f(M\langle Z \rangle)$$

with

$$M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}$$

for $Z \in \mathfrak{H}_n$ and $M \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

The \mathbf{C} -vector space of $V^{(l)}$ -valued Siegel modular forms of degree n and type k, l with respect to Γ_n is defined by

$$(1.3) \quad M_{k,l}^n(V^{(l)}) := \{f: \mathfrak{H}_n \rightarrow V^{(l)} \mid f \text{ is holomorphic on } \mathfrak{H}_n \text{ (and at the cusps if } n=1), \text{ and } f|_{k,l}^n M = f \text{ for all } M \in \Gamma_n\},$$

and the space of cuspforms by

$$(1.4) \quad S_{k,l}^n(V^{(l)}) := \{f \in M_{k,l}^n(V^{(l)}) \mid \lim_{\lambda \rightarrow \infty} f \begin{pmatrix} z & 0 \\ 0 & \sqrt{-1}\lambda \end{pmatrix} = 0 \text{ for all } z \in \mathfrak{H}_{n-1}\}.$$

In the notations (1.3) and (1.4), we omit $(V^{(l)})$ whenever V is obvious. For $l=0$, $M_{k,0}^n(V^{(0)}) = M_{k,0}^n(\mathbf{C})$ is the space of Siegel modular forms of weight k .

Each $f \in M_{k,l}^n(V^{(l)})$ has a Fourier expansion of the following type:

$$(1.5) \quad f(Z) = \sum_{R \geq 0} a(R; f) \mathbf{e}(RZ) \quad (a(R; f) \in V^{(l)}, Z \in \mathfrak{H}_n)$$

where $\mathbf{e}(\cdot) := \exp 2\pi\sqrt{-1} \operatorname{trace}(\cdot)$, and R runs through symmetric, semiintegral, semipositive matrices of size n (We denote such R by “ $R \geq 0$ ” or by “ $R^{(n)} \geq 0$ ”). If f is a cuspform, then $a(R; f) \neq 0$ only for $R > 0$. Throughout this paper, $a(R; f)$ denotes the Fourier coefficient of f at R .

Let $\operatorname{Aut}(\mathbf{C})$ be the group of all field automorphisms of \mathbf{C} . For $\tau \in \operatorname{Aut}(\mathbf{C})$ and a function $f(Z) = \sum_{R \geq 0} a(R; f) \mathbf{e}(RZ)$, set

$$(1.6) \quad f^\tau(Z) := \sum_{R \geq 0} a(R; f) \tau \mathbf{e}(RZ).$$

Let K be any subfield of \mathbf{C} . Put

$$V_K := Kx_1 \oplus \cdots \oplus Kx_n$$

$$M_{k,l}^n(V^{(l)})_K := \{f \in M_{k,l}^n \mid a(R; f) \in V_K^{(l)} \text{ for all } R^{(n)} \geq 0\}$$

and for any subset X of $M_{k,l}^n(V^{(l)})$, set

$$(1.7) \quad X_K := X \cap M_{k,l}^n(V^{(l)})_K.$$

Let $r \leq n$ and put $V_r := \mathbf{C}x_{n-r+1} \oplus \cdots \oplus \mathbf{C}x_n$.

For $1 \leq r \leq n$ with even $k > n + r + 1$, the Langlands-Klingen type Eisenstein series $[f]_r^n \in M_{k,l}^n(V^{(l)})$ is attached to $f \in S_{k,l}^r(V_r^{(l)})$ by

$$(1.8) \quad [f]_r^n(Z) = \sum_{M \in P_{n,r} \backslash \Gamma_n} \rho(CZ + D)^{-1} f(M\langle Z \rangle^*)$$

where $Z \in \mathfrak{S}_n$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $M \langle Z \rangle^*$ denotes the lower-right $r \times r$ block of $M \langle Z \rangle$, and $P_{n,r} = \left\{ \begin{pmatrix} * & * \\ 0 \langle n-r, n+r \rangle & * \end{pmatrix} \in \Gamma_n \right\}$ which is a subgroup of Γ_n . The linear map $[\]_r^n: S_{k,l}^r \rightarrow M_{k,l}^n$ is called the Eisenstein lifting. We define $[\]_0^n$ as the identity map on $S_{k,l}^n$. When $l=0$, the Eisenstein lifting is also defined for $r=0$. In this case, we understand that $M_{k,0}^0(C) = S_{k,0}^0(C) = C$, and the Eisenstein lift of $f=1$

$$(1.9) \quad E_k^n(Z) := [1]_0^n(Z) = \sum_{M \in P_{n,0} \backslash \Gamma_n} \det(CZ + D)^{-k}$$

is Siegel's original Eisenstein series [8].

