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PRIME, IRREDUCIBLE ELEMENTS AND COATOMS IN POSETS

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ABSTRACT. In this paper, some properties of prime elements, pseudoprime elements, irreducible elements and coatoms in posets are investigated. We show that the four kinds of elements are equivalent to each other in finite Boolean posets. Furthermore, we demonstrate that every element of a finite Boolean poset can be represented by one kind of them. The example presented in this paper indicates that this result may not hold in every finite poset, but all the irreducible elements are proved to be contained in each order generating set. Finally, the multiplicative auxiliary relation on posets and the notion of arithmetic poset are introduced, and some properties about them are generalized to posets.

1. Introduction

Prime elements, irreducible elements and coatoms are the basis of representation theory, which play an important role in order structure. It was shown in [4] that every element of a Boolean algebra can be represented by atoms. In order to better investigate the properties of partial order structures, Hofmann and Lawson [13] study the prime elements and irreducible elements in semilattice with continuous property. They show that every element of a continuous semilattice can be written as an infimum of some irreducible elements. Subsequently, Hofmann and Lawson established spectral theory by means of endowing the set that consists of all the prime elements of a lattice with a topology. They expected to find the algebraic properties from the viewpoint of topology. Until now, a great deal of ingenious results about spectral theory have sprung up, which can be seen in [9]. Therefore, the relations among prime elements, irreducible elements and coatoms become a problem deserving study. The corrections among them

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will contribute to exploring the properties of the order structure in topology and representation theory terms flexibly. In [9], the links about these elements are discussed in detail, and it was shown that prime elements, irreducible elements and coatoms are the same in Boolean lattices. Further, the pseudoprime elements, which is more general than prime elements, conincide with prime elements in arithmetic semilattices. Recently, in order to investigate the general representation theory for posets, Hofmann and Lawson have extended the definitions of the prime and irreducible element to posets. With the introduction of distributivity in posets by Larmerová and Rachůnek [15], Boolean posets were well defined by Chajda [2]. So whether the four kinds of elements agree in Boolean posets, and under what conditions properties in semilattices can be generalized to posets are natural questions.

This paper mainly investigates the properties of these elements, as well as the connections among them. In this paper, the relevant results in semilattices presented in [9] are generalized to posets. We show that prime, pseudoprime, irreducible elements and coatoms are equivalent to each other in finite Boolean posets. Based on this assertion, every element of a finite Boolean poset can be represented by atoms, which is an improvement result compared with that proposed in [4]. Finally, we introduce the multiplicative auxiliary relation on posets and the notion of arithmetic posets and further we show that the way-below relation on a poset is multiplicative iff the poset is arithmetic. Thus pseudoprime elements agree with prime elements in arithmetic posets, which is a mild generalization of the relevant result in semilattice given in [9].

The remainder of this paper is arranged as follows. In Section 2, we recall the necessary definitions and notations. Section 3 mainly discusses the conditions under which the coatoms agree with the prime elements. The conditions allowing for the equivalence between coatoms and irreducible elements are investigated in Section 4. Section 5 elaborates the relation among prime elements, irreducible elements and coatoms. In Section 6, we firstly investigate some other properties of the four kinds of elements, then the notions of the multiplicative auxiliary relation on posets and arithmetic posets are introduced and the properties of them are discussed. Finally, conclusion is made in Section 7.

2. Preliminary

Let P be a poset, $A \subseteq P$. The set $A^u = \{x \in P \mid a \leqslant x \text{ for any } a \in A\}$ is called the *upper cone* of A. Dually, $A^l = \{x \in P \mid a \geqslant x \text{ for all } a \in A\}$ is called the *lower cone* of A. A^{ul} means $\{A^u\}^l$ and A^{lu} means $\{A^l\}^u$. The lower set $\{a\}^l$ is simply denoted by a^l and $\{a,b\}^l$ is denoted by $(a,b)^l$. Furthermore, $\{A \cup B\}^u$ is denoted by $\{A,B\}^u$ for $A,B \subseteq P$. And for $x \in P$, the set $\big\{A \cup \{x\}\big\}^u$ is denoted by $\{A,x\}^u$. Similar notations are used for the dual situations. Note that, $A \subseteq A^{ul}$ and $A \subseteq A^{lu}$. In particular, $\{a^u\}^l = \{a\}^l$ and $A^{lul} = A^l$, $A^{ulu} = A^u$. If $A \subseteq B$, then $B^u \subseteq A^u$ and $B^l \subseteq A^l$. For any subset $A \subseteq P$ with $\bigwedge A$ exists, $A^l = \downarrow \bigwedge A$. Dually, for any subset $A \subseteq P$ with $\bigvee A$ exists, we have $A^u = \uparrow \bigvee A$, where $\bigvee A$ and $\bigwedge A$ denote the least upper bound and greatest lower bound of A, respectively.

Definition 2.1. ([9]) Let P be a poset. A nonempty subset D of P is called a *directed set*, provided every finite subset of D has an upper bound in D. Dually, a nonempty subset F of P is called *filtered* if every finite subset of F has a lower bound in F.

Definition 2.2. ([1]) Let P be a poset. $X \subseteq P$. Then X is called a *lower set*, if $X = \downarrow X$, where $\downarrow X = \{y \in P \mid y \leq x \text{ for some } x \in X\}$. The *upper set* is defined dually.

Definition 2.3. ([9]) A subset I of a poset P is called *an ideal* if it is a directed lower set. And it is said to be *proper*, if $I \neq P$. A *maximal ideal* of P is a proper ideal I such that $I \subset J \subseteq P$ implies $J = P$ for the ideal J. Dually, a subset $X \subseteq P$ is called *a filter* if it is a filtered upper set. Further, if an ideal I of P satisfies that $P \backslash I$ is a filter or is empty, then I is called a *prime ideal* .

