GENERALIZED RELATION BETWEEN LIQUID PRESSURE AND SOLID COMPRESSIVE PRESSURE IN NON-UNIDIMENSIONAL FILTER CAKES

Tsutomu ARAGAKI, Setsuro HIRAOKA, and Ikuho YAMADA
Department of Applied Chemistry, Nagoya Institute of Technology, Nagoya 466
Toshiro MURASE, Eiji IRITANI and Mompei SHIRATO
Department of Chemical Engineering, Nagoya University, Nagoya 464

Key Words: Filtration, Highly Compacted Cake, Generalized Equation, Non-unidimensional Filtration, Liquid Pressure, Solid Compressive Pressure, Orthogonal Curvilinear Coordinate

In recent studies of cake filtration, a new deliquoring technique called as "filtration consolidation" was proposed. It is principally based on the mechanism of non-unidimensional filtration and gives a highly compacted cake without any additional mechanical load. To investigate the deliquoring mechanism of filtration consolidation in detail, it is essential to find a relation between liquid pressure P_L and solid compressive pressure P_S in non-unidimensional filter cake.

Accordingly, a generalized P_L–P_S relation applicable to any cake profile and filtrate flow pattern of complicated geometry is derived in terms of an orthogonal curvilinear coordinate system. It is also shown that, starting from the generalized equation, all the P_L–P_S relations already reported for non-unidimensional filter cakes of relatively simple geometries can be easily reduced.

Introduction

In the filtration of solid-liquid suspensions, deliquoring of filter cake is increasingly important in many fields, such as chemical process industries, sewage sludge treatments, fruit pulps processing and fermentation industries. In these fields, almost all methods of reducing the liquid content of cakes have been based on mechanical pressing systems that involve considerable investment expense.

In a recent study[5,2)] of cake filtration, a new deliquoring principle called filtration consolidation, quite different from traditional principles, was proposed. Based on this new principle, a horizontal plate filter was designed, having an impermeable rubber membrane at the top and slurry inlets in the side wall of the filter chamber, as illustrated in **Fig. 1**. In this filter, once the cake fills the filter chamber, the stream line of filtrate flow changes from a straight line to a non-unidimensional curve, as shown in Fig. 1(b). This change of flow pattern causes both the lowering of liquid pressure P_L and the rising of solid compressive pressure P_S in almost all parts of the filter cake. Thus, it becomes possible to obtain a much more compacted cake than that in normal filtration.

In filtration consolidation, there is no doubt that the non-unidimensional flow of filtrate through the cake plays an important role. To investigate the mechanism of these consolidation phenomena in detail, it is essential to set up a generalized relation between P_L and P_S, taking non-unidimensional flow of filtrate into account.

In the field of soil mechanics, the famous piston-spring analogy of Terzaghi,[8,9)] which lead to Eq. (1), has been used:

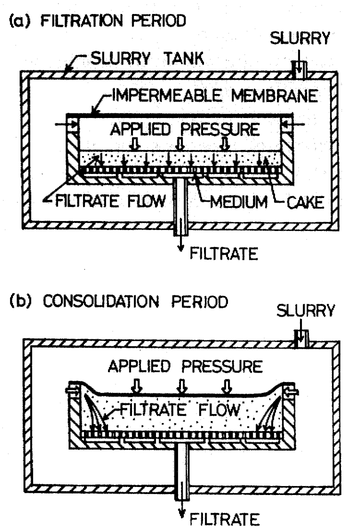

Fig. 1. Filtrate flow pattern in filtration and consolidation period in a horizontal plate filter having an impermeable rubber membrane

Received November 2, 1988. Correspondence concerning this article should be addressed to T. Aragaki. M. Shirato is now at Nagoya Industrial Science Research Institute.

$$P_L+P_S=P \tag{1}$$

where P is the pressure applied to the system.

In the field of filtration, Collins[2)] and Tiller *et al.*[10)] derived the following equation from the force balance acting on a differential slice dx in filter cake:

$$\frac{\partial P_L}{\partial x}+\frac{\partial P_S}{\partial x}=0 \tag{2}$$

On integration, Eq. (2) reduces to Eq. (1).