For $f, g \in M_{k,l}^n$ (at least one in $S_{k,l}^n$), their Petersson inner product (f, g) is defined by:

$$(1.10) \quad (f, g) := \int_{\Gamma_n \backslash \mathfrak{H}_n} (\rho(\sqrt{Y})f(Z), \rho(\sqrt{Y})g(Z)) (\det Y)^{-n-1} dX dY$$

with $Z = X + \sqrt{-1}Y$, X, Y real and $(\ , \)$ in the right-hand side is the inner product (1.2) defined on $V^{(\iota)}$.

We note that if $r < n$, then

$$(1.11) \quad (f, [\phi]_r^n) = 0 \quad \text{for all } f \in S_{k,l}^n \text{ and } \phi \in S_{k,l}^r.$$

Let $L_{\mathcal{C}}^{(n)}$ (resp. $L_{\mathcal{Q}}^{(n)}$) be the abstract Hecke algebra of degree n over \mathcal{C} (resp. \mathcal{Q}) and let

$$t: L_{\mathcal{C}}^{(n)} \longrightarrow \text{End}_{\mathcal{C}}(S_{k,l}^n).$$

be the \mathcal{C} -algebra homomorphism defined as in [1].

We put $T_{\mathcal{C}} := t(L_{\mathcal{C}}^{(n)})$ and $T_{\mathcal{Q}} := t(L_{\mathcal{Q}}^{(n)})$. Let $f \neq 0 \in S_{k,l}^n$ be a common eigenfunction to all $T \in T_{\mathcal{C}}$ (such f is called an eigenform), and for each T , let $\lambda(T) \in \mathcal{C}$ be the eigenvalue on f :

$$(1.12) \quad Tf = \lambda(T)f \quad \text{for all } T \in T_{\mathcal{C}}.$$

Then λ is a \mathcal{C} -algebra homomorphism $\lambda: T_{\mathcal{C}} \rightarrow \mathcal{C}$ and each element of $\widehat{T_{\mathcal{C}}} := \text{Hom}_{\mathcal{C}\text{-alg}}(T_{\mathcal{C}}, \mathcal{C})$ is obtained in this way.

For each $\lambda \in \widehat{T_{\mathcal{C}}}$, put

$$S_{k,l}^n(\lambda) := \{f \in S_{k,l}^n \mid Tf = \lambda(T)f \text{ for all } T \in T_{\mathcal{C}}\}.$$

Then the space of cuspforms decomposes into eigenspaces:

$$S_{k,l}^n = \bigoplus_{\lambda \in \widehat{T_{\mathcal{C}}}} S_{k,l}^n(\lambda).$$

We note that for any $f_1 \in S_{k,l}^n(\lambda_1)$ and $f_2 \in S_{k,l}^n(\lambda_2)$, $(f_1, f_2) = 0$ if $\lambda_1 \neq \lambda_2$.

For each $\lambda \in \widehat{T_{\mathcal{C}}}$, define an extension field of \mathcal{Q} by

$$(1.13) \quad \mathbf{Q}(\lambda) := \mathbf{Q}(\lambda(T) | T \in \mathbf{T}_Q),$$

and for $f \in S_{k,l}^q(\lambda)$ put $\mathbf{Q}(f) := \mathbf{Q}(\lambda)$.

2. Statement of the Theorems.

THEOREM 1. Let $q \geq 1$ be an integer, let $k, l \geq 0$ be even integers satisfying

$$k \geq 2q + 2.$$

Then, the following holds.

$$(1) \quad S_{k,l}^q = S_{k,l}^q \otimes_{\mathbf{Q}} \mathbf{C}.$$

In particular, $\text{Aut}(\mathbf{C})$ acts on $S_{k,l}^q$ by $f \mapsto f^\tau$ in the notation (1.6).

$$(2) \quad \text{Let } \lambda \in \widehat{\mathbf{T}}_c \text{ and } f \neq 0 \in S_{k,l}^q(\lambda)_{\mathbf{Q}(\lambda)}.$$

(i) $\mathbf{Q}(\lambda)$ is a totally real finite extension of \mathbf{Q} with

$$[\mathbf{Q}(\lambda) : \mathbf{Q}] \leq \dim_{\mathbf{C}} S_{k,l}^q.$$

(ii) Let $c(f)$ be the constant of (3.5) below. Then,

$$\left(\frac{c(f)}{(f, f)} \right)^\tau = \frac{c(f^\tau)}{(f^\tau, f^\tau)} \quad \text{for all } \tau \in \text{Aut}(\mathbf{C}).$$

