A DESIGN METHOD FOR A CLASS OF ROBUST NONLINEAR OBSERVERS

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The Luenberger observer design method is extended by a further interesting consideration. This paper addresses the design method of robust observers for linear systems with uncertainty and a class of nonlinear systems by applying the concept of system roubstness degree proposed by the authors.[4] The uncertainty in linear systems and nonlinear distortion in nonlinear systems may lead the observer to be unstable. The Luenberger observer design approach combined with the system robustness degree will construct a robust asymptotic observer. Reduced-order robust observer design is also studied. Finally, a practical example of nonlinear robust observer design for a biochemical reactor is illustrated to show the validity of the proposed method.

Introduction

A well known topic, observer design, is again studied with a further interesting consideration. Usually, observers are used as state estimators for deterministic linear systems. For such a system, the observer can be designed so as to reconstruct the state of the system.[6] It is often the case, however, that the system is not deterministic, but with uncertain function and even nonlinear properties. Therefore, an observer should be designed with the properties of robustness to system uncertainties and adaptivity to system nonlinearities.

Bhattacharyya[2] considered a robust observer to reconstruct a linear function of the state for arbitrarily small perturbations of parameters of the observer. Akashi and Imai[1] discussed the insensitive observer design by the geometric approach. Carroll and Lindorf,[3] and Luders and Narendra[6] considered the design problem of an adaptive observer that had adjustable parameters to be updated for state estimate. Mita[8] proposed a design procedure of a zero-sensitivity observer of a linear functional for single-

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input single-output systems.

This paper considers the design method of robust observers for linear systems with uncertainty and a class of nonlinear systems by applying the concept of system robustness degree proposed by Chu *et al.*[4] The concept of system robustness degree is briefly introduced in Section 1. Section 2 considers robust observer design for linear systems with uncertainty. Section 3 discusses robust observer design for a class of nonlinear systems. Reduced-order nonlinear robust observer design is discussed in Section 4. Finally, a practical example of the nonlinear robust observer for a biochemical reactor will be illustrated to show the feasibility of the proposed method.

1. Robustness Degree

Consider the following linear deterministic system:

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in R^n$ denotes the system state vector. $u \in R^r$ denotes the system input vector. Suppose (A, B) is a controllable pair, and we have derived a feedback control law

$$u = -Kx \tag{2}$$

which can stabilize system (1) as

$$\dot{x} = (A - BK)x \tag{3}$$

It is evident that it is asymptotically stable, so that all of the possible matrix $(A - BK)$ can form an operator set which generates an asymptotically stable semigroup T_t such that

$$\| T_t \| = \| \exp((A - BK)t) \| \leqq M \exp(\omega t) \tag{4}$$

with $\omega < 0$, $M \geqq 1$, $t \geqq 0$ as shown by Chu *et al.*[4] It is also given the following useful definition regarding the system robustness.

Definition 1. Robustness Degree

The robustness degree of system (1) with feedback control Eq. (2) is defined by

$$\rho = -\omega / M > 0 \tag{5}$$

where ω, M are shown by Eq. (4).

If $K = 0$, i.e. no feedback control acts on the system, then ρ is called the inherent robustness degree of the system. If the system is of the form

$$\dot{x} = Ax + Bu + g(x, u, t) \tag{6}$$

where $g(x, u, t)$ denotes an uncertain function, then we have the following theorem.

Theorem 1.

The sufficient condition for system (6) to be asymptotically stable under the feedback control Eq. (2) is that the inequality

$$\| g(x, u, t) \| < \rho \| x \| \tag{7}$$

holds for all allowable control.

Proof. See Chu *et al.*[4]

2. Robust Stability Condition of Observers for Linear Systems with Uncertainty

Here we will consider a dynamical system described by the following differential equation:

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u + \Delta \mathrm{F}(x, u) \tag{10}$$

$$x(0) = x_0$$

$$y = Cx \tag{11}$$

on $[0, T]$, a real finite time interval, where x is an $n \times 1$ state vector, u an $r \times 1$ control vector, y an $m \times 1$ output vector, A an $n \times n$ constant system matrix, and B an $n \times r$ constant input matrix. The term ΔA represents the uncertainty of the plant, and $\Delta \mathrm{F}(x, u)$ symbolizes the disturbance to the plant, while ΔB can be construed as the nonlinearities or disturbances in the input.

