CENTRIFUGAL INSTABILITY OF TURBULENT BOUNDARY LAYERS ALONG CONCAVE SURFACES

NOBUYUKI FUJISAWA AND HIROYUKI SHIRAI
Department of Mechanical Engineering, Gunma University, Kiryu 376

Key Words: Fluid Mechanics, Stability, Turbulent Boundary Layer, Görtler Vortex, Numerical Analysis, Concave Surface

The structure of turbulence in boundary layers along concave surfaces was investigated in relation to the appearance of Taylor-Görtler vortices caused by centrifugal instability.

In the present paper, the neutral stability condition for the appearance of Taylor-Görtler vortices in concave turbulent boundary layers is studied theoretically by a numerical solution. The importance of the normal velocity component on the stability of turbulent boundary layers is noted.

1. Theory

To formulate the present stability problem in curvilinear coordinates (x, y, z) (as shown in **Fig. 1**), the following assumptions are made: (1) The concept of eddy viscosity is used to handle the Reynolds stress in the basic equations of incompressible turbulent flows. (2) The radius R of the concave surface is uniform in x direction and is much larger than the thickness δ of the boundary layer. (3) The velocity components in the perturbed motion are expressed as follows[8]:

$$u = U(x, y) + u_1 \exp\{\textstyle\int \beta(x)dx\} \cos(2\pi z/\lambda) \tag{1}$$

$$v = V(x, y) + v_1 \exp\{\textstyle\int \beta(x)dx\} \cos(2\pi z/\lambda) \tag{2}$$

$$w = w_1 \exp\{\textstyle\int \beta(x)dx\} \sin(2\pi z/\lambda) \tag{3}$$

where $u_1, v_1, w_1 \ll U, V$. (4) The perturbed motion is assumed to be steady in space and independent of the coherent structures of the large eddies in the outer layer and the bursts near the surface. Therefore, the wavelength λ of the perturbed motion is considered to be much larger than the spatial periodity of such structures (which is smaller than or equal to δ). (5) The principle of exchange of stability is used. This has been usually assumed, based on the experimental indications for the Görtler stability.[1,9] Then the perturbed equations are obtained from the momentum equations of incompressible turbulent flows with Eqs. (1) to (3). In a neutral stability state ($\beta = 0$), the nondimensional perturbed equations are written as follows:

$$M\bar{u}_{\eta\eta} - (\bar{V} - M_\eta)\bar{u}_\eta + (\bar{V}_\eta - \sigma^2 M)\bar{u} = \bar{U}_\eta \bar{v} \tag{4}$$

$$\begin{aligned} M\bar{v}_{\eta\eta\eta\eta} &- (\bar{V} - M_\eta)\bar{v}_{\eta\eta\eta} - (\bar{V}_\eta - M_{\eta\eta} + 2\sigma^2 M)\bar{v}_{\eta\eta} \\ &+ \sigma^2(\bar{V} - M_\eta)\bar{v}_\eta + \sigma^2(\bar{V}_\eta + M_{\eta\eta} + \sigma^2 M)\bar{v} \\ &= -2G^2\sigma^2 \bar{U}\bar{u} \end{aligned} \tag{5}$$

$$\sigma\bar{w} = -\bar{v}_\eta \tag{6}$$

where $G = Re\sqrt{\delta/R}$, $M = 1 + \mu_t/\mu$, $Re = U_m\delta/\nu$, $\bar{U} = U/U_m$, $\bar{V} = Re \cdot V/U_m$, $\bar{u} = u_1/U_m$, $\bar{v} = Re \cdot v_1/U_m$, $\bar{w} = Re \cdot w_1/U_m$, $\eta = y/\delta$, $\sigma = 2\pi\delta/\lambda$, and subscript η implies differentiation with respect to η. Boundary conditions for Eqs. (4) to (6) are expressed as $\bar{u} = \bar{v} = \bar{w} = 0$ at $\eta = 0$ and $\eta \to \infty$. In addition, $M = 1$ and $d\bar{U}/d\eta = 0$ at $\eta \to \infty$.

The flow parameters, $\bar{U}$, $\bar{V}$, M and their derivatives with respect to η are given empirically as a function of η and Re.[6] The velocity profile $\bar{U}$ is expressed by the formula of van Driest[3] corrected by the wake function by Coles.[2] The velocity $\bar{V}$ normal to the surface is taken from the continuity equation. Nondimensional effective viscosity M is obtained from the eddy viscosity expression, where the shear stress is calculated from the momentum-integral equation of boundary-layer theory and the empirical formula for the skin friction C_f.[7]

2. Results and Discussion

Figure 2 shows typical numerical results for the neutral stability curves of the turbulent boundary layers at $Re = 10^4$ and the experimental neutral condition by Tani[9] at the same Re. Here the successive approximation procedure and finite difference tech-

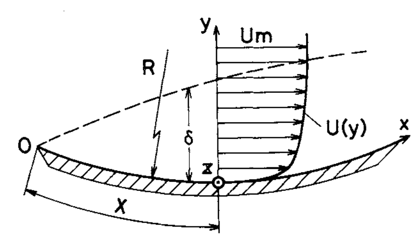

Fig. 1. Turbulent boundary layers and coordinate system.

Received June 28, 1985. Correspondence concerning this article should be addressed to N. Fujisawa.