(iii) Let $m := \dim_{\mathbf{C}} S_{k,l}^q(\lambda)$. There exists an orthogonal basis $\{f_j\}_{j=1}^m$ of $S_{k,l}^q(\lambda)$ such that

$$f_1 = f \quad \text{and} \quad f_j \in S_{k,l}^q(\lambda)_{\mathbf{Q}(\lambda)} \quad (1 \leq j \leq m).$$

THEOREM 2. Let $p \geq q \geq 1$ be integers, let $k, l \geq 0$ be even integers satisfying

$$k > p + q + 1.$$

Let $\lambda \in \widehat{\mathbf{T}}_c$ and $f \neq 0 \in S_{k,l}^q(\lambda)_{\mathbf{Q}(\lambda)}$. Then,

$$([f]_q^p)^\tau = [f^\tau]_q^p \quad \text{for all } \tau \in \text{Aut}(\mathbf{C}).$$

3. Differential operators and the Pullback formula.

The first part of this section is a brief description of the “Pullback Formula” of Böcherer-Satoh-Yamazaki [2].

Let $p, q \geq 1$ be integers. Put

$$V_{\mathbf{x}} := \mathbf{C}x_1 \oplus \cdots \oplus \mathbf{C}x_p, \quad \mathbf{x} := (x_1, \dots, x_p)$$

$$V_{\mathbf{y}} := \mathbf{C}y_1 \oplus \cdots \oplus \mathbf{C}y_q, \quad \mathbf{y} := (y_1, \dots, y_q).$$

and for $r \leq \min(p, q)$, put

$$V_{x,r} := Cx_{p-r+1} \oplus \cdots \oplus Cx_p,$$

$$V_{y,r} := Cy_{q-r+1} \oplus \cdots \oplus Cy_q.$$

and define an isomorphism $\sigma: V_{x,r} \rightarrow V_{y,r}$ by $\sigma(x_{p-j}) = y_{q-j}$ ($j < \min(p, q)$).

Let $\mathfrak{Z} = (\mathfrak{Z}_{ij})_{1 \leq i, j \leq p+q}$ be a variable on \mathfrak{H}_{p+q} and

$$\left(\frac{\partial}{\partial \mathfrak{Z}} \right) = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \mathfrak{Z}_{ij}} \right)_{1 \leq i, j \leq p+q}.$$

For a holomorphic function $f: \mathfrak{H}_{p+q} \rightarrow (V_x \oplus V_y)^{(\iota)}$, we define the operators

$$Df := \frac{1}{2\pi\sqrt{-1}} (x \ y) \left(\frac{\partial}{\partial \mathfrak{Z}} \right) f \begin{pmatrix} x \\ y \end{pmatrix},$$

$$D_{\uparrow} f := \frac{1}{2\pi\sqrt{-1}} (x \ 0) \left(\frac{\partial}{\partial \mathfrak{Z}} \right) f \begin{pmatrix} x \\ 0 \end{pmatrix},$$

$$D_{\downarrow} f := \frac{1}{2\pi\sqrt{-1}} (0 \ y) \left(\frac{\partial}{\partial \mathfrak{Z}} \right) f \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Let d be the diagonal embedding

$$\begin{aligned} d: \mathfrak{H}_p \times \mathfrak{H}_q &\longrightarrow \mathfrak{H}_{p+q} \\ (Z, W) &\longmapsto \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} \end{aligned}$$

and let d^* be the pullback of d .

The differential operator $L^{(\iota)}$ is defined in [2] as follows:

$$\begin{aligned} L^{(\iota)} &= d^* \frac{1}{k^{[\iota]}} \\ &\times \sum_{0 \leq 2\nu \leq \iota} \frac{1}{\nu! (l-2\nu)! (2-k-l)^{[\nu]}} (D_{\uparrow} D_{\downarrow})^{2\nu} (D - D_{\uparrow} - D_{\downarrow})^{l-2\nu}, \end{aligned}$$

where

$$a^{[\iota]} := \begin{cases} \frac{(a+b-1)!}{(a-1)!} & \text{for } a > 0, \\ 1 & \text{otherwise.} \end{cases}$$

for integers a and b .