The system described by Eq. (10) is quite popular for industrial processes, especially chemical processes. Because ΔA, ΔB and $\Delta \mathrm{F}$ are uncertain functions, rearrange system (10) in the following form:

It is difficult to find the precise version of robustness degree shown by Chu *et al.*[4] But an approximate and useful measure, called matrix measure, can be applied to express the system robustness degree as concluded in Corollary 1.

Corollary 1.

The robustness degree ρ, in one of its possible precise forms, of system (1) with feedback control Eq. (2) is equal to the negative of the measure of the matrix $(A - BK)$ as shown by

$$\rho = -\mu(A - BK) \tag{8}$$

where

$$\mu[Z] = \lambda_{\max}[(Z + Z^*)/2] \tag{9}$$

$\lambda_{\max}[\cdot]$ denotes the maximum eigenvalue of $[\cdot]$.

In the following sections, we will discuss the problem of robust observer design for linear and nonlinear systems by applying the concept of the system robustness degree.

$$\dot{x} = Ax + Bu + g(x, u, t) \tag{12}$$

where

$$g(x, u, t) = \Delta Ax + \Delta Bu + \Delta \mathrm{F}(x, u) \tag{13}$$

Usually, we cannot have a detailed knowledge of $g(x, u, t)$, but only its approximate range:

$$\| g(x, u, t) \| \leqq g \tag{14}$$

where g is some constant value. Obviously, the purpose of robust observer design is to observe the real unmeasurable system state through the measurable system output undergoing the presence of uncertainties or disturbances. The observer is driven by the input as well as the output of the original system. And we assume throughout that the pair (A, C) is observable.

In the well-known Luenberger observer design approach, the output $y=Cx$ of the real system is compared with $\hat{y}=C\hat{x}$ where $\hat{x}$ is the output of the observer, and this difference is used as a correcting term. The difference $\tilde{y}(t)- =y(t)-\hat{y}(t)$ is multiplied by an $n\times m$ real constant matrix P and fed into the input of the integrators of the observer. This observer will be called the asymptotic estimator.

The dynamical equation of the estimator is given by

$$\dot{\hat{x}}(t)=A\hat{x}(t)+Bu(t)-P\tilde{y}(t) \tag{15}$$

The state error vector

$$\tilde{x}(t)=x(t)-\hat{x}(t) \tag{16}$$

Then we have

$$\dot{\tilde{x}}=(A+PC)\tilde{x}+g(x, u, t) \tag{17}$$

The classical observer design approach does not consider the effect of uncertain function $g(x, u, t)$, and only if all of the eigenvalues of $(A+PC)$ are located in the left half plane by selecting the proper matrix P will the designed observer be asymptotically stable. But in the presence of uncertainty $g(x, u, t)$ we must design a robust observer to undertake the effect of system uncertainty. Based on the knowledge in section 1, if the matrix P can be determined so as to let the robustness degree of system matrix $(A+PC)$ satisfy

$$\rho(A+PC)>\|g(x, u, t)\|/\|\tilde{x}\| \tag{18}$$

then this observer will be robustly stable in the presence of uncertainty.

By the property of the semigroup, we know that

$$\|\exp[(A+PC)t]\|\leqq M\exp(\omega t)$$

for some $M\geqq 1$, $\omega<0$; then by the definition of ρ

$$\rho(A+PC)=-\frac{\omega}{M} \tag{19}$$

we have the solution of Eq. (15) as

$$\tilde{x}(t)=\exp[(A+PC)t]\tilde{x}(0) + \int_0^t \exp[(A+PC)(t-\tau)]g(x, u, \tau)d\tau \tag{20}$$

Taking a norm to Eq. (20):

$$\|\tilde{x}(t)\|\leqq M\exp(\omega t)\|\tilde{x}(0)\| + \int_0^t M\exp[\omega(t-\tau)]\|g(x, u, \tau)\|d\tau \tag{21}$$

By applying the Gronwall Lemma[9] (see **Appendix**), we derive

$$\|\tilde{x}(t)\|\leqq M\|\tilde{x}(0)\|\exp\left[\int_0^t (\omega+M\|g(x, u, \tau)\|/\|\tilde{x}\|)d\tau\right] \tag{22}$$

Thus, the state of system (10) can be estimated with an n-dimensional observer of the form

$$\dot{\hat{x}}=(A+PC)\hat{x}+Bu-P\hat{y} \tag{23}$$
$$\hat{x}(0)=\hat{x}_0$$

From Eqs. (18), (19), (22) and Corollary 1, we can obtain the following theorem for robust observer design.

Theorem 2.

