A ROBUSTNESS STUDY FOR CHEMICAL PROCESS CONTROL

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Generally, chemical processes behave nonlinearly. But existing well-developed controller design techniques are based on linearized models of nonlinear processes. The neglected nonlinear effects affect the stability of control systems. Robustness degree, a new measure for robust control, is introduced and studied from the engineering point of view. The concept of robustness degree, proposed by applying the semigroup theory of functional analysis, can play a role in analysis and design of control systems. It is a sufficient and quantitative measure of the margin of stability within which certain nonlinear functions and modelling errors can be tolerated. Robustness degree of an open-loop or closed-loop system is linked with system matrix measure as an estimation that is straightforward and easy to calculate. A chemical reactor is studied as an example of robust control system design by this approach.

Introduction

Consider a continuous stirred-tank reactor (CSTR) in which a single first-order exothermic reaction is taking place. The nonlinear system model[9] is

$$V\frac{dC}{dt}=F(C_{Af}-C_A)-Vr$$

$$V\rho C_p\frac{dT}{dt}=\rho C_pF(T_f-T)+V(-\Delta H)r-hA(T-T_c)$$

where r is the reaction rate and equals $k_0\exp(-E/RT)C_A$. We assume that the coolant temperature, T_c, is the only manipulated variable. The dimensionless model can be written as

$$\begin{aligned}\dot{x}_1&=-x_1+D_a(1-x_1)\exp(x_2/(1+x_2/\nu))\\ \dot{x}_2&=-x_2+BD_a(1-x_1)\exp(x_2/(1+x_2/\nu))\\ &\quad-\beta(x_2-x_{20})+\beta u\end{aligned}\tag{1}$$

The dimensionless parameters are given in **Table 1**.

The tranditional method of control system design for a nonlinear system is to linearize the nonlinear model into a linear model about some steady-state operating point so that well-developed linear design procedures may be applied; e.g. system (1) is controlled by a state feedback control law

$$u=-k_1\Delta x_1-k_2\Delta x_2\tag{2}$$

designed on the basis of

$$\begin{aligned}\Delta\dot{x}_1&=\boldsymbol{a}_{11}\Delta x_1+\boldsymbol{a}_{12}\Delta x_2\\ \Delta\dot{x}_2&=\boldsymbol{a}_{21}\Delta x_1+\boldsymbol{a}_{22}\Delta x_2+b_2u\end{aligned}\tag{3}$$

where $\boldsymbol{a}_{11}$, $\boldsymbol{a}_{12}$, $\boldsymbol{a}_{21}$, $\boldsymbol{a}_{22}$ and b_2 are some constant parameters of the linearized system.

Mathematically, system (3) can be stabilized by control (2) if system (3) is controllable. However, it is well known that depending on different parameter values, the reactor shows different unforced (i.e. $u=0$) dynamic behavior corresponding to the multiple steady states (with stable and unstable ones). In addition, the reactor is disturbed by many effects, e.g. the feed temperature, feed flowrate and coolant flowrate. Will the system remain stable after application of control (2), designed on the basis of a linear model that has discrepancies with the system, and in the presence of the uncertainties noted above?

This is a simple case, but the problem is the same

Table 1. Dimensionless parameters for the CSTR model

$\nu=\dfrac{E}{RT_{f0}}$, $\quad B=\dfrac{(-\Delta H)C_{Af0}}{\rho c_pT_{f0}}\nu$
$D_a=\dfrac{k_0\exp(-\nu)V}{F}$, $\quad \beta=\dfrac{hA}{F\rho c_p}$
$x_{20}=\dfrac{T_{c0}-T_{f0}}{T_{f0}}\nu$
Dimensionless time: $t'=t\dfrac{F}{V}$
Dimensionless composition: $x_1=\dfrac{C_{Af0}-C_A}{C_{Af0}}$
Dimensionless temperature: $x_2=\dfrac{T-T_{f0}}{T_{f0}}\nu$
Dimensionless control: $u=\dfrac{T_c-T_{c0}}{T_{f0}}\nu$

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for more general cases. The main problem is to design a controller with which the system is stabilized robustly to some bounded uncertainties, disturbances and possible modeling mismatch. The meaning of robustness can be viewed as tolerance of uncertainty, disturbance and any perturbation of the system before a particular feedback design becomes unstable.