These relations, Eq. (1) and (2), hold in the strict sense only for normal unidimensional filtration, where stream lines of filtrate flow are straight and parallel to each other, and the surface of the cake is always flat.

The geometry of cake profile and the flow pattern of filtrate depend generally on both the shape of the filter medium and the constraining wall, as illustrated in **Fig. 2**. Filtration on the tubular (cylindrical) filter, shown in Fig. 2(a), and on the rectangular leaf, shown in Fig. 2(b), both having two constraining walls, is 2-dimensional. On the other hand, the circular leaf and the spherical filter without constraining wall will lead us to 3-dimensional filtration as shown in Figs. 2(c) and (d). Shirato and co-workers[4,6)] have derived the P_L-P_S relations for several simple geometries as shown in Fig. 2:

(i) for 2-dimensional filtration on tubular filter

$$P_L+P_S=P+\int_r^{r_0}(1-k)P_S\frac{1}{r}dr \tag{3a}$$

(ii) for 3-dimensional filtration on spherical surface

$$P_L+P_S=P+2\int_r^{r_0}(1-k)P_S\frac{1}{r}dr \tag{3b}$$

(iii) for 3-dimensional filtration on circular leaf

$$P_L+P_S=P+\int_\xi^{\xi_0}(1-k)P_S\left(\frac{\xi}{\xi^2+\eta^2}+\frac{\xi}{\xi^2+1}\right)d\xi \tag{3c}$$

and

(iv) for 2-dimensional filtration on rectangular leaf

$$P_L+P_S=P+\int_\xi^{\xi_0}(1-k)P_S\frac{\xi}{(\xi^2+\eta^2)}d\xi \tag{3d}$$

Recently Tiller *et al.*[11)] also derived Eq. (3a). In the equations above, k denotes the coefficient of earth pressure at rest in soil mechanics, as will be mentioned again below, and variables r, ξ, η and the coordinate systems used are shown in Fig. 2.

In filtration consolidation, however, more complicated flow patterns may be encountered than those in the cases (i) to (iv) of non-unidimensional filtrations cited above.

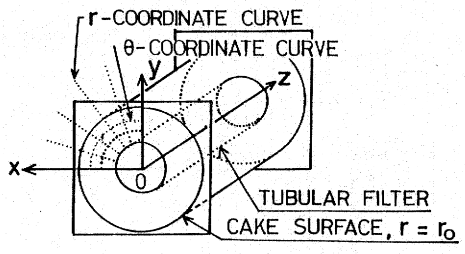

Fig. 2(a). Filtration on tubular leaf (cylindrical coordinate)

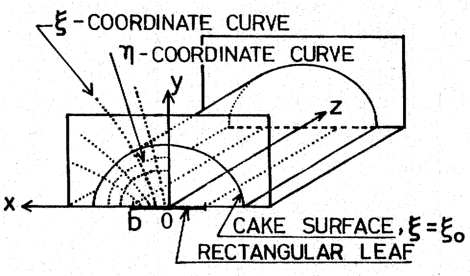

Fig. 2(b). Filtration on rectangular leaf (elliptic cylindrical coordinate)

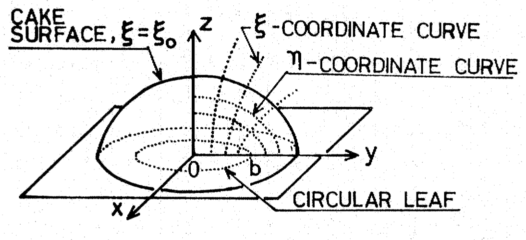

Fig. 2(c). Filtration on circular leaf (oblate spherical coordinate)

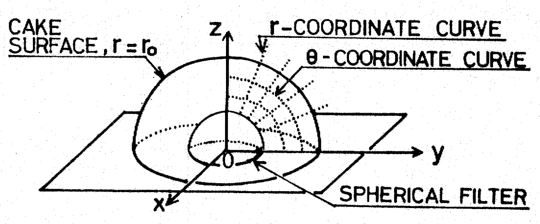

Fig. 2(d). Filtration on spherical leaf (spherical coordinate)