This defines a linear map

$$L^{(\iota)}: M_{k,0}^{p+q}(C) \longrightarrow M_{k,l}^p(V_x^{(\iota)}) \otimes M_{k,l}^q(V_y^{(\iota)}).$$

THEOREM A [2, Prop. 4.4]. *Let $p, q \geq 1$ be integers and $k, l \geq 2$ be even integers satisfying $k > p+q+1$. For each $1 \leq r \leq \min(p, q)$, let $d(r) := \dim_C S_{k,l}^r(V_{y,r}^{(\iota)})$ and $\{f_{j,r}\}_{j=1}^{d(r)}$ an orthonormal basis of $S_{k,l}^r(V_{y,r}^{(\iota)})$ consisting of*

eigenforms.

Let $E_k^{p+q} \in M_{k,0}^{p+q}(\mathbf{C})$ be Siegel's Eisenstein series (1.10) of degree $p+q$ and weight k . Let $\alpha_{k,l}$ and $C_{k,l,r}$ be the constants

$$(3.1) \quad \alpha_{k,l} = \left(-\frac{1}{2\pi\sqrt{-1}} \right)^l \frac{(2k-2)^{[l]}}{l!(k-1)^{[l]}},$$

$$C_{k,l,r} = 2^{r(r-k+1)-l+1} \sqrt{-1}^{rk+l} \frac{\pi^{r(r+1)/2}}{k+l-1} \times \prod_{u=1}^{r-1} \frac{\Gamma(2k-2r+2u-1)(2k-r+u-2)^{[l]}}{(k-r-1+u)\Gamma(2k+u+l-r-1)}.$$

For an eigenform $f \in S_{k,l}^r(V_{\mathbf{y},r}^{(l)})$, put

$$\theta f(Z) := \overline{f(-\bar{Z})},$$

$$\Lambda(f) := \left(\zeta(k)^{-1} \prod_{i=1}^r \zeta(2k-2i)^{-1} \right) L(k-r, f, St),$$

where ζ denotes Riemann zeta function and $L(*, f, St)$ denotes the standard L -function attached to f , respectively.

Then, following equation holds

$$(3.2) \quad L^{(l)} E_k^{p+q}(Z, W) = \alpha_{k,l} \sum_{r=1}^{min(p,q)} C_{k,l,r} \sum_{j=1}^{d(r)} \Lambda(f_{j,r}) [\theta \sigma^{-1} f_{j,r}]_r^p(Z) [f_{j,r}]_r^q(W).$$

In the rest of this section, we study a connection between Fourier coefficients of eigenforms and the partial Fourier expansion of $L^{(l)} E_k^{p+q}$.

Let p, q, k be as in the assumption of Theorem A and suppose also $p \geq q$. Let $R = R^{(p)} \geq 0$ be a symmetric, semiintegral, semipositive matrix of size p . Let $X_p = \{\xi = \prod_{i=1}^p x_i^{\alpha_i} \mid \alpha_i \in \mathbf{Z} \geq 0, \sum_i \alpha_i = l\}$, which is an orthonormal basis of \mathbf{C} -vector space $V_{\mathbf{x}}^{(l)}$.

We attach a $V_{\mathbf{y}}^{(l)}$ -valued modular form $g_{R,\xi}^{p,q} \in M_{k,l}^q(V_{\mathbf{y}}^{(l)})$ for each $R \geq 0$ and $\xi \in X_p$ through the partial Fourier expansion of $L^{(l)} E_k^{p+q}$:

$$(3.3) \quad L^{(l)} E_k^{p+q}(Z, W) = \sum_{R \geq 0} \sum_{\xi \in X_p} g_{R,\xi}^{p,q}(W) \xi \mathbf{e}(RZ).$$

Since the Fourier coefficients of Siegel's Eisenstein series are rational, and $L^{(l)}$ preserves rationality of Fourier coefficients, we have

$$(3.4) \quad g_{R,\xi}^{p,q} \in S_{k,l}^q(V_{\mathbf{y}}^{(l)})_{\mathbf{Q}}.$$

For $F \in M_{k,l}^p(V_{\mathbf{x}}^{(l)})$ and $R = R^{(p)} \geq 0$ and $\xi \in X_p$, let $a(R; F; \xi)$ denote the ξ component of the Fourier coefficient $a(R; F)$.