The sufficient condition for observer (21) to be able to robustly estimate the state of system (10)–(11) is that the matrix P is determined so as to let the robustness degree of system matrix $(A+PC)$ satisfy inequality (18).

The design procedure for the robust observer is to select a matrix P which satisfies inequality (18). Corollary 1 in Section 1 shows the relation between the system robustness degree and pole assignment by Eqs. (8)–(9). Thus it is necessary to know the magnitude of the right-hand side of inequality (18), i.e. the norms of $g(x, u, t)$ and the desired $\tilde{x}(\infty)$ in order to define all the eigenvalues of the observer. The norm of $g(x, u, t)$ is defined by inequality (14). From inequality (22), we know that

$$\|\tilde{x}(t)\|\leqq M\|\tilde{x}(0)\| \tag{24}$$

and if $\|g(x, u, t)\|\to 0$ when $t\to\infty$, then

$$\lim_{t\to\infty}\|x(t)\|=0 \tag{25}$$

which means that the robust observer is asymptotically stable with zero steady state.

3. Robust Stability Condition of Observers for a Class of Nonlinear Systems

In Section 2 we discussed the robust observer design problem for linear systems with uncertainty. We will now consider the robust observer design for a class of nonlinear systems with or without uncertainty. Usually, an observer is designed to estimate the state of a linear system; there are few papers concerning nonlinear observer design problems.

Consider the dynamical nonlinear system that satisfies

$$\dot{x}=(A+\Delta A)x+(B+\Delta B)u+\mathrm{F}(x, u)+\Delta\mathrm{F}(x, u) \tag{26}$$

$$y=Cx \tag{27}$$

All symbols have the same meaning as in system (10), (11) except that $\mathrm{F}(x, u)$ denotes the nonlinear portion of the plant and is assumed to be twice differentiable. The modelling error has been included in $\Delta \mathbf{F}(\boldsymbol{x}, \boldsymbol{u})$. All of the uncertain terms are combined by $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)$ as in Eq. (13) as

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)=\Delta \boldsymbol{A}\boldsymbol{x}+\Delta \boldsymbol{B}\boldsymbol{u}+\Delta \mathrm{F}(\boldsymbol{x}, \boldsymbol{u}) \tag{28}$$

Also, we assume that detailed knowledge of $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)$ is not available, but that as for the approximate norm range

$$\|\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)\| \leqq g \tag{29}$$

for some value of g. Thus, system (26) becomes

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u}+\mathrm{F}(\boldsymbol{x}, \boldsymbol{u})+\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{30}$$

Suppose we have the nonlinear observer as

$$\dot{\hat{\boldsymbol{x}}}=\boldsymbol{A}\hat{\boldsymbol{x}}+\boldsymbol{B}\boldsymbol{u}+\mathrm{F}(\hat{\boldsymbol{x}}, \boldsymbol{u})-\boldsymbol{P}\tilde{\boldsymbol{y}} \tag{31}$$

where

$$\tilde{\boldsymbol{y}}(t)=\boldsymbol{y}(t)-\hat{\boldsymbol{y}}(t)=\boldsymbol{C}(\boldsymbol{x}-\hat{\boldsymbol{x}})=\boldsymbol{C}\tilde{\boldsymbol{x}} \tag{32}$$

where $\tilde{\boldsymbol{x}}$ is the state error vector, then we can derive

$$\dot{\tilde{\boldsymbol{x}}}=(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})\tilde{\boldsymbol{x}}+\mathrm{F}(\boldsymbol{x}, \boldsymbol{u})-\mathrm{F}(\hat{\boldsymbol{x}}, \boldsymbol{u})+\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{33}$$

Take the Taylor series for the first order to nonlinear function $\mathbf{F}(\boldsymbol{x}, \boldsymbol{u})$, around the trajectory $\hat{\boldsymbol{x}}$, (because it is twice differentiable), i.e.

$$\mathbf{F}(\boldsymbol{x}, \boldsymbol{u})-\mathbf{F}(\hat{\boldsymbol{x}}, \boldsymbol{u})=\frac{\partial \mathbf{F}}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}, \boldsymbol{u})\tilde{\boldsymbol{x}} \tag{34}$$

Thus we have

$$\dot{\tilde{\boldsymbol{x}}}=\left(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C}+\frac{\partial \mathbf{F}(\hat{\boldsymbol{x}}, \boldsymbol{u})}{\partial \boldsymbol{x}}\right)\tilde{\boldsymbol{x}}+\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{35}$$

