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SELECTIONS AND COUNTABLE COMPACTNESS

David Buhagiar — Valentin Gutev

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ABSTRACT. The present paper deals with continuous extreme-like selections for the Vietoris hyperspace of countably compact spaces. Several new results and applications are established, along with some known results which are obtained under minimal hypotheses. The paper contains also a number of examples clarifying the role of countable compactness.

1. Introduction

Throughout this paper, all spaces are assumed to be Hausdorff. For a space X, let $\mathscr{F}(X)$ be the set of all nonempty closed subsets of X, and let τ_V be the *Vietoris topology* on $\mathscr{F}(X)$. We refer to $(\mathscr{F}(X), \tau_V)$ as the *Vietoris hyperspace*. Recall that τ_V is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \Big\{ S \in \mathscr{F}(X) : \ S \subset \bigcup \mathscr{V} \ \text{ and } \ S \cap V \neq \varnothing, \ \text{ whenever } \ V \in \mathscr{V} \Big\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X.

A map $f\colon \mathscr{D} \to X$ is a *selection* for $\mathscr{D} \subset \mathscr{F}(X)$ if $f(S) \in S$ for every $S \in \mathscr{D}$. A selection $f\colon \mathscr{D} \to X$ is *continuous*, called also *Vietoris continuous*, if it is continuous with respect to the relative Vietoris topology τ_V on $\mathscr{D}$, and $\mathscr{sel}[\mathscr{D}]$ is used to denote the set of all continuous selections for $\mathscr{D}$. Given $p \in X$, a selection $f\colon \mathscr{F}(X) \to X$ is *p-maximal* [6, 13] if $f(S) = p$ whenever $p \in S \in \mathscr{F}(X)$; and f is called *p-minimal* [6] if $f(S) \neq p$ for every $S \in \mathscr{F}(X)$ with $S \neq \{p\}$. A point $p \in X$ is *selection maximal* (*selection minimal*) if $\mathscr{F}(X)$ has a continuous p-maximal (respectively, p-minimal) selection.

A point p of a connected space X is a *cut point* of X if $X \setminus \{p\}$ is not connected. In this case, $X \setminus \{p\} = U \cup V$ for some nonempty disjoint open sets $U, V \subset X$.

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Since X is connected, we also have that $\overline{U} \cap \overline{V} = \{p\}$ which can be used as an extra condition to define such points in spaces which are not necessarily connected. Namely, for an arbitrary space X, we shall say that $p \in X$ is a *cut point* of X (see, [15]) if $X \setminus \{p\} = U \cup V$ where U and V are disjoint subsets of X such that $\overline{U} \cap \overline{V} = \{p\}$. Note that such sets U and V must be open because $U = X \setminus \overline{V}$ and $V = X \setminus \overline{U}$. A concept similar to this played an important role for first countability of spaces X with $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$, see [8, 14].

Points which are both selection maximal and cut are a useful tool in classifying local properties of spaces, the following theorem illustrates some of the results obtained in [15]. In this theorem, and in what follows, a space X is *zero-dimensional* (at $p \in X$) if it has a base of clopen sets (at the point p); and X is *selection pointwise-maximal* if each point of X is selection maximal [15].

Theorem 1.1. ([15]) *For a space X with $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$, the following holds:*

(a) *If $p \in X$ is a non-isolated point, then X is zero-dimensional and first countable at p if and only if p is a selection maximal cut point.*

(b) *If X is a selection pointwise-maximal space, then it is zero-dimensional and the set $\{p \in X : X \text{ is first countable at } p\}$ is dense in X.*

Selection minimal points behave differently with respect to local properties of spaces, see Example 2.1. In contrast to Theorem 1.1, they were used for extension of continuous selections to one-point compactifications of locally compact spaces ([14: Theorem 3.1]), also for constructing new continuous selections from given ones ([6: Lemma 6.4]).