For each eigenform $f \in S_{k,l}^q(V_{\mathbf{y}}^{(l)})$, put

$$(3.5) \quad c(f) := \alpha_{k,l} C_{k,l,q} \Lambda(f).$$

We note that $c(f)$ is a nonzero constant depending only on $\lambda \in \widehat{T}_c$ such that $S_{k,l}^q(V_{\mathbf{y}}^{(l)}; \lambda) \ni f$. We occasionally write $c(f)$ as $c(\lambda)$ for such λ . By Theorem A, taking inner product of f and $L^{(l)}E_k^{p+q}(-Z, *)$ on $S_{k,l}^q(V_{\mathbf{y}}^{(l)})$, we obtain

$$(3.6) \quad (f, g_{R,\xi}^{p,q}) = c(f)a(R; [\sigma^{-1}f]_q^p; \xi) \quad (R^{(p)} \geq 0, \xi \in X_p).$$

In the rest of the paper, we simply write $M_{k,l}^q(V_{\mathbf{y}}^{(l)})$ (resp. $S_{k,l}^q(V_{\mathbf{y}}^{(l)})$) as $M_{k,l}^q$ (resp. $S_{k,l}^q$).

Let $h_{R,\xi}^{p,q}$ be the projection of $g_{R,\xi}^{p,q}$ to $S_{k,l}^q$. Then, for each eigenform $f \in S_{k,l}^q$, we get

$$(3.7) \quad (f, h_{R,\xi}^{p,q}) = c(f)a(R; [\sigma^{-1}f]_q^p; \xi) \quad (R^{(p)} \geq 0, \xi \in X_p).$$

In particular, when $p=q$,

$$(3.8) \quad (f, h_{R,\xi}^{q,q}) = c(f)a(R; \sigma^{-1}f; \xi) \quad (R^{(q)} \geq 0, \xi \in X_q).$$

PROPOSITION. Let $q \geq 1$ be an integer and $k, l \geq 0$ be even integers satisfying

$$(3.9) \quad k \geq 2q+2.$$

Then,

$$(3.10) \quad S_{k,l}^q = \langle h_{R,\xi}^{q,q} | R^{(q)} > 0, \xi \in X_q \rangle_c,$$

where $\langle \rangle_c$ means the \mathbb{C} -linear span.

Proof. Let S be the space in the right-hand side of (3.10), and S^\perp be its orthogonal complement in $S_{k,l}^q$. Let f be any eigenform in $S_{k,l}^q$. By (3.8), $f \in S^\perp$ if and only if $f=0$. Since $S_{k,l}^q$ has an orthogonal basis consisting of eigenforms, we see that $S^\perp=0$. ■

4. Proof of Theorems.

We shall prove Theorems 1, 2 by similar way as in [7]. First, we introduce a condition on (p, q) .

Condition $C(p, q)$:

$$h_{R,\xi}^{p,q} \in S_{k,l}^q \quad \text{for all } R=R^{(p)} \geq 0 \text{ and } \xi \in X_p.$$

We first show that Theorems 1, 2 are valid for (p, q) which satisfy $C(p, q)$ and $C(q, q)$.

We write the assertion of Theorem 1 for q as $A(q)$ and the assertion of Theorem 2 for (p, q) as $B(p, q)$.

PROPOSITION 4.1. (I) Suppose $q \geq 1$ satisfies $C(q, q)$. Then $A(q)$ holds.

(II) Suppose $p \geq q \geq 1$ satisfy $C(p, q)$ and $C(q, q)$. Then $B(p, q)$ holds.

Proof. (I) Suppose that $k \geq 2q+2$. From (3.10) and $C(q, q)$, $A(q)$ (1) fol-

lows immediately. Next, we show $A(q)(2)(i)$ following [6]. There exists the action of $\text{Aut}(\mathcal{C})$ on $\widehat{T}_{\mathcal{C}}$ which is defined by

$$\lambda^{\tau}(T) := \lambda(T)^{\tau} \quad (T \in T_{\mathcal{Q}})$$

with $\lambda \in \widehat{T}_{\mathcal{C}, \tau} \in \text{Aut}(\mathcal{C})$ and by

$$T_{\mathcal{C}} = T_{\mathcal{Q}} \otimes_{\mathcal{Q}} \mathcal{C}.$$

By $A(q)(1)$ and similar argument to [6, Theorem 1], we have

$$(Tf)^{\tau} = T(f^{\tau}) \quad \text{for all } f \in S_{k, l}^{\mathcal{Q}}, T \in T_{\mathcal{Q}}, \tau \in \text{Aut}(\mathcal{C}).$$

In particular, for all $\tau \in \text{Aut}(\mathcal{C})$ we have

$$(4.1) \quad f^{\tau} \in S_{k, l}^{\mathcal{Q}}(\lambda^{\tau}) \quad \text{for } f \in S_{k, l}^{\mathcal{Q}}(\lambda) \text{ and } \tau \in \text{Aut}(\mathcal{C})$$

and

$$\mathcal{Q}(\lambda^{\tau}) = \mathcal{Q}(\lambda)^{\tau}.$$

Since $\text{Aut}(\mathcal{C})$ acts on $\widehat{T}_{\mathcal{C}}$ whose cardinality $\leq \dim_{\mathcal{C}} S_{k, l}^{\mathcal{Q}}$, we get $[\mathcal{Q}(\lambda) : \mathcal{Q}] \leq \dim_{\mathcal{C}} S_{k, l}^{\mathcal{Q}}$. The field $\mathcal{Q}(\lambda)$ is totally real since all $T \in T_{\mathcal{Q}}$ are Hermitian.