This is a nonlinear time-varying error system. It is difficult to find its robustness degree compared with that for linear time-invariant systems. Eq (35) is then rewritten in another form as

$$\dot{\tilde{\boldsymbol{x}}}=(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})\tilde{\boldsymbol{x}}+\frac{\partial \mathbf{F}}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}, \boldsymbol{u})\tilde{\boldsymbol{x}}+\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{36}$$

The solution of Eq. (36) is of the form:

$$\tilde{\boldsymbol{x}}(t)=\exp[(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})t]\tilde{\boldsymbol{x}}(0) + \int_0^t \exp[(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})(t-\tau)]\left(\frac{\partial \mathbf{F}}{\partial \boldsymbol{x}}\tilde{\boldsymbol{x}}+\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)\right)d\tau \tag{37}$$

Suppose the robustness degree of system matrix $(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})$ (where $\boldsymbol{P}$ is a matrix to be determined) has the form

$$\rho(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})=-\frac{\omega}{M} \tag{38}$$

then taking the norm to Eq. (37)

$$\|\tilde{\boldsymbol{x}}(t)\| \leqq M\exp(\omega t)\|\tilde{\boldsymbol{x}}(0)\| + \int_0^t M\exp[\omega(t-\tau)]\left\|\frac{\partial \mathbf{F}(\hat{\boldsymbol{x}}, \boldsymbol{u})}{\partial \boldsymbol{x}}\tilde{\boldsymbol{x}}(t)+\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)\right\|d\tau \tag{39}$$

By applying the Gronwall Lemma, we have

$$\|\tilde{\boldsymbol{x}}(t)\| \leqq M\|\tilde{\boldsymbol{x}}(0)\| \times \exp\left[\int_0^t \left(\omega+M\left(\left\|\frac{\partial \mathbf{F}}{\partial \boldsymbol{x}}\right\|+g/\|\tilde{\boldsymbol{x}}\|\right)\right)d\tau\right] \tag{40}$$

To design a robust observer against the system nonlinear distortion for estimation and the system uncertainties, the sufficient condition is

$$\omega+M\left(\left\|\frac{\partial \mathbf{F}}{\partial \boldsymbol{x}}\right\|+g/\|\tilde{\boldsymbol{x}}\|\right)<0 \tag{41}$$

i.e.

$$\rho(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})>\left\|\frac{\partial \mathbf{F}}{\partial \boldsymbol{x}}\right\|+g/\|\tilde{\boldsymbol{x}}\| \tag{42}$$

Comparing inequality (18) for linear robust observer design, we know the difference for nonlinear robust observer design. If the nonlinear function $\mathbf{F}(\boldsymbol{x}, \boldsymbol{u})$ is not a function of state $\boldsymbol{x}$, then the robustness degrees for observers needed for both of the linear and the nonlinear system are the same. The remaining design procedures are the same as discussed in Section 2.

Now the nonlinear robust observer can be described by

$$\dot{\hat{\boldsymbol{x}}}=(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})\hat{\boldsymbol{x}}+\boldsymbol{B}\boldsymbol{u}+\mathrm{F}(\hat{\boldsymbol{x}}, \boldsymbol{u})-\boldsymbol{P}\boldsymbol{y} \tag{43}$$

The discussion above can be concluded with the following theorem.

Theorem 3.

The sufficient condition for nonlinear observer (43) to be able to robustly estimate the state of nonlinear system (26), (27) is that the matrix $\boldsymbol{P}$ is determined so as to let the robustness degree of system matrix $(\boldsymbol{A}+\boldsymbol{P}\boldsymbol{C})$ satisfy inequality (42).

4. Robust Stability Condition of a Nonlinear Reduced-Order Observer

The full-order robust observer, designed and discussed in Section 3, was derived by setting up a model of the plant and feeding back a 'correction term' proportional to the difference between the actual and estimated outputs. Such an observer contains redundancy because m state variables can be directly obtained from m outputs which are available for measurement and need not be estimated. The remaining $(n-m)$ state variables can be estimated using an observer of order $(n-m)$.

For a classical reduced-order observer, since there is a direct link from the observed variable $\boldsymbol{y}(t)$ to the estimated state $\hat{x}(t)$, the estimated $\hat{x}(t)$ will be more sensitive to measurement errors in $y(t)$ than the estimate generated by a full-order observer. This is because the noise bypasses the natural filtering action of the observer dynamics.[5]

In this section, we will discuss reduced-order observer design for a class of nonlinear systems that preserves the property of robustness in the presence of system uncertainty or modelling error and measurement noise.