In the last two decades, many papers have been published on this topic. Molander and Willems[6] studied the design of cone-bounded feedback nonlinearities that preserve the stability of a linear open-loop system. Noldus[7] also studied dynamic systems containing cone-bounded nonlinearities in the open-loop equation, and developed some results regarding the robustness of the systems relative to the uncertainties in both the state matrix and in the control matrix. Davison and Ferguson[4] considered the problem of designing realistic multivariable controllers for a servomechanism system so as to achieve closed-loop stability and asymptotic regulation with the property of robustness. Chen and Desoer[2] developed some necessary and sufficient conditions for robust stability of linear distributed feedback systems. However, there was lack of a simple measure for system robustness not only in frequency domain analysis but also in time domain analysis.

The present paper introduces a new concept of robustness degree of control systems. A useful but approximate estimate for robustness degree is discussed. Finally, the robust stability of a chemical reactor is considered. The results reported here are very useful for system analysis and control system synthesis. Compared with other measures for system robustness, robustness degree of the control system, is a simple concept featuring ease of calculation, good estimation results estimate and a clear geometric meaning.

1. Robustness Analysis of Control Systems

Suppose a dynamical system satisfies

$$\begin{aligned}&\dot{\boldsymbol{X}}=(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{X}+(\boldsymbol{B}+\Delta\boldsymbol{B})\boldsymbol{U}+g(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\alpha})\\&\boldsymbol{X}(0)=\boldsymbol{X}_0\end{aligned} \tag{4}$$

on $[0, T]$, a real finite time interval, where $\boldsymbol{X}$ is an $n\times 1$ state vector, $\boldsymbol{U}$ an $r\times 1$ control vector, $\boldsymbol{\alpha}$ a $p\times 1$ parameter vector, $\boldsymbol{A}$ an $n\times n$ constant system matrix, $\boldsymbol{B}$, an $n\times r$ constant input matrix, g an $n\times 1$ known or unknown nonlinear function vector (e.g. the high-order truncation of Taylor's series) assumed to be continuously differentiable in its arguments. The terms $\Delta\boldsymbol{A}$, $\Delta\boldsymbol{B}$ and $g(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\alpha})$ can represent the uncertainties, disturbances, nonlinearities and modelling error of the plant. We can rearrange system (4) in the following form:

$$\dot{\boldsymbol{X}}=\boldsymbol{A}\boldsymbol{X}+\boldsymbol{B}\boldsymbol{U}+F(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\alpha}) \tag{5}$$

where

$$F(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\alpha})=\Delta\boldsymbol{A}\boldsymbol{X}+\Delta\boldsymbol{B}\boldsymbol{U}+g(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\alpha}) \tag{6}$$

The problem of robustness analysis of a control system arises because in practical cases we would like to design a state feedback control law based only on the linear part of system (5) where detailed information about the nonlinear function $F(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\alpha})$ lacking; that is, the system

$$\dot{\boldsymbol{X}}=\boldsymbol{A}\boldsymbol{X}+\boldsymbol{B}\boldsymbol{U} \tag{7}$$

The state feedback control law can be derived by any of several techniques, such that

$$\boldsymbol{U}=-K\boldsymbol{X} \tag{8}$$

which can stabilize system (7); i.e. the closed system

$$\dot{\boldsymbol{X}}=(\boldsymbol{A}-\boldsymbol{B}K)\boldsymbol{X} \tag{9}$$

is asymptotically stable. But actually the real system is in the form of Eq. (5). We now consider the stability and robustness of system (5) with linear feedback control (8):

$$\dot{\boldsymbol{X}}=(\boldsymbol{A}-\boldsymbol{B}K)\boldsymbol{X}+F(\boldsymbol{X},-K\boldsymbol{X},\boldsymbol{\alpha}) \tag{10}$$

To investigate the stability, functional analysis is applied to deal with the problem of analyzing the stability and robustness of system (10).

It is not difficult to show that all of the possible matrix $(\boldsymbol{A}-\boldsymbol{B}K)$ can form an operator set that generates an asymptotically stable linear semigroup T_t,[3] such that

$$\| T_t\| \leqq M\exp(\omega t) \tag{11}$$

with the constants $M\geqq 1$, $\omega<0$, for $t\geqq 0$.

Suppose that the nonlinear function $F(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\alpha})$ in the system forms another nonlinear operator set which is bounded on Range (F). We can expect the solution for Eq. (10) to have the form

$$\boldsymbol{X}(t)=T_t\boldsymbol{X}(0)+\int_0^t T_{t-\tau}F(\boldsymbol{X},-K\boldsymbol{X},\boldsymbol{\alpha})d\tau \tag{12}$$

where $\boldsymbol{X}(0)$ is the initial condition.