Next, we shall show $A(q)(2)(iii)$. Put $d = \dim_{\mathcal{C}} S_{k, l}^{\mathcal{Q}}$. We choose $\{(R_i, \xi_i) \mid R_i > 0, \xi_i \in X_{\mathcal{Q}}, 1 \leq i \leq d\}$ so that

$$\{h_{R_i, \xi_i}^{q, q} \mid i = 1, \dots, d\}$$

is a \mathcal{C} -basis of $S_{k, l}^{\mathcal{Q}}$. We claim that this is also a \mathcal{Q} -basis of $S_{k, l, \mathcal{Q}}^{\mathcal{Q}}$. For any $h \in S_{k, l, \mathcal{Q}}^{\mathcal{Q}}$, there exists unique $(\alpha_1, \dots, \alpha_d) \in \mathcal{C}^d$ such that

$$h = \sum_{i=1}^d \alpha_i h_{R_i, \xi_i}^{q, q}.$$

Since $h, h_{R_i, \xi_i}^{q, q} \in S_{k, l, \mathcal{Q}}^{\mathcal{Q}}$ by the assumption, we get

$$h = \sum_{i=1}^d \alpha_i^{\tau} h_{R_i, \xi_i}^{q, q} \quad \text{for all } \tau \in \text{Aut}(\mathcal{C}),$$

but by the uniqueness of $(\alpha_1, \dots, \alpha_d)$, we get $(\alpha_1, \dots, \alpha_d)^{\tau} = (\alpha_1, \dots, \alpha_d)$ for all $\tau \in \text{Aut}(\mathcal{C})$.

Hence, $(\alpha_1, \dots, \alpha_d) \in \mathcal{Q}^d$, and we see that

$$(4.2) \quad \{h_{R_i, \xi_i}^{q, q} \mid i = 1, \dots, d\}$$

is a \mathcal{Q} -basis of $S_{k, l, \mathcal{Q}}^{\mathcal{Q}}$.

For $T \in T_{\mathcal{Q}}$ let $B(T) \in M_d(\mathcal{C})$ be the representation matrix of T with respect to the basis (4.2). Since $S_{k, l, \mathcal{Q}}^{\mathcal{Q}}$ is $T_{\mathcal{Q}}$ -stable, $B(T)$ lies in $M_d(\mathcal{Q})$.

For $\lambda \in \widehat{T}_{\mathcal{C}}$, put $m = m(\lambda) := \dim_{\mathcal{C}} S_{k, l}^{\mathcal{Q}}(\lambda)$. Let $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ be column vectors in \mathcal{C}^d which spans $\{\mathbf{a} \in \mathcal{C}^d \mid (B(T) - \lambda(T)1_d)\mathbf{a} = 0 \text{ for all } T \in T_{\mathcal{Q}}\}$. Since $B(T) \in M_d(\mathcal{Q})$ and $\lambda(T) \in \mathcal{Q}(\lambda)$, we can take such $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ in $\mathcal{Q}(\lambda)^d$. Put

$$\phi_j = (h_{R_1, \xi_1}^{q,q} \cdots h_{R_d, \xi_d}^{q,q}) \mathbf{a}_j \quad (1 \leq j \leq m).$$

Then, $\{\phi_j\}_{j=1}^m$ is a \mathbf{C} -basis of $S_{k,l}^q(\lambda)$, which is also a $\mathbf{Q}(\lambda)$ -basis of $S_{k,l}^q(\lambda)_{\mathbf{Q}(\lambda)}$.

For given $f \neq 0 \in S_{k,l}^q(\lambda)_{\mathbf{Q}(\lambda)}$, we choose an index j_0 so that $\{\phi_j | 1 \leq j \leq m, j \neq j_0\} \cup \{f\}$ is a $\mathbf{Q}(\lambda)$ -(resp. \mathbf{C} -) basis of $S_{k,l}^q(\lambda)_{\mathbf{Q}(\lambda)}$ (resp. $S_{k,l}^q(\lambda)$). Let $j_0 = m$ by changing order.