Suppose the nonlinear system (26), (27) can be partitioned into the following form through proper rearrangement:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} f_1(x, u) \\ f_2(x, u) \end{bmatrix} + \begin{bmatrix} g_1(x, u, t) \\ g_2(x, u, t) \end{bmatrix} \tag{44}$$

$$y = [\mathbf{0}\, C_m][x_1^T x_2^T]^T \tag{45}$$

where m state variables x_2 (R^m) can be directly obtained from Eq. (45):

$$x_2 = C_m^{-1} y \tag{46}$$

The remaining $(n-m)$ state variables require an observer for estimation. Constructing a subsystem with dimension of $n-m$ for x_1

$$\dot{x}_1 = A_{11} x_1 + v \tag{47}$$

$$z = A_{21} x_1 \tag{48}$$

where

$$\begin{aligned} v &= A_{12} x_2 + B_1 u + f_1(x, u) + g_1(x, u, t) \\ &= A_{12} C_m^{-1} y + B_1 u + f_1(x, u) + g_1(x, u, t) \end{aligned} \tag{49}$$

$$\begin{aligned} z &= A_{21} x_1 \\ &= \dot{x}_2 - A_{22} x_2 - B_2 u - f_2(x, u) - g_2(x, u, t) \\ &= C_m^{-1} \dot{y} - A_{22} C_m^{-1} y - B_2 u - f_2(x, u) \\ &\quad - g_2(x, u, t) \end{aligned} \tag{50}$$

Now we estimate x_1 with an observer

$$\begin{aligned} \dot{\hat{x}}_1 &= (A_{11} + PA_{21}) \hat{x}_1 + v - Pz \\ &= (A_{11} + PA_{21}) \hat{x}_1 + A_{12} C_m^{-1} y + B_1 u \\ &\quad + f_1(\hat{x}, u) - PC_m^{-1} \dot{y} + PA_{22} C_m^{-1} y \\ &\quad + PB_2 u + Pf_2(\hat{x}, u) \end{aligned} \tag{51}$$

By setting

$$\bar{x}(t) = \hat{x}_1(t) + PC_m^{-1} y \tag{52}$$

then we have the observer

$$\begin{aligned} \dot{\bar{x}} &= (A_{11} + PA_{21}) \bar{x} + (B_1 + PB_2) u \\ &\quad + [f_1(\hat{x}, u) + Pf_2(\hat{x}, u)] \\ &\quad + [A_{12} + PA_{22} - (A_{11} + PA_{21}) P] C_m^{-1} y \end{aligned} \tag{53}$$

Let

$$\tilde{x} = x_1 - \hat{x}_1$$

then

$$\begin{aligned} \dot{\tilde{x}} &= (A_{11} + PA_{21}) \tilde{x} + f_1(x, u) - f_1(\hat{x}, u) \\ &\quad + P[f_2(x, u) - f_2(\hat{x}, u)] + g_1(x, u, t) \\ &\quad + Pg_2(x, u, t) \end{aligned} \tag{54}$$

Similarly to Eq. (34), we assume

$$f_1(x, u) - f_1(\hat{x}, u) = \frac{\partial f_1(\hat{x}, u)}{\partial x} \begin{bmatrix} \tilde{x} \\ \mathbf{0} \end{bmatrix} \tag{55}$$

where the Jacobian of f_1 with respect to its augments is an $(n-m) \times n$ matrix, if its first $(n-m) \times (n-m)$ square submatrix is denoted by $\partial f_1 / \partial x_1$, taking the similar procedures for $[f_2(x, u) - f_2(\hat{x}, u)]$, then Eq. (54) becomes

$$\begin{aligned} \dot{\tilde{x}} &= (A_{11} + PA_{21}) \tilde{x} + \frac{\partial f_1}{\partial x_1} \tilde{x} + P \frac{\partial f_2}{\partial x_1} \tilde{x} \\ &\quad + g_1(x, u, t) + Pg_2(x, u, t) \end{aligned} \tag{56}$$

As discussed in Section 3, we should have:

$$\begin{aligned} \rho(A_{11} + PA_{21}) &> \left\| \frac{\partial f_1}{\partial x_1} + P \frac{\partial f_2}{\partial x_1} \right\| \\ &\quad + \| g_1(x, u, t) + Pg_2(x, u, t) \| / \| \tilde{x} \| \end{aligned} \tag{57}$$

It should be pointed out that P appears in both sides of the inequality (57). The detailed design procedures are stated in the next section, referred to the example.