Selection minimal and maximal points have also some common properties. Turning to this, let $\mathscr{F}_2(X) = \{S \subset X : 1 \leq |S| \leq 2\}$. A selection $f \colon \mathscr{F}_2(X) \to X$ is called a *weak selection* for X. Every weak selection f for X generates an order-like relation $\preceq_f$ on X [21: Definition 7.1] defined by $x \preceq_f y$ if $f(\{x, y\}) = x$. The relation $\preceq_f$ is both total and antisymmetric, but not necessarily transitive. Weak selections exist in pairs, namely to every selection $f \colon \mathscr{F}_2(X) \to X$ one can associate another one $f^c \colon \mathscr{F}_2(X) \to X$ defined by $S = \{f(S), f^c(S)\}$, $S \in \mathscr{F}_2(X)$. Then, f is continuous if and only if so is f^c (see, for instance, [12: Theorem 3.5]), while the $\preceq_{f^c}$-relation is reverse to the $\preceq_f$-one. Motivated by this, we shall say that a point $p \in X$ is *weakly extreme* if X has a continuous weak selection f such that $p \preceq_f x$ for every $x \in X$ (or, equivalently, $x \preceq_f p$ for every $x \in X$). Clearly, every selection maximal or selection minimal point is weakly extreme.

We are also ready to state the main purpose of this paper. In the next section we demonstrate that, in the setting of arbitrary spaces, there are many examples of mutually excluding local properties of spaces which possess the same extreme-like selections, see, for instance, Examples 2.1, 2.4, 2.5 and 2.7. However, the situation is quite favourable in the realm of countably compact spaces. In Section 3 we introduce the greatest lower bound property for order-like relations generated by weak selections, and show that it holds for countable subsets of countably compact spaces, see Theorem 3.4. As a consequence, we generalise a result of Eric van Douwen [24] that every countably compact space with a continuous weak selection is sequentially compact, see Corollary 3.7. In Section 4 we deal with weakly extreme cut points of countably compact spaces, see Theorem 4.1. The last Sections 5 and 6 are devoted to some further results and applications about special points in countably compact spaces defined by means of extreme-like selections, see Lemma 5.1 and Theorem 6.1, also Corollaries 5.4 and 6.5.

2. Totally disconnected spaces and weakly extreme points

Every continuous weak selection f for X can be considered as a continuous map $f\colon X \times X \to X$ such that $f(x,y) = f(y,x)$ and $f(x,y) \in \{x,y\}$ for every $x, y \in X$, see [1]. Another way is to look at f as the relation $\preceq_f$ on X which is both total and antisymmetric, i.e. a *selection relation* in the terminology of [16]. According to [11: Proposition 2.1], the continuity of f is equivalent to $\preceq_f$ being closed in $X \times X$. This implies immediately that f remains continuous with respect to any other topology on X which is finer than the original one. The interested reader is referred to [18] for other alternative characterisations of continuity of weak selections.

In the sequel, we will often write $\preceq_{\mathbf{s}}$ for a selection relation on X, and $x \prec_{\mathbf{s}} y$ to express that $x \preceq_{\mathbf{s}} y$ and $x \neq y$. Whenever $x \in X$, let

$$(\leftarrow, x]_{\preceq_{\mathbf{s}}} = \{y \in X : \ y \preceq_{\mathbf{s}} x\} \quad \text{and} \quad [x, \rightarrow)_{\preceq_{\mathbf{s}}} = \{y \in X : \ x \preceq_{\mathbf{s}} y\}.$$

We will refer to these sets as $\preceq_{\mathbf{s}}$-*closed intervals*. Similarly, we consider the corresponding $\preceq_{\mathbf{s}}$-*open intervals*:

$$(\leftarrow, x)_{\preceq_{\mathbf{s}}} = \{y \in X : \ y \prec_{\mathbf{s}} x\} \quad \text{and} \quad (x, \rightarrow)_{\preceq_{\mathbf{s}}} = \{y \in X : \ x \prec_{\mathbf{s}} y\}.$$