Now if we assume that the solution is given by

$$\boldsymbol{X}(t)=S_t\boldsymbol{X}(0) \tag{13}$$

i.e.

$$S_t\boldsymbol{X}(0)=T_t\boldsymbol{X}(0)+\int_0^t T_{t-\tau}F(\boldsymbol{X},-K\boldsymbol{X},\boldsymbol{\alpha})d\tau \tag{14}$$

we can prove that $S_t\boldsymbol{X}$ satisfy those properties of a nonlinear semigroup.[1] If S_t is a stable semigroup, the closed-loop system (10) is stable.

Taking the norm on Eq. (12) (for simplicity, $F(\boldsymbol{X},-K\boldsymbol{X},\boldsymbol{\alpha})$ is denoted by F):

$$\|X(t)\| \leqq \|T_t X(0)\| + \int_0^t \|T_{t-\tau} F\| d\tau \tag{15}$$

$$\leqq Ma\exp(\omega t) + \int_0^t M\exp[\omega(t-\tau)]\|F\| d\tau \tag{16}$$

where $\|X(0)\| = a$. We can also write

$$\|X(t)\|\exp(-\omega t) \leqq Ma + \int_0^t M\|F\|\exp(-\omega\tau)d\tau \tag{17}$$

Here we would like to introduce an important lemma without proof. It is a version of the Gronwall Lemma.[8)]

Lemma 1

If $\phi(t)$, $\psi(t)$ and $\mu(t)$ are each nonnegative continuous functions for $t \geqq 0$ and λ is a positive constant such that

$$\phi(t) \leqq \lambda + \int_0^t [\psi(s)\phi(s) + \mu(s)]ds \qquad \text{for all} \quad t \geqq 0$$

then

$$\phi(t) \leqq \lambda\exp\left(\int_0^t (\psi(s) + \mu(s)/\lambda)ds\right) \qquad \text{for all} \quad t \geqq 0$$

Rewrite Eq. (17) in the following form:

$$\|X(t)\|\exp(-\omega t) \leqq Ma + \int_0^t M\|X(\tau)\|\exp(-\omega\tau)\|F\|/\|X(\tau)\| d\tau \tag{18}$$

By applying Lemma 1, we have

$$\|X(t)\|\exp(-\omega t) \leqq Ma\exp\left(\int_0^t M\|F\|/\|X\| d\tau\right) \tag{19}$$

i.e.

$$\|X(t)\| \leqq Ma\exp\left(\omega t + \int_0^t M\|F\|/\|X\| d\tau\right) \tag{20}$$

By Eq. (13), we find

$$\|S_t X(0)\| \leqq Ma\exp\left(\omega t + \int_0^t M\|F\|/\|X\| d\tau\right) \tag{21}$$

or

$$\|S_t X(0)\| \leqq Ma\exp\left(\int_0^t (\omega + M\|F\|/\|X\|)d\tau\right) \tag{22}$$

Obviously, $S_t X$ is a stable semigroup if

$$\omega + M\|F\|/\|X\| < 0 \tag{23}$$

holds for $t \geqq 0$, that is to say the closed-loop system (10) is robustly stable in the sense of Lyapunov.

We can see, in another form of inequality (23), that nonlinear operator F forms a conic sector

$$\|F(X, -KX, \alpha)\| < -\frac{\omega}{M}\|X\| \tag{24}$$

We define

$$\rho = -\frac{\omega}{M} > 0 \tag{25}$$

Then we call the parameter ρ the robustness degree of nonlinear system (5) with linear state feedback control law (8). The following definition is more formal.

Definition 1 Robustness Degree

For a nonlinear system with the form of Eq. (5), if the system is stabilized by the linear state feedback control law Eq. (7) when the nonlinear function vector is located within the conic sector

$$\|F(X, U, \alpha)\| < \rho\|X\| \tag{26}$$

then ρ is called robustness degree of the controlled system. That is to say, linear system (7) with feedback Eq. (8) preserves a robustness degree ρ to suffer the effect of the nonlinear term $F(X, U, \alpha)$ with relation (26).

Here we give the definition of robustness degree for a system in the form of Eq. (5), but actually almost the same definition can be given for a general system. Robustness degree can be regarded as a quantitative measure of the capability to reject or eliminate the effects or distortion of the nonlinear term and the parameter-induced variations. It should be emphasized that the robustness degree ρ here is related only to the linear system Eq. (7) and the feedback control Eq. (8), and not to the nonlinear term $F(X, U, \alpha)$. Robustness degree ρ is a ratio of two parameters in operator T_t; that is, ρ is related only to the properties of matrix $(A - BK)$. The larger the robustness degree ρ, the larger the norm of $F(X, U, \alpha)$ can be without destroying the stability of the closed-loop system.