1. Derivation of Generalized P_L–P_S Relation

We assume that the stream line of the filtrate in filter cake is perpendicular to the isopotential surface as is usual with the flow through porous media, and introduce an orthogonal curvilinear coordinate system (x^1, x^2, x^3). We denote the stream line of filtrate by x^1-coordinate curve, and represent the isopotential surface by x^1-coordinate surface, to which both x^2- and x^3-coordinate curves are perpendicular. The volume element formed by the $(x^i-(1/2)dx^i)$- and $(x^i+(1/2)dx^i)$-coordinate surfaces has the shape of a rectangular parallelepiped enclosing a central point $M(x^1, x^2, x^3)$ as illustrated in **Fig. 3**, where $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, $\boldsymbol{g}_3$ are covariant base vectors in the direction of increasing x^1, x^2 and x^3, respectively.

The force per unit area, F, due to the hydraulic and

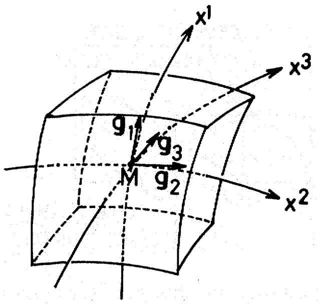

Fig. 3. Volume element in orthogonal curvilinear coordinate system

the solid compressive pressures, on any surface element normal to the direction of $\boldsymbol{g}_1$ is simply evaluated as:

$$F=P_L+P_S \tag{4}$$

On the other hand, the force per unit area, F_S, acting on the surface element normal to the direction of $\boldsymbol{g}_2$ or $\boldsymbol{g}_3$ is:

$$F_S=P_L+kP_S \tag{5}$$

This expression is different from Eq. (4) because the lateral value of the solid compressive pressure is not equal to the normal P_S, as known in soil mechanics, and k in Eq. (5) denotes again the ratio of lateral to normal solid compressive pressure.[1,4,11)]

The generalized P_L–P_S relation is developed from a force balance over the volume element shown in Fig. 3. We begin by considering the pair of surface elements perpendicular to the x^1-curve. The forces acting on the $(x^1-(1/2)dx^1)$ and $(x^1+(1/2)dx^1)$ coordinate surfaces are:

$$F(\boldsymbol{g}_2\times\boldsymbol{g}_3)dx^2dx^3-\frac{\partial}{\partial x^1}\{F(\boldsymbol{g}_2\times\boldsymbol{g}_3)dx^2dx^3\}\frac{1}{2}dx^1,$$

and

$$-\left[F(\boldsymbol{g}_2\times\boldsymbol{g}_3)dx^2dx^3+\frac{\partial}{\partial x^1}\{F(\boldsymbol{g}_2\times\boldsymbol{g}_3)dx^2dx^3\}\frac{1}{2}dx^1\right]$$

respectively. The second expression has a negative sign because the force acts in the direction of the interior normal, whereas $(\boldsymbol{g}_2\times\boldsymbol{g}_3)$ points to the exterior normal. Thus, the resultant $d\boldsymbol{F}_1$ of these forces acting on the pair of surfaces is

$$d\boldsymbol{F}_1=-\frac{\partial}{\partial x^1}\{F(\boldsymbol{g}_2\times\boldsymbol{g}_3)\}dx^1dx^2dx^3 \tag{6}$$

Similar expressions can be written for the other two pairs of surface elements perpendicular respectively to the x^2- and the x^3-curve in the forms.

$$d\boldsymbol{F}_2=-\frac{\partial}{\partial x^2}\{F_S(\boldsymbol{g}_3\times\boldsymbol{g}_1)\}dx^1dx^2dx^3 \tag{7}$$

$$d\boldsymbol{F}_3=-\frac{\partial}{\partial x^3}\{F_S(\boldsymbol{g}_1\times\boldsymbol{g}_2)\}dx^1dx^2dx^3 \tag{8}$$