Notice that the ideal defined above is different from the Frink ideal [8].

Definition 2.4. ([15]) A poset P is said to be *distributive*, if for all $a, b, c \in P$, $\{(a,b)^u, c\}^l = \{(a,c)^l, (b,c)^l\}^{ul}$ holds.

Larmerová and Rachůnek [15] have proved that every distributive poset is dually distributive. That is for all $a, b, c \in P$, $\{(a,b)^u, c\}^l = \{(a,c)^l, (b,c)^l\}^{ul}$ holds iff $\{(a,b)^l, c\}^u = \{(a,c)^u, (b,c)^u\}^{lu}$ holds. Larmerová [16] characterized the semilattice with distributive properties defined above. Based on these results, Rachůnek [19] investigated the larger classes of semilattices.

Definition 2.5. ([2]) An element $y \in P$ is called a *complement* of $x \in P$, if $(x,y)^{ul} = (x,y)^{lu} = P$. P is said to be *complemented* if each element of it has a complement in P; and P is said to be *uniquely complemented* if each element $x \in P$ has a unique complement, denoted by x' in P. A distributive complemented poset is called a *Boolean poset*.

Note that each element has at most one complement element in a distributive poset, so Boolean posets must be uniquely complemented. A Boolean poset may have no bottom and top element, but once it has one of them, it must have the other [18].

Definition 2.6. ([11]) An element $x^* \in P$ is called the *pseudocomplement* of $x \in P$, if $(x, x^*)^l = P$ and for $y \in P$, $(x,y)^l = P$ implies $y \leqslant x^*$.

Obviously, if the pseudocomplement exists, then it is unique.

Definition 2.7. ([9]) An element $x \in P$ is said to be *irreducible* if x is maximal or $\uparrow x \backslash \{x\}$ is a filter. Dually, we have the definition of *co-irreducible*. The sets of all irreducible and co-irreducible elements are denoted by $\mathrm{IRR}(P)$ and $\mathrm{COIRR}(P)$, respectively. An element $x \in P$ is called *completely irreducible* if either x is maximal but different from the top element or $\uparrow x \backslash \{x\}$ has a least element, denoted by x^+.

Definition 2.8. ([9]) An element $x \in P$ is said to be *prime* if $x = 1$ or $P \backslash \downarrow x$ is a filter. Dually, we have the definition of *co-prime element*. The sets of all prime and co-prime elements are denoted by $\mathrm{PRI}(P)$ and $\mathrm{COPRI}(P)$, respectively. An element $x \in P$ is said to be *completely prime* if $P \backslash \downarrow x$ has a least element, denoted by x_+. An element $x \in P$ is said to be *completely co-prime* if $P \backslash \uparrow x$ has a greatest element, denoted by x_-.

Definition 2.9. ([2]) Let P be a poset. An element a of P is called an *atom* whenever:

(1) If P has the least element 0 and $0 < x \leqslant a$ for some $x \in P$, then $x = a$;

(2) If P has not the least element, then a is a minimal element of P.

Dually, we have the definition of *coatom*, denoted by $A(P)$ and $\mathrm{COA}(P)$, respectively.

Theorem 2.1. ([2]) *Let P be a Boolean poset. Then for each $a, b \in P$, $a \leqslant b$ implies $b' \leqslant a'$.*

In this paper, we specialize our discussion to bounded posets. Furthermore, the results also can be generalized to the unbounded posets.

3. Relations between coatoms and prime elements

Generally, a coatom may not be prime, even in a lattice. See the lattice M_3 depicted in Figure 1, where the central element v is a coatom but not a prime element. However, a coatom coincides with a prime element in a Boolean lattice (see [9]). In this section, we extend this property to finite Boolean posets.

Proposition 3.1. *In a distributive poset P, every coatom is prime.*

Proof. Let x be a coatom. Then $x \neq 1$. To verify $P \backslash \downarrow x$ is a filter, it is sufficient to show that for all $a, b \in P \backslash \downarrow x$ and $a \neq 1$, $b \neq 1$, $P \backslash \downarrow x \bigcap (a, b)^l \neq \emptyset$. Suppose $P \backslash \downarrow x \bigcap (a, b)^l = \emptyset$, then for all $y \in (a, b)^l$, we have $y \leqslant x$. It follows that $\{(a, b)^l, x\}^u = x^u$. Since $a \in P \setminus \downarrow x, b \in P \setminus \downarrow x$ and x is a coatom, we conclude that $a \ngeqslant x$, $a \nleqslant x$ and $b \ngeqslant x$, $b \nleqslant x$. Hence, $(a, x)^u = \{1\}$ and $(b, x)^u = \{1\}$. Then $\{(a, x)^u, (b, x)^u\}^{lu} = \{1\}$. Therefore, $x \notin \{(a, x)^u, (b, x)^u\}^{lu}$, but $x \in x^u = \{(a, b)^l, x\}^u$. A contradiction to that P is a distributive poset, which satisfies $\{(a, b)^l, x\}^u = \{(a, x)^u, (b, x)^u\}^{lu}$. Hence, $P \setminus \downarrow x$ is a filter. This proves that x is prime. □

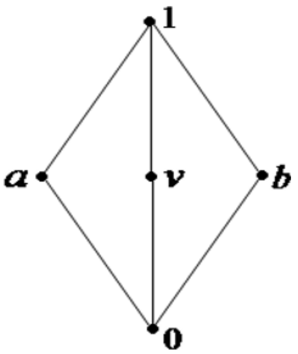

FIGURE 1.