Finally, for points $x, y \in X$, we have the following composite intervals:

$$\begin{aligned} (x,y)_{\preceq_{\mathbf{s}}} &= (x,\rightarrow)_{\preceq_{\mathbf{s}}} \cap (\leftarrow, y)_{\preceq_{\mathbf{s}}}, \\ [x,y]_{\preceq_{\mathbf{s}}} &= [x,\rightarrow)_{\preceq_{\mathbf{s}}} \cap (\leftarrow, y]_{\preceq_{\mathbf{s}}}, \\ (x,y]_{\preceq_{\mathbf{s}}} &= (x,\rightarrow)_{\preceq_{\mathbf{s}}} \cap (\leftarrow, y]_{\preceq_{\mathbf{s}}}, \\ [x,y)_{\preceq_{\mathbf{s}}} &= [x,\rightarrow)_{\preceq_{\mathbf{s}}} \cap (\leftarrow, y)_{\preceq_{\mathbf{s}}}. \end{aligned}$$

Since a selection relation $\preceq_{\mathbf{s}}$ is not necessarily transitive, both intervals $(x,y)_{\preceq_{\mathbf{s}}}$ and $(y,x)_{\preceq_{\mathbf{s}}}$ could be nonempty. Let us also remark that if $\preceq_{\mathbf{s}}$ is closed in $X \times X$ (i.e., corresponding to a continuous weak selection for X), then all $\preceq_{\mathbf{s}}$-open intervals are open in X [21] (see, also, [18: Corollary 4.2]); the converse is not necessarily true [12: Example 3.6] (see, also, [18: Example 4.3]).

In this section we show that, in the setting of arbitrary spaces, there are many examples of spaces possessing weakly extreme cut points with mutually excluding local properties in them. To this end, we first furnish the following example showing that, in contrast to selection maximal points, selection minimal points behave differently with respect to local properties (see, Theorem 1.1).

Example 2.1. There is a space X with $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$, and a selection minimal point $p \in X$ which is a cut point of X but X is not zero-dimensional at p.

P r o o f. Let $\mathfrak{C}$ be the Cantor set in the interval $[0,1]$, $p = 1 \in \mathfrak{C}$, $\leq$ be the linear ordering on $\mathfrak{C}$ as a subset of $[0,1]$, and let $D = \{d_n : n < \omega\} \subset \mathfrak{C}$ be a strictly increasing sequence convergent to p. Define another topology on $\mathfrak{C}$ in which a set $U \subset \mathfrak{C}$ is open if $p \in U$ and $U = V \setminus D$ for some open set V in $\mathfrak{C}$, or $p \notin U$ and U is open in $\mathfrak{C}$. Call the resulting topological space as X. In fact, the topology of X is obtained from the topology of $\mathfrak{C}$ by making D to be a closed discrete subset of X. As shown in [17: Example 4.4], the space X is not regular at p, hence it also fails to be zero-dimensional at p. However, $g(S) = \min_{\leq} S$, $S \in \mathscr{F}(X)$, is a continuous selection for $\mathscr{F}(X)$ such that $g(S) = p$ iff $S = \{p\}$. To show finally that p is a cut point of X, let $\{W_n : n < \omega\}$ be a strictly decreasing clopen base at p in $\mathfrak{C}$ such that $W_0 = \mathfrak{C}$ and $d_n \in S_n = W_n \setminus W_{n+1}$, $n < \omega$. Then, $U = \bigcup\{S_{2n} : n < \omega\}$ and $V = \bigcup\{S_{2n+1} : n < \omega\}$ are disjoint open subsets of X such that $U \cup V = X \setminus \{p\}$ and $\overline{U} \cap \overline{V} = \{p\}$. □

The situation with weakly extreme points is very similar to that of selection minimal points.