Since inertial forces have been shown to be negligible in filtration,[12)] the resultant total force of $d\boldsymbol{F}_1$, $d\boldsymbol{F}_2$ and $d\boldsymbol{F}_3$ must be equal to zero, i.e.,

$$d\boldsymbol{F}_1+d\boldsymbol{F}_2+d\boldsymbol{F}_3=0 \tag{9}$$

Substituting Eqs. (6), (7) and (8) into Eq. (9) and considering the following formulas:

$$\begin{aligned}&\boldsymbol{g}_2\times\boldsymbol{g}_3=(J/g_{11})\boldsymbol{g}_1;\quad \boldsymbol{g}_3\times\boldsymbol{g}_1=(J/g_{22})\boldsymbol{g}_2;\\&\boldsymbol{g}_1\times\boldsymbol{g}_2=(J/g_{33})\boldsymbol{g}_3;\\&J\equiv\boldsymbol{g}_1\cdot(\boldsymbol{g}_2\times\boldsymbol{g}_3)\equiv[\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{g}_3],\\&g_{ij}\equiv\boldsymbol{g}_i\cdot\boldsymbol{g}_j\end{aligned} \tag{10}$$

we obtain

$$\{F(J/g_{11})\boldsymbol{g}_1\}_{,1}+\{F_S(J/g_{22})\boldsymbol{g}_2\}_{,2}+\{F_S(J/g_{33})\boldsymbol{g}_3\}_{,3}=0 \tag{11}$$

where the subscript ",i" denotes differentiation with respect to x^i. Introducing the Christoffel symbol of the second kind, the differentiation of base vector $\boldsymbol{g}_i$ with respect to x^j can be written in the form

$$\boldsymbol{g}_{i,j}\equiv\frac{\partial\boldsymbol{g}_i}{\partial x^j}=\Gamma^m_{ij}\boldsymbol{g}_m \tag{12}$$

with the summation convention implied over the repeated index m. Applying this to Eq. (11), we have

$$\begin{aligned}&\{F(J/g_{11})\}_{,1}\boldsymbol{g}_1+\{F_S(J/g_{22})\}_{,2}\boldsymbol{g}_2\\&\quad+\{F_S(J/g_{33})\}_{,3}\boldsymbol{g}_3+J[F(\Gamma^m_{11}/g_{11})\\&\quad+F_S(\Gamma^m_{22}/g_{22})+F_S(\Gamma^m_{33}/g_{33})]\boldsymbol{g}_m=0\end{aligned} \tag{13}$$

This vectorial form, Eq. (13), consists of three scalar relations. Among them, the most interesting relation is a force balance in the $\boldsymbol{g}_1$-direction, i.e., the direction of the flow of filtrate. Multiplying Eq. (13) by $\boldsymbol{g}_1$ and noting that $\boldsymbol{g}_1\cdot\boldsymbol{g}_2=\boldsymbol{g}_1\cdot\boldsymbol{g}_3=0$ due to the orthogonality, we have

$$\begin{aligned}&F_{,1}+F\left(\frac{g_{11}}{J}\right)\left(\frac{J}{g_{11}}\right)_{,1}+F\Gamma^1_{11}\\&\quad+F_S\{\Gamma^1_{22}(g_{11}/g_{22})+\Gamma^1_{33}(g_{11}/g_{33})\}=0\end{aligned} \tag{14}$$

The orther two scalar relations are given in Appendix. Taking account of $J\equiv\boldsymbol{g}_1\cdot\boldsymbol{g}_2\times\boldsymbol{g}_3=\sqrt{g_{11}g_{22}g_{33}}$, the second term in Eq. (14) can be rearranged to

$$\begin{aligned}\left(\frac{g_{11}}{J}\right)\left(\frac{J}{g_{11}}\right)_{,1}&=[\ln\sqrt{g_{22}g_{33}/g_{11}}]_{,1}\\&=\frac{1}{2}\left\{\frac{g_{22,1}}{g_{22}}+\frac{g_{33,1}}{g_{33}}-\frac{g_{11,1}}{g_{11}}\right\}\end{aligned} \tag{15}$$

while differentiation of the metric tensor g_{22} with respect to x^1 is

$$g_{22,1} \equiv (\boldsymbol{g}_2 \cdot \boldsymbol{g}_2)_{,1} = 2\boldsymbol{g}_{2,1} \cdot \boldsymbol{g}_2 = 2\Gamma^2_{21} g_{22} \tag{16a}$$