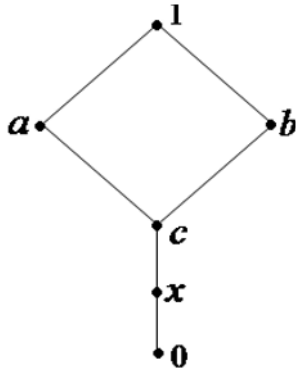

FIGURE 2.

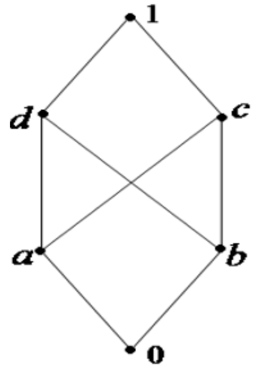

FIGURE 3.

However, the converse of the above proposition does not hold, even for the distributive lattices. Moreover, a completely prime element may also not be a coatom. See the distributive lattice depicted in Figure 2, in which x is a completely prime element, since c is the least element of $P \backslash \downarrow x$, but clearly it is not a coatom. Nevertheless, if we consider a Boolean poset, then every completely prime element is a coatom.

PROPOSITION 3.2. *Every completely prime element of a Boolean poset P is a coatom.*

Proof. Suppose $x < y \leqslant 1$. We need to prove $y = 1$. Since $x^u = (x, y)^{lu} = \{(x, y)^l, (y', y)^l\}^u$, then $x^l = x^{ul} = (x, y)^{lul} = \{(x, y)^l, (y', y)^l\}^{ul}$. By distributivity, $x^l = \{(x, y')^u, y\}^l$. For all $a \in (x, y')^u$, we have $a \geqslant x$ and $a \geqslant y'$. In fact, there exists $a \in (x, y')^u$ such that $a = x$. Suppose not, that is $a > x$ for all $a \in (x, y')^u$, then $a \nleqslant x$. It follows that $(x, y')^u \subseteq P \setminus \downarrow x$. Since $y > x$, then $y \in P \setminus \downarrow x$. Hence, $\big\{(x, y')^u, y\big\} \subseteq P \setminus \downarrow x$. By the hypothesis, x is a completely prime element, then $P \setminus \downarrow x$ has a least element z. So $z \in \big\{(x, y')^u, y\big\}^l = x^l$, which means that $z \leqslant x$. A contradiction to $z \in P \setminus \downarrow x$. Consequently, there is an element $a \in (x, y')^u$ such that $a = x$ and $a \geqslant y'$. Then $x \geqslant y'$. By Theorem 1.1, $y \geqslant x'$. So $y \in (x', x)^u = \{1\}$. □

PROPOSITION 3.3. *Each prime element of a Boolean poset with no infinite chains is a coatom.*

COROLLARY 3.3.1. *Every prime element of a finite Boolean poset is a coatom.*

Combining Propositions 3.1 and 3.3, we obtain the following theorem.

THEOREM 3.4. *In a Boolean poset with no infinite chains, an element is prime iff it is a coatom.*

COROLLARY 3.4.1. *In a finite Boolean poset, an element is prime if and only if it is a coatom.*

4. Relations between coatoms and irreducible elements

Irreducible elements coincide with coatoms in Boolean lattices (see [9]). Consequently, the related questions for Boolean posets are considered in this section.

Let A be a subset of a poset P. In Refs. [6, 20], if $x = \bigwedge A$ implies $x \in A$, then x is called a $\bigwedge$-irreducible element, denoted by $M(P)$. The $\bigvee$-irreducible element is defined dually. Generally, the $\bigwedge$-irreducible element is different from the irreducible element. It can be demonstrated from Figure 3, in which the element a is $\bigwedge$-irreducible, but not irreducible. However, it is easy to check that in a finite semilattice, x is irreducible iff x is $\bigwedge$-irreducible. And $\bigwedge$-irreducible is also different from the completely irreducible, but in terms of the complete semilattice, they are the same.

Lemma 4.1. *In a poset, $\bigwedge A$ exists iff $\bigwedge A^{lu}$ exists. Once one exists, then $\bigwedge A^{lu} = \bigwedge A$.*

Proof. Suppose $\bigwedge A$ exists, then $A^{lu} = (\bigwedge A)^{lu} = \uparrow \bigwedge A$. Hence, $\bigwedge A^{lu}$ exists and $\bigwedge A^{lu} = \bigwedge A$. Conversely, if $\bigwedge A^{lu}$ exists, then $(A^{lu})^l = \left\{\bigwedge(A^{lu})\right\}^l$. Since $A^l = A^{lul}$, then $\left\{\bigwedge(A^{lu})\right\}^l = A^l$. It is obvious that $\bigwedge(A^{lu})$ is a lower bound of A. Since $x \in \left\{\bigwedge(A^{lu})\right\}^l$ for all $x \in A^l$, then $x \leqslant \bigwedge(A^{lu})$, which means that $\bigwedge(A^{lu})$ is the greatest lower bound of A. Therefore, $\bigwedge A = \bigwedge(A^{lu})$. $\square$

Theorem 4.1. *In a finite Boolean poset P, x is irreducible iff x is $\bigwedge$-irreducible.*

Proof. Suppose x is irreducible and $x \neq 1$, then $\uparrow x \backslash \{x\}$ is a filter. Suppose $x = \bigwedge A$ for some $A \subseteq P$. Then for all $a \in A$, we have $x \leqslant a$. In fact, there exists some $a \in A$ such that $x = a$. Otherwise, if for all $a \in A$, $x < a$. We can get that for all $a \in A$, $a \in \uparrow x \backslash \{x\}$. That is $A \subseteq \uparrow x \backslash \{x\}$. Since P is finite and $\uparrow x \backslash \{x\}$ is a filter, then $x = \bigwedge A \in \uparrow x \backslash \{x\}$, which is impossible. Hence, there exists some $a \in A$ such that $x = a$. It follows that x is $\bigwedge$-irreducible.