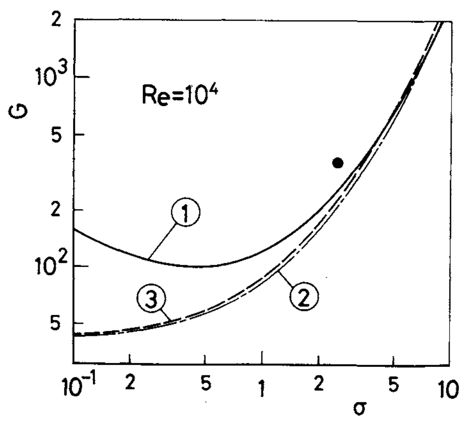

Fig. 2. Neutral stability diagram for Görtler parameter G and wavenumber σ. ① with $\bar{V}$ terms, ② without $\bar{V}$ terms, ③ without $\bar{V}$ terms (Sandmayr's analysis[6]), experiment ($Re=10^4$).[9]

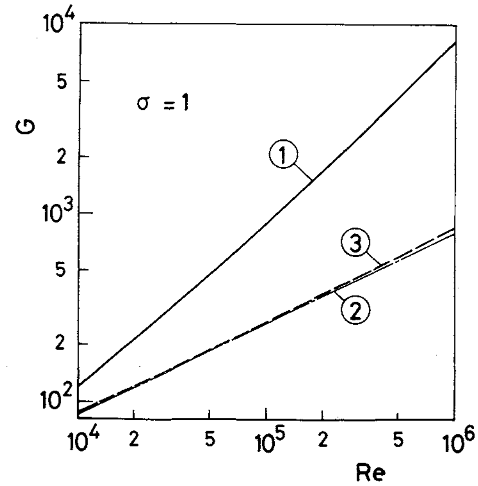

Fig. 3. Neutral stability relations for Görtler parameter G and Reynolds number Re. Captions are the same as in Fig. 2.

nique are adopted as a numerical method. Curve ① in this figure indicates the present numerical result. The result on the assumption of $\bar{V}=0$ in Eq. (2), which is often used,[4] is also shown by curve ②. The present numerical method was also applied to the perturbed equations of Sandmayr,[6] where the velocity component $\bar{V}$ is neglected and the growth rate of the perturbation is treated as a function of time. This result, shown as curve ③, agrees closely with curve ②. Therefore, the neutral stability condition is little affected by the assumption for the way of growth of the perturbation. It is noted from this figure that the neutral stability curve ① has a minimum value of G at $\sigma \simeq 0.4$ and the Görtler parameter G becomes large in a range of the wavenumber $\sigma < 2$ compared with the other curves ② and ③. These results are apparently due to the contribution of the $\bar{V}$ terms (which means the terms including $\bar{V}$ and $\bar{V}_\eta$ in Eqs. (4) and (5)). On the contrary, for laminar boundary layers, the parameter G becomes small in a small wavenumber range when the $\bar{V}$ terms are included in the analysis. This difference between the turbulent and laminar stability results is mainly due to the fact that the magnitude of the eddy viscosity in the outer layer is larger than that in the inner layer of the turbulent boundary layers.

Figure 3 shows the neutral stability relations of Görtler parameter G and the Reynolds number Re at the wavenumber $\sigma=1$. Curves ①, ② and ③ in this figure are obtained for the same conditions as for curves ①, ② and ③ respectively in Fig. 2. It is apparent that by the contribution of the $\bar{V}$ terms the Görtler parameter increases more rapidly as the Reynolds number increases.

Nomenclature

G	=	Görtler parameter	[—]
M	=	nondimensional effective viscosity	[—]
R	=	radius of surface curvature	[m]
Re	=	Reynolds number ($=U_m\delta/\nu$)	[—]
U, V	=	velocities of basic flow in x, y direction	[m/s]
$\bar{U}, \bar{V}$	=	nondimensional values of U, V	[—]
U_m	=	velocity of main flow	[m/s]
u, v, w	=	velocity components in x, y and z direction	[m/s]
u_1, v_1, w_1	=	perturbed velocity components in x, y and z direction	[m/s]
u, v, w	=	nondimensional values of u_1, v_1 and w_1	[—]
X	=	streamwise distance from leading edge of surface	[m]
x, y, z	=	orthogonal curvilinear coordinate	[m]
β	=	streamwise growth rate of perturbation	[m^{-1}]
δ	=	boundary layer thickness, where $U=0.99\,U_m$	[m]
η	=	nondimensional distance ($=y/\delta$)	[—]
λ	=	wavelength of perturbation	[m]
μ	=	viscosity of fluid	[Pa·s]
μ_t	=	eddy viscosity	[Pa·s]
ν	=	kinematic viscosity of fluid	[m^2/s]
σ	=	wavenumber ($=2\pi\delta/\lambda$)	[—]

Literature Cited

1) Aihara, Y.: *Bull. Aero. Res. Inst., Univ. of Tokyo*, **3**, 195 (1962).
2) Coles, D.: *J. Fluid Mech.*, **1**, 191 (1956).
3) Driest, E. R. van: *J. Aeronaut. Sci.*, **23**, 1007 (1956).
4) Görtler, H.: *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Klasse, Neue Folge, Fachgruppe I*, **2**, 1 (1940).
5) Hinze, J. O.: "Turbulence," McGraw-Hill Co., New York (1975).
6) Sandmayr, G.: *Deutsche Luft- und Raumfahrt Forschungsbericht*, 66–41 (1966).
7) Schlichting, H.: "Boundary-Layer Theory," McGraw-Hill Co., New York (1968).
8) Smith, A. M. O.: *Quart. Appl. Math.*, **13**, 233 (1955).
9) Tani, I.: *J. Geophys. Res.*, **67**, 3075 (1962).
10) Tomita, Y., K. Funatsu, S. Tsuzuki and K. Miyazaki: *Trans. Jpn. Soc. Mech. Eng.*, **51**, 479 (1985).

(Presented at the 50th Annual Meeting of The Society of Chemical Engineers, Japan, March 28, 1985.)