Similarly we have

$$g_{33,1} \equiv 2\Gamma^3_{31} g_{33} \tag{16b}$$

$$g_{11.1} = 2\Gamma^1_{11} g_{11} \tag{16c}$$

Moreover, differentiation of the relation $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = 0$ $(i \neq j)$ with respect to x^k gives

$$\Gamma^i_{ik} = -\Gamma^i_{jk} g_{ii}/g_{jj} \text{ (no summation over } i \text{ and } j) \tag{17}$$

and then

$$\Gamma^1_{22}(g_{11}/g_{22}) = -\Gamma^2_{12} = -\Gamma^2_{21} \tag{18a}$$

$$\Gamma^1_{33}(g_{11}/g_{33}) = -\Gamma^3_{13} = -\Gamma^3_{31} \tag{18b}$$

Using Eqs. (15) to (18) and then substituting the definitions of F and F_S into Eq. (14), we finally obtain the relation of force balance in the $\boldsymbol{g}_1$-direction as

$$\frac{\partial}{\partial x^1}(P_L + P_S) + (1-k)P_S(\Gamma^2_{21} + \Gamma^3_{31}) = 0 \tag{19}$$

This equation, relating P_L to P_S, is of the generalized form for any geometry of cake profile and any flow pattern in the filter cake. It is here emphasized that the coordinate must be chosen properly so that the x^1-coordinate curves do represent the stream line of filtrate flow.

In the case of usual filtration, i.e., unidimensional filtration, therefore, it is natural to use the Cartesian coordinate system (x, y, z), since the stream lines of filtrate flow are always straight and parallel to each other, and isopotential surfaces are flat. Noting that in the Cartesian coordinate system all components of Γ^k_{ij} are zero, it is obvious that Eq. (19) reduces to Eq. (2).

In the case of non-unidimensional filtration, however, the second term in Eq. (19) does not vanish, and it follows that some different expressions relating P_L to P_S must be obtained.

2. Application of Eq. (19)

2.1 Filtration on spherical surface

Taking account of the geometry concerned, it is easy to see that the spherical coordinate system (r, θ, ϕ) is appropriate in handling the present problem. Since the r-coordinate curves imply the flow pattern of filtrate, the variable x^1 must be equivalent to r, and then we read $(x^1, x^2, x^3) \equiv (r, \theta, \phi)$. If we denote unit vectors pointing in the positive directions of the r-, θ-, ϕ-coordinate curves by $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$, respectively, we have $(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \equiv (\boldsymbol{e}_r, r\boldsymbol{e}_\theta, r\sin\theta\boldsymbol{e}_\phi)$ and $(\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3) \equiv (\boldsymbol{e}_r, \boldsymbol{e}_\theta/r, \boldsymbol{e}_\phi/(r\sin\theta))$, where $\boldsymbol{g}^i$ is contravariant base vector. Using the formula $\Gamma^k_{ij} = \boldsymbol{g}^k \cdot \boldsymbol{g}_{i,j}$, we have $\Gamma^2_{21} = \Gamma^3_{31} = 1/r$, and it follows from Eq. (19) that

$$\frac{\partial}{\partial r}(P_L + P_S) + (1-k)P_S\frac{2}{r} = 0 \tag{20}$$

Noting that $P_L = P$ and $P_S = 0$ at the cake surface $r = r_0$, and integrating Eq. (20) over the range of $[r, r_0]$, we obtain Eq. (3b).

2.2 Filtration on tubular (cylindrical) surface

The cylindrical coordinate system (r, θ, z) can be used in this case. By a similar way to that mentioned above it can be easily shown that $\Gamma^2_{21} = 1/r$ and $\Gamma^3_{31} = 0$, and substituting them into Eq. (19) we have

$$\frac{\partial}{\partial r}(P_L + P_S) + (1-k)P_S\frac{1}{r} = 0 \tag{21}$$

On integration, this equation reduces to Eq. (3a).