Conversely, if x is $\bigwedge$-irreducible and $x \neq 1$, we need to prove $\uparrow x \backslash \{x\}$ is a filter. If $\uparrow x \backslash \{x\}$ is not a filter, then there must exist $a, b \in \uparrow x \backslash \{x\}$ such that for every $c \in \uparrow x \backslash \{x\}$, $a \ngeqslant c$ or $b \ngeqslant c$. Then c is not the lower bound of a and b. In the following, we prove that x is the greatest lower bound of a and b. If not, there must exist some $m \in (a, b)^l$ such that $m \ngeqslant x$ and $m \nleqslant x$. By the property that $(m, m')^l = \{0\}$, we obtain $\{(m, m')^l, x\}^u = x^u$. Further, by distributivity, $\{(m, m')^l, x\}^u = \{(m, x)^u, (m', x)^u\}^{lu}$. Applying Lemma 4.1, we have $x = \bigwedge x^u = \bigwedge\{(m, x)^u, (m', x)^u\}^{lu} = \bigwedge\{(m, x)^u, (m', x)^u\}$. By the hypothesis, $x \in \{(m, x)^u, (m', x)^u\}$. Hence, $x \in (m, x)^u$, or $x \in (m', x)^u$.

If $x \in (m,x)^u$, then $x \geqslant m$, a contradiction to $x \ngeqslant m$. If $x \in (m',x)^u$, then $x \geqslant m'$. Since $a > x$ and $b > x$, then $a > m'$ and $b > m'$, which means that $a, b \in (m,m')^u$. A contradiction to $(m,m')^u = \{1\}$. Hence, x is the greatest lower bound of a and b. That is $x = a \bigwedge b$. By the definition of $\bigwedge$-irreducible, we have $x = a$ or $x = b$, which is a contradiction to $a, b \in {\uparrow}x\backslash\{x\}$. Consequently, there exists some $c \in {\uparrow}x\backslash\{x\}$, such that $a \geqslant c$ and $b \geqslant c$. Then ${\uparrow}x\backslash\{x\}$ is a filter. We conclude that x is irreducible. □

Theorem 4.2. *In a Boolean poset P, $M(P) = \mathrm{COA}(P)$.*

Proof. Let $x \in \mathrm{COA}(P)$. Then $x \neq 1$. Assume $x = \bigwedge B$, $B \subseteq P$. We claim $x \in B$. If $x \notin B$, then for all $b \in B$, we have $x < b$. Since x is a coatom, then $b = 1$ for all $b \in B$. It follows that $x = \bigwedge B = 1$, which is a contradiction to $x \neq 1$. Hence, $x \in M(P)$.

For the converse, suppose $x \in M(P)$ and $x < c \leqslant 1$. We need to prove $c = 1$. By the relation among x, c, and c', we have $x^{lu} = (x,c)^{lu} = \{(x,c)^l, (c',c)^l\}^u$. Hence, $x^l = x^{lul} = \{(x,c)^l, (c',c)^l\}^{ul}$. By the distributivity, $x^l = \{(x,c')^u, c\}^l$. Then $x^{lu} = \{(x,c')^u, c\}^{lu}$. By Lemma 4.1, $x = \bigwedge x^{lu} = \bigwedge\{(x,c')^u, c\}^{lu} = \bigwedge\{(x,c')^u, c\}$. Since $x \in M(P)$ and $x \neq c$, we get $x \in (x,c')^u$. Hence, $x \geqslant c'$. $c^u = (x,c)^u \subseteq (c,c')^u = \{1\}$. Then $c = 1$. □

Remark 1. The $\bigwedge$-irreducible element may not be a coatom if the poset is merely a distributive poset but not a Boolean poset. Just as the element a shown in Figure 3, where a is $\bigwedge$-irreducible but not a coatom. And even in a distributive lattice, an irreducible element need not be a coatom. The lattice displayed in Figure 2 is distributive, and ${\uparrow}x\backslash\{x\}$ is a filter. Hence, x is irreducible, but not a coatom.

The following theorem is a direct consequence of Theorem 4.1 and Theorem 4.2.

Theorem 4.3. *In a finite Boolean poset, x is irreducible if and only if x is a coatom.*

5. Relations between prime and irreducible elements

In a semilattice, an element x is prime iff for all $a, b \in P$, $a \bigwedge b \leqslant x$ implies $a \leqslant x$ or $b \leqslant x$ (see [9]). We extend this result to the posets.

Proposition 5.1. *Let P be a poset. Then x is prime iff for all $a, b \in P$, $x \in (a,b)^{lu}$ implies $a \leqslant x$ or $b \leqslant x$.*

Proof. Suppose x is prime, $a, b \in P$ and $x \in (a,b)^{lu}$. If $a \nleqslant x$ and $b \nleqslant x$, then $a, b \in P\backslash{\downarrow}x$. Since $P\backslash{\downarrow}x$ is a filter, there is an element $c \in P\backslash{\downarrow}x$ such that $c \in (a,b)^l$ and $c \nleqslant x$. It follows that $x \notin (a,b)^{lu}$, which is a contradiction to the hypothesis. Hence, $a \leqslant x$ or $b \leqslant x$.