Then the estimate of the full state x is given by

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ C_m y \end{bmatrix} = \begin{bmatrix} I \\ O \end{bmatrix} \bar{x} + \begin{bmatrix} -P \\ I \end{bmatrix} C_m^{-1} y$$

5. Application of a Nonlinear Observer for a Biochemical Reactor

It is well known that the microorganism in a biochemical process is very difficult to be detected for monitoring and operating purposes. In this paper we describe design of a robust nonlinear observer to estimate the biomass concentration of *Pichia magi* IFO 06-2.

5.1 Modelling

The continuous process can be described by the following nonlinear equations proposed by Takamatsu *et al.*[10]

$$\frac{dX_c}{dt} = \frac{0.2945 a S X_c}{0.001 + S} + U X_c$$

$$\frac{da}{dt} = 0.1353 \exp\left(-\frac{0.001 a}{1 - a} \right) \frac{S}{0.25 + S} - 0.2945(a + 0.385) \left(\frac{S}{0.001 + S} - \frac{S}{0.25 + S} \right)$$

$$\frac{dS}{dt}=-\frac{0.0409aSX_c}{0.001+S}-\frac{0.04SX_c}{1.15+S}+U(1-S)$$

$$\frac{dc}{dt}=83.53(7.6-c)-\left(\frac{258.98aS}{0.001+S}+3.3\right)X_c$$

where
X_c: dry cell concentration [g/l]
a: a 'dummy' state denoting DNA [—]
S: substrate concentration [Vol.%]
c: dissolved oxygen concentration [ppm]
U: dilute rate [h^{-1}]

For such a nonlinear system, we have no effective method to design an observer. In this section we use a simplified nonlinear model by Taylor's series:

$$\dot{x}=Ax+Bu+\mathrm{F}(x^Tx, xu, u^2)$$

and a linear model:

$$\dot{x}=Ax+Bu$$

where

$$x^T=[X_c-\bar{X}_c, a-\bar{a}, S-\bar{S}, c-\bar{c}]$$
$$u=U-\bar{U}$$

where $\bar{X}_c$, $\bar{a}$, $\bar{S}$, $\bar{c}$, $\bar{U}$ are the operating points of X_c, a, S, c, U.

Obviously, the substrate concentration S and the dissolved oxygen concentration c are easy to measure. We would like to observe the cell concentration X_c and the 'dummy' state a through S, c and input U. The design procedures for the nonlinear and linear observers can be referred to the above results. And we will consider the difference between these two kinds of observers.

5.2 Robustness consideration

We consider the original nonlinear model as a real system. Then there exist truncating errors for the simplified nonlinear model:

$$g_{SNL}(x, u)=\mathrm{R}_3(x, u)$$

and for the linear model:

$$g_L(x, u)=\mathrm{R}_2(x, u)$$

where $\mathrm{R}_i(x, u)$ $(i=2, 3)$ denotes the truncated terms from the i-th order term. We can find the conservative maximum values of $g_{SNL}(x, u)$ and $g_L(x, u)$, denoted by g_{SNL} and g_L, by some calculation. Thus the conservative robustness degree for the nonlinear observer can be calculated by

$$\rho_{NLO}(A_{11}+PA_{21})\geqq\left\|\frac{\partial f_1}{\partial x_1}+P\frac{\partial f_2}{\partial x_1}\right\| + \|g_{SNL1}+Pg_{SNL2}\|/\|\tilde{x}\|$$

and for the linear observer:

$$\rho_{LO}(A_{11}+PA_{21})\geqq\|g_{L1}+Pg_{L2}\|/\|\tilde{x}\|$$

Because both sides of the above inequalities include matrix P, we must choose P carefully.

For example, in the range of $|u|=0.02$, if P is chosen to be

$$P=\begin{bmatrix}6 & 0.0005\\ 5 & 0.01\end{bmatrix}$$

then

$$\rho_{NLO}=0.143$$

thus the possible maximum estimating error

$$\|\tilde{x}\|=0.064$$

For the linear case, if with the same P, the possible maximum estimating error becomes

$$\|\tilde{x}\|=0.34$$

It is obvious that the difference between the two observers is more than fivefold according to the discussion above, although this is very conservative. If a smaller estimating error is desired, we should increase the robustness degree.