2.3 Filtration on circular leaf

Using the oblate spherical coordinate system (ξ, η, ζ), the stream line of filtrate and the isopotential line are represented by

$$\frac{x^2 + y^2}{b^2(1-\eta^2)} - \frac{z^2}{b^2\eta^2} = 1$$

and

$$\frac{x^2 + y^2}{b^2(1+\xi^2)} + \frac{z^2}{b^2\xi^2} = 1$$

respectively, where b denotes the radius of the circular leaf. These relations can be transformed as

$$\left\{\begin{array}{l} x = b\sqrt{(1+\xi^2)(1-\eta^2)}\cos\zeta \\ y = b\sqrt{(1+\xi^2)(1-\eta^2)}\sin\zeta \\ z = b\xi\eta \\ (0 \leqq \xi < \infty,\ 0 \leqq \eta < 1) \end{array}\right. \tag{22}$$

In the present situation it is convenient to start from position vector $\boldsymbol{R}$ in the form

$$\boldsymbol{R} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z \tag{23}$$

where $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ are unit vectors in Cartesian coordinates, and x, y, z are functions of ξ, η, ζ as given by Eq. (22). Noting the definition of covariant base vector, $\boldsymbol{g}_i \equiv \partial\boldsymbol{R}/\partial x^i$ $(i = 1, 2, 3)$ where $(x^1, x^2, x^3) \equiv (\xi, \eta, \zeta)$, we have the following formulae:

$$\left\{\begin{array}{l} \boldsymbol{g}_1 \equiv \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}\,\boldsymbol{e}_\xi \\ \boldsymbol{g}_2 \equiv \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2}\,\boldsymbol{e}_\eta \\ \boldsymbol{g}_3 \equiv \sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}\,\boldsymbol{e}_\zeta \end{array}\right. \tag{24}$$

where $\boldsymbol{e}_\xi$, $\boldsymbol{e}_\eta$, $\boldsymbol{e}_\zeta$ denote unit vectors in the directions of the ξ-, η-, ζ-coordinate curves respectively. Substituting Eq. (22) into (24) and then multiplying each $\boldsymbol{g}_i$ by itself, we get

$$\begin{cases} \boldsymbol{g}_2 \cdot \boldsymbol{g}_2 \equiv g_{22} = b^2(\xi^2+\eta^2)/(1-\eta^2) \\ \boldsymbol{g}_3 \cdot \boldsymbol{g}_3 \equiv g_{33} = b^2(1+\xi^2)/(1-\eta^2) \end{cases} \tag{25}$$

Differentiating Eq. (25) with respect to $x^1(\equiv\xi)$, and dividing by g_{22} or g_{33}, we have $\Gamma^2_{21}=\xi/(\xi^2+\eta^2)$ and $\Gamma^3_{31}=\xi/(1+\xi^2)$. Thus from Eq. (19) the following equation results:

$$\frac{\partial}{\partial\xi}(P_L+P_S)+(1-k)P_S\left[\frac{\xi}{\xi^2+\eta^2}+\frac{\xi}{1+\xi^2}\right]=0 \tag{26}$$

This equation is of the differential form corresponding to Eq. (3c).

2.4 Filtration on rectangular leaf

This problem can be easily handled by the elliptic cylindrical coordinate system (ξ, η, ζ), which is written in the form

$$\begin{cases} x=b\sqrt{(1+\xi^2)(1-\eta^2)} \\ y=b\xi\eta \\ z=\xi \end{cases} \tag{27}$$

Application of a similar method to that used above yields the relations $\Gamma^2_{21}=\xi/(\xi^2+\eta^2)$ and $\Gamma^3_{31}=0$. Thus,

$$\frac{\partial}{\partial\xi}(P_L+P_S)+(1-k)P_S\left(\frac{\xi}{\xi^2+\eta^2}\right)=0 \tag{28}$$

On integration, this reduces to Eq. (3d).