Proposition 2.2. *Let X be a space with $\mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$, and let $p \in X$ be a non-isolated point such that $\{p\}$ is a countable intersection of clopen sets. Then, p is weakly extreme. Moreover, if X is first countable and zero-dimensional at p, then p is also a cut point of X.*

P r o o f. The first part of this statement follows by [16: Theorem 3.1]. The fact that p is a cut point provided X is first countable and zero-dimensional at p was actually established in [15: Corollary 3.2]. □

Proposition 2.3. *Let X be a space with $\mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$, and let $p \in X$ be a weakly extreme cut point. Then, X is totally disconnected at p.*

P r o o f. By hypothesis, $X \setminus \{p\} = U \cup V$ for some disjoint sets $U, V \subset X$ such that $\overline{U} \cap \overline{V} = \{p\}$. Since p is weakly extreme, there exists $g \in \mathscr{sel}[\mathscr{F}_2(X)]$ such that $p \preceq_g x$ for every $x \in X$. Take a point $q \in U$. Then, $(\leftarrow, q)_{\preceq_g}$ is a neighbourhood of $p \in \overline{U} \cap \overline{V} \subset \overline{V}$ and, therefore, there exists $z \in (\leftarrow, q)_{\preceq_g} \cap V$.

It remains to observe that $G = (z, \to)_{\preceq_g} \cap U$ is clopen in X because $G = [z, \to)_{\preceq_g} \cap \overline{U}$ (indeed, $z \notin \overline{U}$), and it contains q because $z \prec_g q$ and $q \in U$. However, $p \notin G$ because $p \prec_g z$. The proof is completed. $\square$

We now proceed with several examples showing that, in general, Propositions 2.2 and 2.3 cannot be improved in the setting of arbitrary spaces.

Example 2.4. There is a space X which has a continuous weak selection and each point $p \in X$ is a weakly extreme cut point with $\{p\}$ being also a countable intersection of clopen sets, but X is not first countable at any of its points.

P r o o f. Let $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ be the irrational numbers with the usual topology. Then, the family of all sets of the form $U \setminus S$, where U is open in $\mathbb{P}$ and S is countable, is a base for a non-regular topology on $\mathbb{P}$. The resulting space X is as required. Since $\mathbb{P}$ is uncountable, X is not first countable at any of its points. Since the topology of X is finer than that of $\mathbb{P}$, X has a continuous weak selection (because so does $\mathbb{P}$) and each point of X is a countable intersection of clopen sets. By Proposition 2.2, each point of X is weakly extreme. According to the definition of the topology, each point of X is also a cut point. $\square$

Example 2.5. There is a Lindelöf space X (hence, paracompact as well) which has a continuous weak selection and a weakly extreme point $p \in X$ such that X is zero-dimensional at p and $\{p\}$ is a countable intersection of clopen sets, but p is not a cut point of X.

P r o o f. Let $\mathscr{A}$ be a free ultrafilter on ω. The important properties for us are that each $A \in \mathscr{A}$ is infinite and $S \subset \omega$ implies that $S \in \mathscr{A}$ or $\omega \setminus S \in \mathscr{A}$. Whenever $n < \omega$, let $\Delta_n = \left(2^{-(n+1)}, 2^{-n}\right)$, and let

$$Y = \{0\} \cup \Big(\bigcup\{\Delta_n : \ n < \omega\}\Big) = [0,1] \setminus \{2^{-n} : \ n < \omega\}\,.$$

Then, Y is zero-dimensional at $p = 0$ (as a subspace of $\mathbb{R}$). Let X be the space obtained from Y by promoting all sets of the form

$$O_A = \{0\} \cup \Big(\bigcup\{\Delta_n : \ n \in A\}\Big), \qquad A \in \mathscr{A},$$

to be open. Since the topology of X is finer than that of Y, $\{p\}$ remains a countable intersection of clopen subsets of X, and X itself has a continuous weak selection. Thus, by Proposition 2.2, p is a weakly extreme point of X. If U is a neighbourhood of p in X, then $O_A \subset U$ for some $A \in \mathscr{A}$. However, the set O_A is closed in X because $X \setminus O_A = \bigcup\{\Delta_n : \ n \in \omega \setminus A\}$. Hence X is zero-dimensional at p and, in particular, a regular space. By the same reason, X is Lindelöf because $X \setminus O_A$ is a countable union of Lindelöf spaces.