5.3 Simulation

The purpose of simulation is to show the feasibility of the robust nonlinear or linear observers and show the difference between the two observers. The abnormal conditions considered are the initial state

$$x_1(0)=0.15$$

(others are zeros) and a constant input disturbance after 40 h. Thus

$$u=\begin{cases}0 & 0\leqq t<40\,\mathrm{h}\\ 0.02 & t\geqq 40\,\mathrm{h}\end{cases}$$

Figures 1 and **2** show the simulation results for different P, i.e. $\rho_1=0.143$, and $\rho_2=0.098$, where we can see that the estimating errors for x_2, i.e. the "dummy" state, are very small by using nonlinear and linear observers as shown in Figs. 1b and 2b. But the estimating errors for x_1, the cell concentration, are completely different, as shown in Figs. 1a and 2a. The x_1 estimated by the nonlinear observer closely follows the real value. But there exists a difference when the linear observer is applied. Moreover, we find that as the robustness degree increases, the estimating error by the linear observer becomes smaller, as shown in Figs. 1c and 2c. This fact is further demonstrated by **Fig. 3**, which is based upon several other simulation results for the linear observer with various values of robustness degree.

Furthermore, we consider the complex input disturbance shown in **Fig. 4a**, and take $\rho=0.143$. Simulation results in Figs. 4b, 4c, 4d show that the performance of the nonlinear observer is better than that of the linear one.

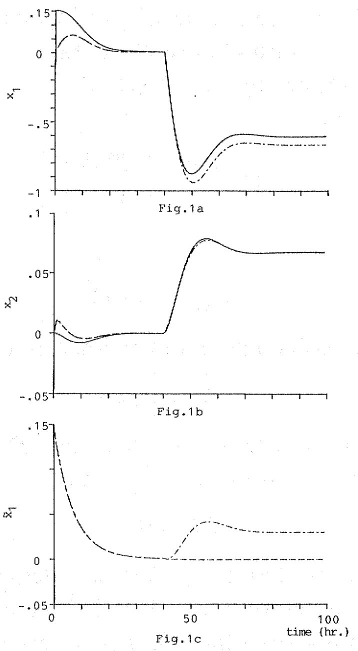

Fig. 1. Simulation results of nonlinear observed (NLO) and linear observer (LO) with x_1 $(0)=0.15$ and $\rho=0.143$ ——: real value; - - - -: by NLO; –·–·–: by LO

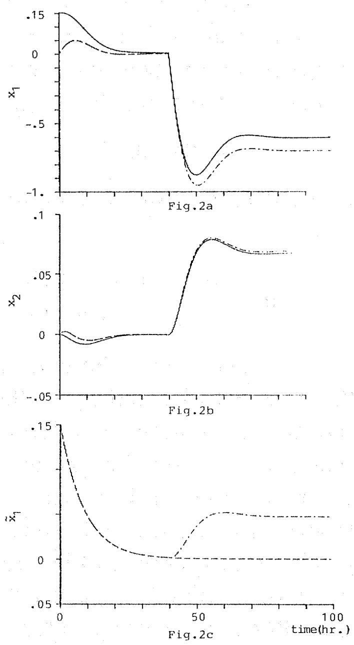

Fig. 2. Simulation results of nonlinear observer (NLO) and linear observer (LO) with $x_1(0)=0.15$ and $\rho=0.098$ ——: real value; - - - -: by NLO; –·–·–: by LO

Conclusions and Discussion

The method of nonlinear robust observer design is introduced and discussed. The key point of this method is the combined consideration of the Luenberger observer with the system robustness degree, and the linear observer is extended to the nonlinear case and to the uncertain case. The design approach for full- and reduced-order nonlinear observers is feasible from the mathematical point of view.

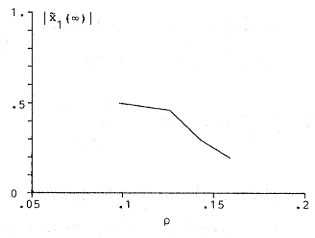

Fig. 3. Maximum $|\tilde{x}_1(\infty)|$ by linear observer vs. robustness degree ρ