3. Filtration Consolidation

Flow pattern of filtrate in consolidation period may be more complicate than that in normal filtration, and it may depend on the structure of filter. However, even if we restrict ourself to the situation where geometries concerned are 3-dimensional but axisymmetric, almost all parts of possible situation will be covered. Therefore, it is presumed that the stream line and the isopotential surface are represented by $\xi=\xi(r,z)$ and $\eta=\eta(r,z)$, respectively, where r, z denote variables in cylindrical coordinate. Inverse transformation of the above relations will give $r=G(\xi,\eta)$ and $z=H(\xi,\eta)$, from which we have

$$\begin{cases} x=G(\xi,\eta)\cos\zeta \\ y=G(\xi,\eta)\sin\zeta \\ z=H(\xi,\eta) \end{cases} \tag{29}$$

Here, the functions, G and H, should be decided properly so as to fit the corresponded geometries. Just the same procedure as used in deriving Eq. (26) can be applied again. Without going into details, the relevant terms are

$$\begin{cases} \Gamma^2_{21}=\dfrac{\partial}{\partial\xi}\ln\sqrt{\left(\dfrac{\partial G}{\partial\eta}\right)^2+\left(\dfrac{\partial H}{\partial\eta}\right)^2} \\ \Gamma^3_{31}=\dfrac{\partial}{\partial\xi}\ln\sqrt{G^2} \end{cases} \tag{30}$$

As a result, according to Eq. (19) we have

$$\frac{\partial}{\partial\xi}(P_L+P_S)+(1-k)P_S\cdot\frac{\partial}{\partial\xi}\times\ln\sqrt{G^2\left\{\left(\frac{\partial G}{\partial\eta}\right)^2+\left(\frac{\partial H}{\partial\eta}\right)^2\right\}}=0 \tag{31}$$

Conclusion

A generalized equation (19) relating the liquid pressure P_L to the solid compressive pressure P_S in filter cake, which holds for any geometry of cake profile and filtrate flow pattern, is presented. Since the P_L–P_S relation is essential in predicting filtration behavior based on compression-permeability cell results, this equation plays an important role in investigating the mechanism of filtration consolidation, including non-unidimensional filtration.

The proposed equation has been applied to several cases of non-unidimensional filtration, and it is shown that all the P_L–P_S relations already reported can be easily reduced from the generalized form.

Appendix

Multiplying Eq. (13) by $\boldsymbol{g}_2$, and rearranging, we have

$$F_{S,2}+F_S\left(\frac{g_{22}}{J}\right)\left(\frac{L}{g_{22}}\right)_{,2} + F\Gamma^2_{22}\frac{g_{22}}{g_{11}}+F_S\Gamma^2_{22}+F_S\Gamma^2_{33}\frac{g_{22}}{g_{33}}=0 \tag{A1}$$

Noting that $J\equiv\sqrt{g_{11}g_{22}g_{33}}$, the second term can be changed to

$$\left(\frac{g_{22}}{J}\right)\left(\frac{J}{g_{22}}\right)_{,2}=\left(\ln\frac{J}{g_{22}}\right)_{,2} =\frac{1}{2}\left(\frac{g_{11,2}}{g_{11}}+\frac{g_{33,2}}{g_{33}}-\frac{g_{22,2}}{g_{22}}\right) \tag{A2}$$

While differentiation of the relation $\boldsymbol{g}_i\cdot\boldsymbol{g}_i\equiv g_{ii}$ with respect to x^j yields

$$\frac{1}{2}\frac{g_{ii,j}}{g_{ii}}=\Gamma^i_{ij} \quad \text{(no summation over } i\text{)} \tag{A3}$$

According to the formula above, we have

$$\frac{1}{2}\frac{g_{11,2}}{g_{11}}=\Gamma^1_{12};\ \frac{1}{2}\frac{g_{33,2}}{g_{33}}=\Gamma^3_{32};\ \frac{1}{2}\frac{g_{22,2}}{g_{22}}=\Gamma^2_{22} \tag{A4}$$

Moreover, using Eq. (17) yields

$$\begin{cases} \Gamma^2_{11}\dfrac{g_{22}}{g_{11}} = -\Gamma^1_{21} = -\Gamma^1_{12} \\ \Gamma^2_{33}\dfrac{g_{22}}{g_{33}} = -\Gamma^3_{23} = -\Gamma^3_{32} \end{cases} \tag{A5}$$

Combining Eq. (A1) with Eqs. (A2), (A4), (A5), and noting the definitions of F and F_S, we finally obtain the relation of force balance in the $\boldsymbol{g}_2$-direction.