For the converse, suppose $x \neq 1$, we need to prove that $P \backslash {\downarrow} x$ is a filter. Let $a, b \in P \backslash {\downarrow} x$. Then by the hypothesis, $x \notin (a, b)^{lu}$. Thus, there exists an element $z \in (a, b)^l$ such that $z \nleqslant x$. That is, there is an element z with $z \leqslant a$ and $z \leqslant b$ such that $z \in P \backslash {\downarrow} x$. Therefore, $P \backslash {\downarrow} x$ is a filter. Then x is a prime element. □

In distributive semilattices, prime elements are exactly irreducible, but it may not be true in posets. See the distributive poset depicted in Figure 3. $P \backslash {\downarrow} a = \{b, c, d, 1\}$ is a filter, so a is a prime element. But ${\uparrow} a \backslash \{a\} = \{c, d, 1\}$ is not a filter, which means that a is not irreducible.

Theorem 5.2. *In a distributive poset P, each completely irreducible element is prime.*

Proof. Suppose x is completely irreducible. By Proposition 5.1, to prove x is prime, it is sufficient to show that for all $a, b \in P$, $x \in (a, b)^{lu}$ implies $x \in a^u$ or $x \in b^u$. If $x \notin a^u$ and $x \notin b^u$, then for all $c \in (a, x)^u$, $c > x$ and for all $d \in (b, x)^u$, $d > x$. Hence, $\{(a, x)^u, (b, x)^u\} \subseteq {\uparrow} x \backslash \{x\}$. Since x is completely irreducible, then ${\uparrow} x \backslash \{x\}$ has a least element x^+. Clearly, $x^+ \in \{(a, x)^u, (b, x)^u\}^l$. By the distributivity and $x \in (a, b)^{lu}$, we have $x^u = \{(a, b)^l, x\}^u = \{(a, x)^u, (b, x)^u\}^{lu}$. Since $x^l = x^{ul} = \{(a, b)^l, x\}^{ul} = \{(a, x)^u, (b, x)^u\}^{lul} = \{(a, x)^u, (b, x)^u\}^l$, then $x^+ \in x^l$, which is a contradiction to $x^+ \in {\uparrow} x \backslash \{x\}$. Consequently, $x \in a^u$ or $x \in b^u$. The proof is complete. □

Theorem 5.3. *In a distributive poset with no infinite chains, each irreducible element is prime.*

Corollary 5.3.1. *In a finite distributive poset, each irreducible element is prime.*

From Theorems 3.1, 4.1, 4.2, 4.3 and Corollary 3.4.1, we can deduce the following two theorems.

Theorem 5.4. *In a finite Boolean poset P,* $\mathrm{COA}(P) = M(P) = \mathrm{IRR}(P) = \mathrm{PRI}(P)$.

Theorem 5.5. *In a Boolean poset P with no infinite chains,* $\mathrm{COA}(P) = M(P) = \mathrm{PRI}(P)$.

6. Other properties and applications

It was proved in [5] that $\mathrm{COA}(P)$ was meet dense in finite Boolean lattices. Motivated by the result in [21], we obtain the corresponding result in finite Boolean posets.

Definition 6.1. ([21]) A poset P is said to be *atomistic* if for all $a, b \in P$ with $a \nleqslant b$, there exists an atom $p \in P$ such that $p \leqslant a$ and $(p, b)^l = \{0\}$.

Lemma 6.1. ([21]) *Let P be a finite poset. If $M(P) = \mathrm{COA}(P)$, then P is atomistic.*

Note that a poset P is atomistic iff for every $a \in P$, $a = \bigvee A_L(a)$, where $A_L(a) = \{x \in A(P) \mid x \leqslant a\}$. Then combining Lemma 6.1 with the duality of Theorem 4.2, we obtain the following theorem.

Theorem 6.1. *Let P be a finite Boolean poset. Then for all $a \in P$, $a = \bigvee A_L(a)$.*

Theorem 5.3 and Theorem 6.1 indicate that coatoms, irreducible elements and prime elements are affluent in finite Boolean posets. Properties of Boolean posets are discussed in detail in [3, 12, 18]. From these papers, we can see that finite Boolean posets can be classified according to the number of atoms, and prime elements are used for describing Prime Ideal Theorem. The following property can be viewed as a particular situation of [12: Theorem 2], but it can be shown in a direct way by Theorem 6.1, which is different from the method used in [18].

Proposition 6.2. *In a finite Boolean poset P, $a^u = (A_L(a))^u$ for every $a \in P$.*

P r o o f. For every $a \in P$, by Theorem 6.1, $a = \bigvee A_L(a)$, then $a^u = \left(\bigvee A_L(a)\right)^u = (A_L(a))^u$. $\square$

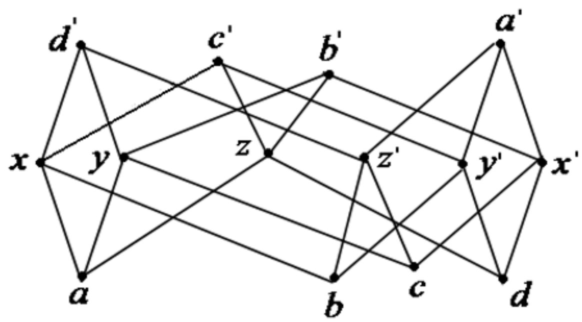

Figure 4.