We finally show that p is not a cut point of X. Contrary to this, suppose that $X \setminus \{p\} = U \cup V$ for some disjoint open sets such that $\overline{U} \cap \overline{V} = \{p\}$. Since all intervals Δ_n, $n < \omega$, are connected and U and V are clopen in $X \setminus \{p\}$, we get that $\Delta_n \subset U$ or $\Delta_n \subset V$ for every $n < \omega$. Hence, $S = \{n < \omega : \Delta_n \subset U\}$ and $T = \{n < \omega : \Delta_n \subset V\}$ define a partition of ω. However $\mathscr{A}$ is an ultrafilter, and we now have that $S \in \mathscr{A}$ or $T \in \mathscr{A}$, say $S \in \mathscr{A}$. Then, O_S is a clopen neighbourhood of p and $O_S \subset \overline{U}$, so $\overline{V} \subset X \setminus O_S$ because $V \subset X \setminus O_S$. That is, $p \notin \overline{V}$ which is a contradiction! Thus, p is not a cut point of X and, by Proposition 2.2, X is not first countable at p. □

In order to prepare for our last example in this section, we need the following simple observation.

Proposition 2.6. *Let X be a space which has a continuous weak selection, and let $p \in X$ be such that there exists a countable set $A \subset X \setminus \{p\}$ with $p \in \overline{A}$. Then, p is a G_δ-point of X provided it is weakly extreme.*

Proof. Suppose that X has a continuous weak selection g such that $p \preceq_g x$ for every $x \in X$. By hypothesis, $p \in \overline{A}$ for some countable $A \subset X \setminus \{p\}$. Hence, by [6: Theorem 4.1], p should be a G_δ-point of $X = [p, \rightarrow)_{\preceq_g}$. □

Following Fujii and Nogura [5], for infinite ordinal numbers ξ and η we will use $L(\xi, \eta)$ to denote the quotient space obtained from the disjoint union of the ordinal spaces $\xi + 1$ and $\eta + 1$ by identifying the points ξ and η into a single point ∞. Note that $L(\xi, \eta)$ is linearly ordered by considering $\xi + 1$ in its usual order and $\eta + 1$ — in the reverse one. With respect to this order, $L(\xi, \eta)$ is a compact orderable space, hence it has a continuous (weak) selection.

Example 2.7. There is an orderable space X which is zero-dimensional, each of its points is a cut point, but X has no weakly extreme points.

Proof. Our construction is based on [2: Example 5.4] that there exists a suborderable space which is not first countable at any point. Let $Y = L(\omega_1, \omega)$, and let Y^ω be the lexicographical product of ω-copies of Y. For convenience, for any pair of distinct elements $f, g \in Y^\omega$, set

$$f \nabla g = \min\{n < \omega : f(n) \neq g(n)\}.$$

Define X as the subset of Y^ω consisting of all $f \in Y^\omega$ such that for some $n < \omega$, $f(k) \neq \infty$ for every $k \leq n$ and $f(k) = \infty$ for every $k > n$. Whenever $f \in X$, let

$$\boldsymbol{i}(f) = \min\{n < \omega : f(n) = \infty\},$$

and note that $\boldsymbol{i}(f) > 0$. In these terms, $f(k) \neq \infty$ for $k < \boldsymbol{i}(f)$ and $f(k) = \infty$ for every $k \geq \boldsymbol{i}(f)$. Also, for $f, g \in X$, we have that $f \prec g$ if and only if $f(k) = g(k)$ for $k < f \nabla g$ and $f(f \nabla g) < g(f \nabla g)$.

Now, endow X with the open interval topology with respect to this order. Thus, we get an orderable space which is zero-dimensional. To show the latter, suppose that $f \prec g$. Then, $f(f \nabla g) < g(f \nabla g)$ which implies that $f(f \nabla g) \neq \infty$