$$\frac{\partial}{\partial x^2}(P_L + kP_S) - (1-k)P_S\Gamma^1_{12} = 0 \tag{A6}$$

Similarly, we have the relation in the $\boldsymbol{g}_3$-direction as follows:

$$\frac{\partial}{\partial x^3}(P_L + kP_S) - (1-k)P_S\Gamma^1_{13} = 0 \tag{A7}$$

Nomenclature

b	= half-width of rectangular leaf or radius of circular leaf	[m]
$\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$	= unit vectors in spherical coordinate system	[—]
$\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$	= Unit vectors in Cartesian coordinate system	[—]
$\boldsymbol{e}_\xi, \boldsymbol{e}_\eta, \boldsymbol{e}_\zeta$	= unit vectors in elliptic cylindrical or oblate spherical coordinate systems	[—]
F	= resultant force per unit area due to P_L and P_S acting on surface element normal to direction of filtrate flow, defined by Eq. (4)	[Pa]
F_S	= resultant force per unit area due to P_L and P_S acting on surface element parallel to direction of filtrate flow, where lateral value of solid compressive pressure is assumed to be kP_S, defined by Eq. (5)	[Pa]
G, H	= functions by which stream line and isopotential surface are specified	
$\boldsymbol{g}^i$	= contravariant base vectors	
$\boldsymbol{g}_i$	= covariant base vectors	
g_{ij}	= metric tensor	
J	= Jacobian, defined by Eq. (10)	
k	= ratio of lateral to normal P_S (coefficient of earth pressure at rest)	[—]
P	= filtration pressure	[Pa]
P_L	= local liquid pressure in filter cake	[Pa]
P_S	= local solid compressive pressure in filter cake	[Pa]
$\boldsymbol{R}$	= position vector	[m]
x^i	= variables in orthogonal curvilinear coordinate system	
Γ^k_{ij}	= Christoffel symbol of the second kind	

⟨Subscripts⟩

0	= denotes surface of filter cake	
$,i$	= denotes differentiation with respect to x^i, i.e., $\partial/\partial x^i$	

⟨Coordinate systems⟩

(x, y, z)	= Cartesian
(r, θ, ϕ)	= spherical
(r, θ, z)	= cylindrical
(ξ, η, ζ)	= elliptic cylindrical or oblate spherical

Literature Cited

1) Brooker, E. W. and H. O. Ireland, *Can. Geotech J.*, **11**, 1 (1965).
2) Collins, R. E., Modern Chem. Engng, Vol. 1, Physical operation, Reinhold, 327 (1963).
3) Murase, T., E. Iritani, J. H. Cho and M. Shirato, *Kagaku Kogaku Ronbunshu*, **12**, 688 (1986).
4) Shirato, M. and T. Aragaki, *Kagaku Kōgaku*, **33**, 205 (1969)
5) Shirao, M., T. Murase and M. Iwata, Deliquoring by expression—Theory and practice, edited by R. J. Wakeman, Progress in Filtration and Separation 4, Elsevier, Amsterdam, 181 (1986)
6) Shirato, M. and K. Kobayashi, Preprint 37E, Presented at the Symposium on filtration-Part II, Sixty-fourth National Meeting (AIChE), (1969)
7) Shirato, M., M. Sambuichi, T. Murase, T. Aragaki, K. Kobayashi and E. Iritani, Memoirs of Faculty of Engng, Nagoya University, Vol. 37, No. 1, 38 (1985)
8) Taylor, D. W., fundamentals of Soil Mechanics, 5th ed., John Wiley & Sons, New York, 225 (1962)
9) Terzaghi, K. and P. B. Peck, Soil Mechanics in Engng Practice, John Wiley & Sons, New York, (1948)
10) Tiller, F. M. and C. J. Huang, *Ind. Eng. Chem.*, **53**, 529 (1961)
11) Tiller, F. M. and C. S. Yeh, *AIChE J.*, **31**, 1241 (1985)
12) Tiller, F. M., C. S. Yeh, W. Chen and C. D. Tsai, Proc. World Congress III of Chem. Eng., Tokyo, Vol. III, 126 (1986)