Example 1. Let a, b, c, d denote the red ball, yellow ball, green ball, and blue ball, respectively. Choosing two balls from the four balls yields six schemes denoted by x, y, z, x', y', z'. And there are four schemes denoted by a', b', c', d' for getting three balls from the four balls. This incident can be modeled as a poset as displayed in Figure 4. Obviously, it is a Boolean poset. Generally speaking, if we want to find all the irreducible elements and prime elements from an arbitrary Boolean poset, it seems less visible. Fortunately, by Theorem 5.3 we have proved, the two kinds of elements all coincide with coatoms. There are four coatoms a', b', c', d' in this Boolean poset, which are exactly all the irreducible elements and prime elements. Moreover, according to Theorem 6.1, every element of this poset can be represented by some atoms. In fact, $x = a \vee b$, $y = a \vee c$, $z = a \vee d$, $z' = b \vee c$, $y' = b \vee d$, $x' = d \vee c$, $d' = a \vee b \vee c$, $c' = a \vee b \vee d$, $b' = a \vee c \vee d$, and $a' = b \vee c \vee d$. As a result, this poset can be regarded as a poset generated (in the sense of $\vee$) by a, b, c, d. Dually, the set $\{a', b', c', d'\}$ is order-generating. Further, denote this poset by P, then $P \cong A_L(P)$. Here, $A_L(P) = \big\{A_L(x) \mid x \in P\big\}$.

Definition 6.2. ([9]) A subset X of a poset P is said to be *order generating* if $x = \inf(\uparrow x \bigcap X)$ for all $x \in P$.

Evidently, Theorems 5.3 and 6.1 imply the following two results.

Corollary 6.2.1. *In a finite Boolean poset P, the set* $\mathrm{IRR}(P)\backslash\{1\}$ *is order generating.*

Corollary 6.2.2. *In a finite Boolean poset P, the set* $\mathrm{PRI}(P)\backslash\{1\}$ *is order generating.*

Remark 2. In Ref. [10], it has been proved that in a continuous semilattice, every element can be generated by the irreducible elements, but it is pitiful that this result can not be extended to continuous posets without additional conditions. It is easy to find this from Figure 3. The element a can not be generated by the only two nonidentity irreducible elements d, c. However, the poset in Figure 3 is generated by all the prime elements.

Although the condition that all the irreducible elements generate continuous posets may not hold, we shall show that in a finite poset, all the irreducible elements except the top are contained in every order generating subset.

Proposition 6.3. *If X is an order generating subset of a finite poset P, then* $\mathrm{IRR}(P) \setminus \{1\} \subseteq X$.

Proof. Since X is order generating, then for every $p \in P$, we have $p = \inf(\uparrow p \bigcap X)$. Now assume that $p \in \mathrm{IRR}(P) \setminus \{1\}$. If p is maximal in P, then it is obvious that $p \in X$. If p is not maximal and $p \notin X$, then $\uparrow p \bigcap X \subseteq \uparrow p\backslash\{p\}$. Since P is finite and $\uparrow p\backslash\{p\}$ is a filter, then $\inf(\uparrow p \bigcap X) > p$, which contradicts $p = \inf(\uparrow p \bigcap X)$. $\square$

Proposition 6.4. *Let P be a poset. If x is completely co-prime, then x_- is prime.*

Proof. Assume x is completely co-prime. Let $a, b \in P\backslash{\downarrow} x_-$. Then $a \nleqslant x_-$ and $b \nleqslant x_-$. Since x_- is the greatest element of $P\backslash{\uparrow} x$, then $a \geqslant x$ and $b \geqslant x$. Again by $x \in P\backslash{\downarrow} x_-$, we have $P\backslash{\downarrow} x_-$ is a filter. Thus, x_- is prime. $\square$

Theorem 6.5. *Let P be a Boolean poset. Then we have the following properties:*

(1) *x is prime (irreducible, completely prime, completely irreducible) iff x' is co-prime (co-irreducible, completely co-prime, completely co-irreducible), respectively.*

(2) *x is prime (irreducible, completely prime, completely irreducible) iff x^* is co-prime (co-irreducible, completely co-prime, completely co-irreducible), respectively.*

Definition 6.3. ([9]) Let P be a poset. For all $x, y \in P$, we say that x *is way-below* y, denoted by $x \ll y$ if for each directed set $D \subseteq P$ with $\sup D$ exists, $y \leqslant \sup D$ implies $x \leqslant d$ for some $d \in D$. An element satisfying $x \ll x$ is said to be *compact*.

Definition 6.4. ([9]) A poset P is called *continuous* if for every $x \in P$, $\Downarrow x = \{u \in P \mid u \ll x\}$ is directed and $x = \sup \Downarrow x$. Moreover, if every element of P can be represented as the supremum of the compact elements less than it, then P is called *algebraic*.

Definition 6.5. ([9]) Let P be a poset. An upper set $U = \uparrow U$ is called an *open set* if for each directed set $D \subseteq P$, $\sup D \in U$ implies $D \cap U \neq \emptyset$.

Definition 6.6. ([9]) A sup semilattice P is called *join continuous* if it is filtered complete and satisfies $x \bigvee \inf D = \inf\{x \bigvee D\}$ for all $x \in P$ and all filtered sets $D \subseteq P$.

It is mentioned in [9] that each completely irreducible element is maximal in $P \backslash \uparrow k$ for some compact elements of join continuous distributive complete lattices. Now, we extend this result to the join continuous distributive sup semilattices. Other assertions about join continuous characterization can be seen in [17].