For any $\phi \in S_{k,l}^q(\lambda)_{\mathbf{Q}(\lambda)}$, (3.8) implies

$$\begin{aligned} (\phi, \phi_j) &= ((\phi, h_{R_1, \xi_1}^{q,q}) \cdots (\phi, h_{R_d, \xi_d}^{q,q})) \mathbf{a}_j \\ &= c(\lambda)(a(R_1; \sigma^{-1}\phi; \xi_1) \cdots a(R_d; \sigma^{-1}\phi; \xi_d)) \mathbf{a}_j \\ &\in c(\lambda) \cdot \mathbf{Q}(\lambda) \quad (1 \leq j \leq m-1) \end{aligned}$$

and in particular,

$$\frac{(\phi, \phi_j)}{(\phi_j, \phi_j)} \in \mathbf{Q}(\lambda) \quad (1 \leq j \leq m-1, \phi \in \{f\} \cup \{\phi_j\}_{j=1}^{m-1}).$$

Hence, by Gram-Schmidt orthogonalization on $\{f\} \cup \{\phi_j\}_{j=1}^{m-1}$, we get the required basis of $S_{k,l}^q(\lambda)$.

Next, we prove $A(q)(2)(ii)$. For given $f \neq 0 \in S_{k,l}^q(\lambda)_{\mathbf{Q}(\lambda)}$, take $R > 0$ and $\xi \in X_q$ so that $a(R; \sigma^{-1}f; \xi) \neq 0$. Let $h(\lambda)$ be the projection of $h_{R,\xi}^{q,q}$ to $S_{k,l}^q(\lambda)$. Using $h_{R,\xi}^{q,q} \in S_{k,l}^q(\lambda)_{\mathbf{Q}}$ and (4.1), we see

$$(4.3) \quad h(\lambda)^\tau = h(\lambda^\tau)$$

for $\tau \in \text{Aut}(\mathbf{C})$. Let $\{f_i (= f), \dots, f_m\}$ be the orthogonal basis of $A(q)(2)(iii)$.

Writing

$$h(\lambda) = \sum_{j=1}^m \beta_j f_j \quad (\beta_j \in \mathbf{Q}(\lambda)),$$

we have

$$(4.4) \quad (f, h(\lambda)) = (f, h_{R,\xi}^{q,q}) = c(f) a(R; \sigma^{-1}f; \xi)$$

and

$$(4.5) \quad (f, h(\lambda)) = \left(f, \sum_{j=1}^m \beta_j f_j \right) = \beta_1 (f, f).$$

On the other hand, we get for $\tau \in \text{Aut}(\mathbf{C})$,

$$(f^\tau, h(\lambda)^\tau) = (f^\tau, h_{R,\xi}^{q,q}) = c(f^\tau) a(R; \sigma^{-1}(f^\tau); \xi)$$

by (4.4) and

$$(f^\tau, h(\lambda^\tau)) = (f^\tau, h(\lambda)^\tau) = \beta_1^\tau (f^\tau, f^\tau)$$

by (4.5).

Therefore

$$\left(\frac{c(f)}{(f, f)}\right)^\tau = \frac{\beta_1^\tau}{a(R; \sigma^{-1}(f^\tau); \xi)} = \frac{c(f^\tau)}{(f^\tau, f^\tau)},$$

Thus the part (I) is proved.

(II) Suppose that $k > p + q + 1$. By $C(q, q)$ and $k \geq 2q + 2$, $A(q)$ is valid. For any $R = R^{(p)} \geq 0$ and $\xi \in X_p$, let $h(\lambda)$ be the projection of $h_{R, \xi}^{p, q}$ to $S_{k, l}^q(\lambda)$. By $C(p, q)$, (4.3) holds again for this $h(\lambda)$, and by the same argument as in (I), we find a $\beta \in Q(\lambda)$ such that

$$\beta \frac{(f, f)}{c(f)} = a(R; [\sigma^{-1}f]_q^p; \xi),$$

$$\beta^\tau \frac{(f^\tau, f^\tau)}{c(f^\tau)} = a(R; [\sigma^{-1}(f^\tau)]_q^p; \xi) \quad \text{for all } \tau \in \text{Aut}(\mathcal{C}).$$

Then, from $A(q)$ (2) (ii) and the expression above,

$$a(R; [\sigma^{-1}f]_q^p; \xi)^\tau = a(R; [\sigma^{-1}(f^\tau)]_q^p; \xi)$$

for any $R = R^{(p)} \geq 0$, $\xi \in X_p$ and $\tau \in \text{Aut}(\mathcal{C})$. Part (II) is proved. ■

Remark. $A(q)$ (ii) and (4.1) imply the existence of an orthogonal basis B_q of $S_{k, l}^q$ such that:

- (1) B_q is permuted by the action of $\text{Aut}(\mathcal{C})$.
- (2) Each $f \in B_q$ satisfies $f \in S_{k, l_{Q(f)}}^q$.