or $g(f\triangledown g) \neq \infty$, and, therefore, $f\triangledown g \leq \boldsymbol{i}(g)$ because $f(k) = g(k) \neq \infty$ for every $k < f\triangledown g$. If $f\triangledown g < \boldsymbol{i}(g)$, take an arbitrary $\alpha < \infty = \omega_1$. Otherwise, if $f\triangledown g = \boldsymbol{i}(g)$, we have that $f(\boldsymbol{i}(g)) < g(\boldsymbol{i}(g)) = \infty$, so there exists an $\alpha < \omega_1 = \infty$ with $f(\boldsymbol{i}(g)) < \alpha < g(\boldsymbol{i}(g))$. Either way, we may define $h \in Y^\omega$ by $h(k) = g(k)$ for $k < \boldsymbol{i}(g)$ and $h(k) = \alpha$ for $k \geq \boldsymbol{i}(g)$. By the choice of α, we have $f \prec h \prec g$, while $h \in Y^\omega \setminus X$ because $h(k) \neq \infty$ for every $k < \omega$. This implies that $U = (h, \rightarrow)_\preceq \cap X$ is a clopen subset of X which contains g and does not contain f. Note that $(h, \rightarrow)_\preceq \cap X = [h, \rightarrow)_\preceq \cap X$ and hence U is clopen in X with respect to the subspace topology on X (i.e., when X is considered as a suborderable space), but we consider X as an orderable space. So, some explicit arguments are required.

In order to show that U is open in X, take $u \in (h, \rightarrow)_\preceq \cap X$. Since $h \prec u$ and $h(k) \neq \infty$ for every $k < \omega$, we have that $h\triangledown u \leq \boldsymbol{i}(u)$. If $h\triangledown u = \boldsymbol{i}(u)$, then $h(\boldsymbol{i}(u)) < u(\boldsymbol{i}(u)) = \infty$, so there is $\beta < \omega_1 = \infty$ with $h(\boldsymbol{i}(u)) < \beta < u(\boldsymbol{i}(u))$. If $h\triangledown u < \boldsymbol{i}(u)$, take an arbitrary $\beta < \omega_1$. Next, define $v \in Y^\omega$ by $v(k) = u(k)$ provided $k \neq \boldsymbol{i}(u)$ and $v(\boldsymbol{i}(u)) = \beta$. Thus, $v \in X$ and $h \prec v \prec u$, which implies that U is open in X. The verification that U is also closed in X is similar. Namely, take $\ell \in (\leftarrow, h)_\preceq \cap X$. Again, we have that $\ell\triangledown h \leq \boldsymbol{i}(\ell)$ because $h(k) \neq \infty$ for every $k < \omega$. If $\ell\triangledown h = \boldsymbol{i}(\ell)$, then $\ell(\boldsymbol{i}(\ell)) = \infty < m < h(\boldsymbol{i}(\ell))$ for some $m < \omega$. If $\ell\triangledown h < \boldsymbol{i}(\ell)$, take an arbitrary $m < \omega$. Define $w \in Y^\omega$ by $w(k) = \ell(k)$ if $k \neq \boldsymbol{i}(\ell)$ and $w(\boldsymbol{i}(\ell)) = m$. Then, $w \in X$ and $\ell \prec w \prec h$ which shows that $X \setminus U = (\leftarrow, h)_\preceq \cap X$ is open in X.

To show finally that X is as required, let $f \in X$ and, for each $\alpha \in Y = L(\omega_1, \omega)$, define $f_\alpha \in X$ by

$$f_\alpha(k) = \begin{cases} f(k) & \text{if } k \neq \boldsymbol{i}(f), \\ \alpha & \text{if } k = \boldsymbol{i}(f). \end{cases}$$

Next, let $L_f = \{f_\alpha :\ \alpha \leq \infty\}$ and $R_f = \{f_n :\ \infty \leq n\}$. In this way, we get that

$$f_\alpha \prec f_\infty = f \prec f_n, \qquad \text{for every} \quad \alpha < \infty < n.$$