Theorem 6.6. *In a join continuous distributive sup semilattice, x is completely irreducible iff x is irreducible and maximal in $P \backslash \uparrow k$ for some compact element k.*

Proof. It is illustrated in Theorem 5.1 that in a distributive poset, an element x is completely irreducible, then x is prime. Therefore, $U = P \backslash \downarrow x$ is a scott open filter. Set $k = \inf U$. Since P is join continuous, then $x \bigvee k = x \bigvee \inf U = \inf(x \bigvee U) \geqslant \min(\uparrow x \backslash \{x\}) = x^+ > x$. It follows that $k \nleqslant x$, i.e. $k \in P \backslash \downarrow x = U$. Therefore, $\uparrow k = U$, which means $\uparrow k$ is an open filter. Then for every directed set D, if $k \leqslant \sup D$, then $\sup D \in \uparrow k$. By the definition of the open filter, there exists some $d \in D$ such that $d \in \uparrow k$. Hence, k is compact. Combining $U = P \backslash \downarrow x$ with $\uparrow k = U$, we have $P \backslash \uparrow k = \downarrow x$. Therefore, x is maximal in $P \backslash \uparrow k$.

Conversely, let x be irreducible and maximal in $P \backslash \uparrow k$ for some compact element k. If x is maximal in P, then it is completely irreducible, as $k \nleqslant x$ implies $x \neq 1$. If x is not maximal in P, then $\emptyset \neq \uparrow x \backslash \{x\} \subseteq \uparrow k$. Hence, $x^+ = \inf(\uparrow x \backslash \{x\})$ exists and $x^+ \in \uparrow x \backslash \{x\}$. Hence, x is completely irreducible. □

The auxiliary relation denoted by $\prec$ is a binary relation which was investigated in [9] to fit a more general framework than way-below relation, so way-below relation becomes the smallest approximating relation. Later, the multiplicative auxiliary relation is well defined in semilattice to characterize the relation between pseudoprime elements and prime elements. In the following, we introduce the multiplicative auxiliary relation on posets.

Definition 6.7. An auxiliary relation on a poset P is called *multiplicative* if for all $x \in P$, the set $\{y \in P \mid x \prec y\}$ is a filter.

Proposition 6.7. *Let P be a poset. The following conditions are all equivalent for any auxiliary relation $\prec$.*

(1) *The auxiliary relation $\prec$ on P is multiplicative.*

(2) *For all $a, x, y \in P$, the relations $a \prec x$ and $a \prec y$ imply there exists an element $v \in (x, y)^l$ such that $a \prec v$.*

(3) *For all $a, b, x, y \in P$, the relations $a \prec x$ and $b \prec y$ imply there exists an element $v \in (x, y)^l$ such that $u \prec v$ for all $u \in (a, b)^l$.*

Proof. According to the definition of the filter, (1) $\iff$ (2) and (2) $\iff$ (3) are straightforward. $\square$

Theorem 6.8. *Let P be a Boolean poset. Then a proper ideal is prime if and only if it is maximal.*

Proof. Let I be a nonempty proper prime ideal of P. Then I is maximal. Otherwise, if J is an ideal such that $I \subset J \subset P$, then there exists an element $a \in J$, but $a \notin I$. Since J and I are proper ideals, then $a' \in P \backslash J \subset P \backslash I$ and $a \in P \backslash I$. This, together with the fact that $P \backslash I$ is a filter yields $P = I$, which is a contradiction.

Conversely, let I be a maximal ideal. Then I must be a prime ideal. Otherwise, suppose I is not a prime ideal, then $P \backslash I$ is not a filter. Then there exists two elements $a, b \in P \backslash I$ such that $c \notin (a, b)^l$ for all $c \in P \backslash I$. The above statement implies that there are two elements $a', b' \in I$ such that the assertion that there exists an element of I more than both a' and b' fails. A contradiction to I is an ideal. Therefore, I is a prime ideal. $\square$

Remark 3. In [7, 18], the properties of prime ideals and maximal ideals are discussed in detail, but the ideals used in the above-mentioned papers are Frink ideals, which are different from the ideals in this paper.

Corollary 6.8.1. *Let P be a Boolean poset and $a \in P$ is prime. Then a^l is a maximal ideal.*

Proposition 6.9. *Let P be a finite Boolean poset with n prime elements. Then there are exactly n prime ideals (maximal ideals).*

Proof. By Theorem 5.3 and Corollary 6.8.1, we get the conclusion easily. $\square$

Proposition 6.10. *Let P be a finite Boolean poset. Then for $a, b \in P$ with $b \nleqslant a$, there exists a prime ideal I of P such that $a \in I$, $b \notin I$.*

Proof. Since $b \nleqslant a$, then $(b', a)^u \neq \{1\}$. There exists a coatom c such that $c \in (b', a)^u$. Therefore, $a \in c^l$ and $b \notin c^l$. This completes the proof. $\square$

Notice that each prime element leads to a prime ideal, but the converse in general fails. Then the pseudoprime element as a more general ingredient was introduced. In a continuous lattice, pseudoprimes are exactly those elements which can be approximated by prime elements from a certain attitude [9].

Definition 6.8. An element p of a poset P is called *pseudoprime* if $p = \sup I$ for some prime ideal $I \subseteq P$. The set of pseudoprime elements are denoted by $\Psi\,\mathrm{PRI}(P)$.

By the definition of the pseudoprime element and Proposition 6.5, pseudoprime elements agree with prime elements in finite Boolean posets. This, together with Theorem 5.3 yields the following theorem.

Theorem 6.11. *Let P be a finite Boolean poset. Then $\mathrm{COA}(P) = M(P) = \mathrm{IRR}(P) = \mathrm{PRI}(P) = \Psi\,\mathrm{PRI}(P)$.*

Proposition 6.12. *Let P be a continuous poset. If $\ll$ is multiplicative, then the following conditions are equivalent for an element $p \in P$:*

(1) *p is pseudoprime.*

(2) *If for all $u \in (a, b)^l$, $u \ll p$ then $a \leqslant p$ or $b \leqslant p$.*

(3) *p is prime.*

Proof. Clearly (3) implies (1).