Now, we shall show that the condition $C(p, q)$ actually holds when k is sufficiently large.

PROPOSITION 4.2. *Let $p \geq q \geq 1$ be integers and $k, l \geq 0$ be even integers such that*

$$k > p + q + 1.$$

Then,

- (1) $C(p', 1)$ holds for $1 \leq p' \leq p$.
- (2) Suppose that

$$C(p, r), C(q, r) \quad \text{and} \quad C(r, r) \quad \text{hold for } 1 \leq r < q.$$

Then, $C(p, q)$ holds.

Proof. (1) Let $R^{(p)} \geq 0$ and $\xi \in X_p$ be arbitrary. In this case, $g_{R, \xi}^{p', 1}$ and $h_{R, \xi}^{p', 1}$ are identified with elliptic modular forms by

$$V_{\mathbf{y}} = \mathcal{C}y_1$$

and

$$M_{k, l}^1(V_{\mathbf{y}^{(l)}}) = M_{k+l, 0}^1(\mathcal{C}) \cdot y_1^l.$$

Therefore

$$g_{R, \xi}^{p', 1} - h_{R, \xi}^{p', 1} = a(0; g_{R, \xi}^{p', 1}; y_1^l) E_{k+l}^1 y_1^l,$$

where $E_{k+l}^1: \mathfrak{H}_1 \rightarrow \mathcal{C}$ is the elliptic Eisenstein series, whose Fourier coefficients

lie in Q . Then, by $g_{k,\xi}^{p',1} \in M_{k,l_Q}^1$, we see $h_{k,\xi}^{p',1} \in S_{k,l_Q}^1$.

(2) By the assumption and Proposition 4.1, we can assume $A(r)$ of Theorem 1, $B(q, r)$ and $B(p, r)$ of Theorem 2 for $1 \leq r < q$, noting that $k > p + r + 1 \geq q + r + 1 > 2r + 1$. In particular by $A(r)$ (2) (iii), for each r , there exists an orthogonal basis B_r of $S_{k,l}^r$ as stated in the Remark above.

By Theorem A of section 3, together with (3.6), (3.7), (3.8), we have

$$(4.6) \quad g_{k,\xi}^{p,q} - h_{k,\xi}^{p,q} = \sum_{r=1}^{q-1} \sum_{f \in B_r} \frac{c(f)}{(f, f)} a(R; [\sigma^{-1}f]^p; \xi)[f]^q$$

for any $R = R^{(p)} \geq 0$ and $\xi \in X_p$.

Since B_r is permuted by $\text{Aut}(C)$, $f \in B_r$ satisfies $f \in S_{k,l_{Q(f)}}^r$, and by $A(r)$ (2) (ii), $B(q, r)$, $B(p, r)$, we see the right-hand side of (4.6) is invariant under $\text{Aut}(C)$. Thus, $h_{k,\xi}^{p,q} \in S_{k,l_Q}^q$. ■

Theorems 1, 2 are proved by induction using Proposition 4.2.

Proof of Theorem 1. Let q, k, l satisfy the assumption. Then, $C(1, 1)$ is valid by Proposition 4.2(1).

Let $1 \leq q' < q$ and suppose that

$$(4.7) \quad C(m, n) \text{ is valid for } (m, n) \text{ with } 1 \leq n \leq m \leq q'.$$

Again by Proposition 4.2(1), $C(q' + 1, 1)$ is valid. By (4.7) and repeated use of Proposition 4.2(2), $C(q' + 1, n)$ holds for $1 \leq n \leq q' + 1$ (Note that $k \geq 2q + 2 \geq 2(q' + 1) + 1$).

Thus we have:

$$(4.8) \quad C(m, n) \text{ is valid for } (m, n) \text{ with } 1 \leq n \leq m \leq q' + 1,$$

and finally we obtain $C(q, q)$, which imply Theorem 1 by Proposition 4.1. ■

Proof of Theorem 2. Let p, q, k, l satisfy the assumption. Then, $C(m, n)$ is valid for $1 \leq n \leq m \leq q$, as seen above. We have $C(p, 1)$, and using Proposition 4.2(2) repeatedly, we get $C(p, q)$ and the assertion of Theorem 2. ■

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