According to the development and discussion above, the following theorem on robust stability can be derived.

Theorem 1

A nonlinear system in the form of Eq. (5) with a linear state feedback control law given by Eq. (8) is robustly stable in the sense of Lyapunov with robustness degree ρ defined by Eq. (25), if the nonlinear term $F(X, U, \alpha)$ satisfies a conic sector type of inequality condition (26).

This theorem shows the relation between the stability of the closed-loop system with a given feedback law for some robustness degree and the effect of the nonlinear function vector $F(X, U, \alpha)$. The concept of robustness degree to measure the stability margin of a closed-loop system is introduced, and will be helpful in analyzing the characteristics of control systems.

Remark It can be seen from relation (17) that if $\|F\|/\|X\|$ goes to zero as $t \to \infty$, then the system is asymptotically stable with the robustness degree ρ.

2. Further Discussion of Robustness Degree

We have now developed robustness degree for control systems. But what is the relationship between robustness degree ρ, i.e. ω and M in Eq. (11), and the properties of matrix $(\boldsymbol{A}-\boldsymbol{BK})$? From the definition of operator T_t we can see that in a general expression the operator function can be a transition matrix of $(\boldsymbol{A}-\boldsymbol{BK})$, if $(\boldsymbol{A}-\boldsymbol{BK})$ is a constant matrix; then

$$T_t=\exp[(\boldsymbol{A}-\boldsymbol{BK})t] \tag{27}$$

i.e.

$$\|T_t\|=\|\exp[(\boldsymbol{A}-\boldsymbol{BK})t]\|\leqq M\exp(\omega t) \tag{28}$$

However, it is difficult to find a precise version of robustness degree. But an approximate and useful measure, called matrix measure, can be used to express system robustness degree.

Definition 2 Matrix measure $\mu(\boldsymbol{Z})$[5)]

$$\mu(\boldsymbol{Z})=\lim_{\delta\to 0^+}(\|I-\delta\boldsymbol{Z}\|-1)\delta^{-1}$$

For the 2-norm concerned

$$\mu(\boldsymbol{Z})=\lambda_{\max}\left[\frac{1}{2}(\boldsymbol{Z}+\boldsymbol{Z}^*)\right] \tag{29}$$

By applying the so-called Coppel's inequality,[5)] we obtain the following inequality for the solution of Eq. (7) with (8):

$$\|\boldsymbol{X}(t)\|\leqq\|\boldsymbol{X}(0)\|\exp\left(\int_0^t \mu(\boldsymbol{A}-\boldsymbol{BK})d\tau\right)$$
$$=\|\boldsymbol{X}(0)\|\exp(\mu(\boldsymbol{A}-\boldsymbol{BK})t) \tag{30}$$

Thus, robustness degree ρ in one of its estimates has the form of

$$\rho=-\mu(\boldsymbol{A}-\boldsymbol{BK}) \tag{31}$$

and the following corollary of the theorem can be derived.

Corollary 1

The robustness degree ρ, in one of its possible precise forms, is equal to the negative of the measure of matrix $(\boldsymbol{A}-\boldsymbol{BK})$ as shown by Eq. (31).

3. Application of the Results to a CSTR

Consider again the CSTR in the introduction section. For $B=8$, $\beta=0.3$, $\nu=20$, $D_a=0.072$ and $x_{20}=0$ in particular, the reactor shows an ignition/extinction behavior corresponding to the multiple steady states.[9)]

$$[x_1^s, x_2^s]_1=[0.144,\ 0.886] \quad \text{(stable)}$$
$$[x_1^s, x_2^s]_2=[0.447,\ 2.752] \quad \text{(unstable)}$$
$$[x_1^s, x_2^s]_3=[0.765,\ 4.705] \quad \text{(stable)}$$

Suppose the unstable equilibrium point 2 [0.447, 2.752] is the desired operating point. The linearized model on this point is

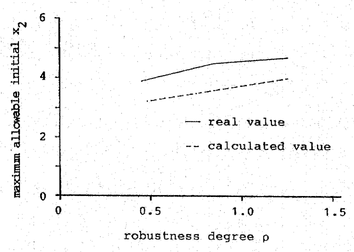

Fig. 1. Robustness degree ρ vs. maximum allowable initial x_2

$$\begin{aligned}\Delta\dot{x}_1&=-1.809\Delta x_1+0.346\Delta x_2\\ \Delta\dot{x}_2&=-6.470\Delta x_1+1.464\Delta x_2+0.3u\end{aligned} \tag{32}$$

where $\Delta x_1=x_1-x_1^s$, $\Delta x_2=x_2-x_2^s$. Obviously, this is an unstable open-loop system.