(1) implies (2): Let p be a pseudoprime element, and suppose that for all $u \in (a, b)^l$, $u \ll p$. Let I be a prime ideal with $\sup I = p$. Since P is continuous, then $\{x \mid x \ll p\} \subseteq I$. By the hypothesis, $(a, b)^l \subseteq I$. Hence, $P \backslash I \subseteq P \backslash (a, b)^l$. Let $a, b \in P \backslash I$. Since $P \backslash I$ is a filter, then there is an element $c \in (a, b)^l \bigcap P \backslash I$, a contradiction to $P \backslash I \subseteq P \backslash (a, b)^l$. Hence, $a \in I$ or $b \in I$.

(2) implies (3): Suppose p is not prime, then $P \backslash {\downarrow} p$ is not a filter. There must exist $x, y \in P \backslash {\downarrow} p$ such that $(x, y)^l \bigcap P \backslash {\downarrow} p = \emptyset$. Then $(x, y)^l \subseteq {\downarrow} p$. By the continuity of P, we find elements $a, b \nleqslant p$ with $a \ll x$ and $b \ll y$. Since $\ll$ is multiplicative, we conclude that for all $u \in (a, b)^l$, there exists $v \in (x, y)^l$ such that $u \ll v \leqslant p$. By the hypothesis, $x \leqslant p$ and $y \leqslant p$. A contradiction to $x, y \in P \backslash {\downarrow} p$. $\square$

The properties of arithmetic semilattice were investigated in [9]. Subsequently, the concept of semiarithmetic lattice was introduced in [14]. In the following, we introduce the concept of arithmetic poset and extend some relative results in semilattices appeared in [9] to posets.

Definition 6.9. A poset P is called an *arithmetic poset* if it is algebraic and for all compact elements x, y, there is a compact element $u \in (x, y)^l$.

Notice that if the arithmetic poset is also a semilattice, the above definition agrees with the definition of arithmetic semilattice in [9].

Proposition 6.13. *Let P be a poset. Then P is arithmetic iff the way-below relation $\ll$ is multiplicative.*

Proof. Let $a \ll x$ and $a \ll y$. Then there are compact elements c, k with $a \leqslant c \leqslant x$ and $a \leqslant k \leqslant y$. Thus, there is an element $u \in (c, k)^l$ such that $a \ll u$, and there is an element $v \in (x, y)^l$ such that $z \ll v$ for all $z \in (c, k)^l$. Hence, $u \ll v$. By the transitivity, $a \ll v$, then $\ll$ is multiplicative.

Conversely, let $x \ll x$ and $y \ll y$. Then for all $u \in (x, y)^l$, there is an element $v \in (x, y)^l$ such that $u \ll v$. Since $v \in (x, y)^l$, then $v \ll v$. So P is arithmetic. $\square$

By Propositions 6.4 and 6.5, we immediately arrive at the following conclusion.

Corollary 6.13.1. *An element of an arithmetic poset is pseudoprime if and only if it is prime.*

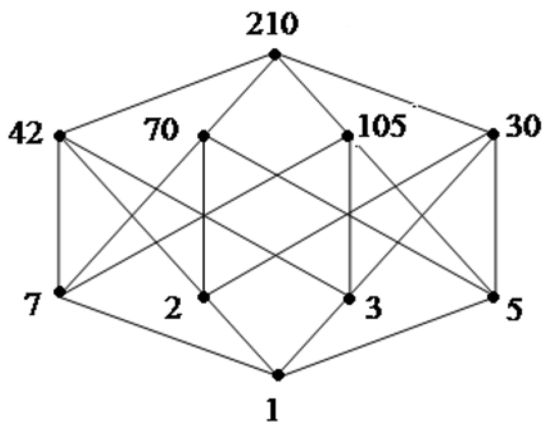

Figure 5.

Example 2. Let $a \vee b$, $a \wedge b$ denote the least common multiple and the greatest common divisor of a and b, respectively. Then the poset displayed in Figure 5 is a Boolean poset. By the definition of the compact element, every element in

Figure 5 is compact. Moreover, $42 = 7 \vee 2 \vee 3$, $70 = 7 \vee 2 \vee 5$, $105 = 7 \vee 3 \vee 5$, $30 = 5 \vee 2 \vee 3$, and $210 = 7 \vee 2 \vee 3 \vee 5$. Therefore, it is an algebraic, which is generated by $2, 3, 5, 7$. Since the least element 1 is compact, then the second condition of Definition 6.9 holds naturally. Thus, the poset displayed in Figure 5 is an arithmetic poset, but not an arithmetic semilattice. Further, there are four prime elements, that is, $42, 70, 105, 30$, which yield exactly four prime ideals (maximal ideals) as follows: $42^l = \{42, 2, 3, 7, 1\}$, $70^l = \{70, 2, 5, 7, 1\}$, $105^l = \{105, 5, 3, 7, 1\}$, and $30^l = \{30, 2, 3, 5, 1\}$.

7. Conclusion

In this paper, we show prime elements, pseudoprime elements, irreducible elements and coatoms are the same in finite Boolean posets, which is a generalization of the results in Boolean lattices. Furthermore, a general representation to finite Boolean posets is deduced. Some other properties of these elements in posets are investigated. With our introduction of multiplicative relation on posets, pseudoprime elements coincide with prime elements in arithmetic posets.

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