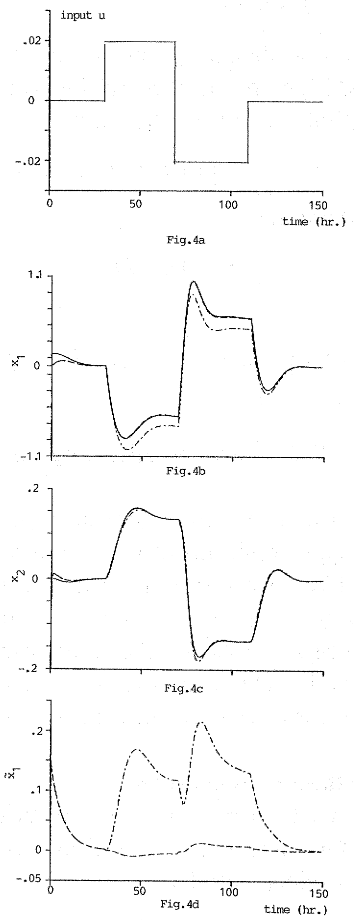

Fig. 4. Simulation results of nonlinear observer (NLO) and linear observer (LO) with $x_1(0)=0.15$ and $\rho=0.143$ ——: real value; - - - -: by NLO; –·–·–: by LO

An application example for a bioreactor is studied. The simulation results show that the performance of the nonlinear observer is better than that of the linear one, and that consideration of robustness can play a role.

However, it is somewhat difficult to determine $\boldsymbol{P}$ for a robust reduced-order observer. Another problem is that the condition of robust stability is only a sufficient one.

Appendix

Lemma. (Gronwall Lemma)

If $\phi(t)$, $\psi(t)$ and $\mu(t)$ are all nonnegative continuous functions for $t \geqq 0$ and λ is a positive constant such that

$$\phi(t) \leqq \lambda + \int_0^t [\psi(s)\phi(s) + \mu(s)]ds \qquad \text{for all} \quad t \geqq 0$$

then

$$\phi(t) \leqq \lambda \exp\left(\int_0^t [\psi(s) + \mu(s)/\lambda]ds\right) \qquad \text{for all} \quad t \geqq 0$$

Nomenclature

a	= 'dummy' state denoting DNA	[—]
$\boldsymbol{A}$	= $n \times n$ system matrix	[—]
$\boldsymbol{B}$	= $n \times r$ system input matrix	[—]
c	= dissolved oxygen concentration	[ppm]
$\boldsymbol{C}$	= $m \times n$ system output matrix	[—]
$\boldsymbol{C}_m$	= submatrix in $\boldsymbol{C}$	[—]
$\mathbf{F}(\boldsymbol{x}, \boldsymbol{u})$	= $n \times i$ nonlinear function vector	[—]
$f_i(\boldsymbol{x}, \boldsymbol{u})$	= i-th sub-vector of $\mathbf{F}(\boldsymbol{x}, \boldsymbol{u})$	[—]
$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)$	= uncertain nonlinear vector	[—]
g	= boundedness of $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)$	[—]
i	= number	[—]
$\boldsymbol{K}$	= feedback gain matrix	[—]
M	= parameter of semigroup	[—]
M'	= parameter of semigroup	[—]
m	= number of outputs	[—]
n	= number of states	[—]
$\boldsymbol{P}$	= gain matrix in an observer design	[—]
R^i	= space with i dimensions	[—]
$\mathbf{R}_i(\boldsymbol{x}, u)$	= truncated error from i-th order term	[—]
r	= number of inputs	[—]
S	= substrate concentration	[Vol.%]
s	= integrating variable	[—]
T_t	= linear semigroup	[—]
t	= time	[—]
$\boldsymbol{u}$	= $r \times 1$ control input vector	[—]
U	= dilute rate	[h^{-1}]
u	= single control variable	[—]
$\boldsymbol{v}$	= new input	[—]
$\boldsymbol{x}$	= $n \times 1$ state vector	[—]
$\boldsymbol{x}_1, \boldsymbol{x}_2$	= sub-vector in $\boldsymbol{x}$	[—]
X_c	= dry cell concentration	[g/l]
$\boldsymbol{y}$	= $m \times 1$ output vector	[—]
z	= vector	[—]
$\boldsymbol{Z}$	= square matrix	[—]
Δ	= variation of a function	[—]
ε	= denoting "belong to"	[—]
λ	= eigenvalue of a matrix	[—]
λ	= constant	[—]
μ	= matrix measure	[—]
$\mu(t)$	= continuous function	[—]
ρ	= robustness degree	[—]
τ	= integral variable denoting time	[—]
$\phi(t)$	= continuous function	[—]
$\psi(t)$	= continuous function	[—]
ω	= parameter of semigroup	[—]

⟨Superscripts and Subscripts⟩

$*$, T	= conjugate, transpose of matrix
max, min	= maximum, minimum
SNL	= simplified nonlinear
L	= linear
NLO	= nonlinear observer
LO	= linear observer
$\hat{}$	= estimated value
$\tilde{}$	= error between real and estimated ones
$\bar{}$	= estimated value

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