The purpose of this simulation study is to show the relation of the closed-loop system robustness degree with the maximum allowable initial state that will not lead the system to instability. As we know, a nonlinear model can be approximately described by a linear model in some range through the Taylor expansion, the higher-order terms being considered to be much higher infinitesimals compared with its incremement $\|\Delta\boldsymbol{X}\|^2$. For this reason and the results developed above, if we design a controller so that the closed-system robustness degree is $\rho=-\mu(\boldsymbol{A}-\boldsymbol{BK})$, then we have the robustly stable range

$$\|\Delta\boldsymbol{X}\|^2<\rho\|\Delta\boldsymbol{X}\| \tag{33}$$

i.e.

$$\|\Delta\boldsymbol{X}\|<\rho \tag{34}$$

This relation of course includes the allowable initial state range. For simplicity, we assume $x_1(0)=0$, then the stable range for $x_2(0)$ will be

$$\rho-x_2^s<x_2(0)<\rho+x_2^s \tag{35}$$

As discussed above, the robustness degree condition is only a sufficient one. We design three controllers with different robustness degrees as follows:

$$K_1=[-15\ \ 13], \quad \rho_1=1.252$$
$$K_2=[-15\ \ 10], \quad \rho_2=0.849$$
$$K_3=[-15\ \ 8], \quad \rho_3=0.450$$

and then the predicted stable range by (35) and the real simulated stable range are shown in **Fig. 1**. We note that x_2 as defined is always greater than zero.

4. Conclusions and Discussion

The problem of robustness analysis of a closed-loop nonlinear system with a linear state feedback control has been discussed, and some new results have been obtained that can be applied to the analysis of practical problems. The definition of robustness degree of a control system has been introduced. This concept is very helpful in practical applications. Robustness degree is a measure of the stability margin of control systems.

Semigroup theory of functional analysis is applied to develop the new algorithm and results, concluding with the theorem and its corollary. Theorem 1 shows the sufficient condition of the nonlinear function in the system for robust stability of the closed-loop system. Corollary 1 provides a nice relation between robustness degree and the matrix measure so that the robustness degree can be easily determined by the system matrix independent of the nonlinear term. All the results developed here can be easily determined and applied, unlike many results derived in the frequency domain, where the robustness problem of a system cannot be measured by a simple value. Most results are linked with the singular values of a transfer matrix function with great difficulty to solve.

The simulation results of reactor control show the reasonableness of the robustness degree analysis method. This simulation is only an example of the application of the results proposed in this paper. Further application can be expected, e.g. omitting the nonlinear term of a weak nonlinear system for satisfactory linear controller design by ensuring a certain robustness degree, reducing the offset of the system response to a constant disturbance by increasing the system robustness degree.

Nomenclature

$\boldsymbol{A}$	= system matrix of a state space equation
a	= norm of the initial state vector
$\boldsymbol{B}$	= input matrix of a state space equation
C_A	= concentration of component A in reactor
C_{Af}	= concentration of component A in feed
c_p	= specific heat of reactant
F	= feed flowrate of the reactor
F	= a nonlinear function of the system
g	= a nonlinear function of the system
$-\Delta H$	= enthalpy of reactant
h, A	= heat transfer coefficient and heat transfer area
k	= reaction coefficient
M	= a constant in a semigroup
r	= reaction rate
S_t	= a nonlinear semigroup
T_t	= a linear semigroup
T	= reactor temperature
T_c	= coolant temperature
T_f	= feed temperature
t	= time
$\boldsymbol{U}$	= control vector
u_i	= control variable i
V	= the reactor volume
$\boldsymbol{X}$	= state vector
x_i	= state variable i
$\boldsymbol{Z}$	= a matrix
$\boldsymbol{\alpha}$	= a variable parameter vector
Δ	= variation
δ	= variation
λ	= a positive constant
$\lambda_{\max}$	= maximum eigenvalue of a matrix
μ	= matrix measure
ρ	= density of reactant
ρ	= system robustness degree
τ	= integral variable
$\phi(t), \psi(t), \mu(t)$	= nonnegative continuous functions
ω	= a negative constant of semigroup

⟨Subscripts⟩

s	= steady state
0	= initial value

⟨Superscripts⟩

*	= conjugate

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