First, let us observe that f is not isolated in L_f. Indeed, let $g \prec f$ for some $g \in X$. Then, as it was already shown, $g\triangledown f \leq \boldsymbol{i}(f)$. If $g\triangledown f < \boldsymbol{i}(f)$, then $g \prec f_\alpha$ for every $\alpha < \infty$. If $g\triangledown f = \boldsymbol{i}(f)$, then $g(\boldsymbol{i}(f)) < \alpha < f(\boldsymbol{i}(f)) = \infty$ for some $\alpha < \infty$, so $g \prec f_\alpha$ for this particular α. Since $\{\alpha \in Y :\ \alpha < \infty\} = \omega_1$, this also implies that f is not a G_δ-point of $(\leftarrow, f]_\preceq$. Concerning how f is situated in R_f, let $g \in X$ with $f \prec g$. Again, $f\triangledown g \leq \boldsymbol{i}(f)$ because $f(k) = g(k) \neq \infty$ for every $k < f\triangledown g$. If $f\triangledown g < \boldsymbol{i}(f)$, then $f_n \prec g$ for every $n > \infty$ because $f_n(k) = f(k)$ for $k < \boldsymbol{i}(f)$. If $f\triangledown g = \boldsymbol{i}(f)$, then $f(\boldsymbol{i}(f)) = \infty < n < g(\boldsymbol{i}(f))$ for some $n < \omega$. Consequently, $f \prec f_n \prec g$ for this particular n. That is, f is not isolated in R_f, and clearly f is a G_δ-point of R_f because R_f is countable. In particular,

$f \in \overline{A}$ for some countable $A \subset X \setminus \{f\}$. According to Proposition 2.6, f cannot be a weakly extreme point of X because it is not a G_δ-point of $(\leftarrow, f]_{\preceq}$. The proof is completed. $\square$

3. Selection relations and the greatest lower bound property

Let X be a set, $\preceq_{\mathbf{s}}$ be a selection relation on X, and let $S \subset X$. The following terminology is standard for linear orders, it is just adapted for the case of selection relations. An element $p \in X$ is a *lower* $\preceq_{\mathbf{s}}$*-bound* (respectively, an *upper* $\preceq_{\mathbf{s}}$*-bound*) for S if $p \preceq_{\mathbf{s}} x$ (respectively, $x \preceq_{\mathbf{s}} p$) for every $x \in S$. In this case, we shall say that S is $\preceq_{\mathbf{s}}$*-bounded below* (respectively, $\preceq_{\mathbf{s}}$*-bounded above*). A lower $\preceq_{\mathbf{s}}$-bound p for S is the *greatest lower* $\preceq_{\mathbf{s}}$*-bound*, or the $\preceq_{\mathbf{s}}$*-infimum* of S, written $p = \inf_{\preceq_{\mathbf{s}}} S$, if $y \preceq_{\mathbf{s}} p$ for any other lower $\preceq_{\mathbf{s}}$-bound y for S. Similarly, an upper $\preceq_{\mathbf{s}}$-bound p for S is the *least upper* $\preceq_{\mathbf{s}}$*-bound*, or the $\preceq_{\mathbf{s}}$*-supremum* of S, written $p = \sup_{\preceq_{\mathbf{s}}} S$, if $p \preceq_{\mathbf{s}} y$ for any other upper $\preceq_{\mathbf{s}}$-bound y for S. In the sequel, we will consider only lower $\preceq_{\mathbf{s}}$-bounds, and $\preceq_{\mathbf{s}}$-infimums. The other case is completely analogous by considering the reverse relation.

First of all, let us explicitly mention that the lack of transitivity of a selection relation $\preceq_{\mathbf{s}}$ may lead to examples of $\preceq_{\mathbf{s}}$-bounded below sets $S \subset X$ which have finitely many lower $\preceq_{\mathbf{s}}$-bounds but have no $\preceq_{\mathbf{s}}$-infimum.

Example 3.1. Let $X = \{0, 1, 2\} \cup (3, +\infty) \subset \mathbb{R}$. Define a selection relation $\preceq_{\mathbf{s}}$ on X by

(a) $x \prec_{\mathbf{s}} y$ for every $x \in \{0, 1, 2\}$ and $y \in (3, +\infty)$,

(b) $0 \prec_{\mathbf{s}} 1 \prec_{\mathbf{s}} 2 \prec_{\mathbf{s}} 0$,

(c) $\preceq_{\mathbf{s}}$ on $(3, +\infty)$ coincides with the usual linear order on $(3, +\infty)$.

Then, $\preceq_{\mathbf{s}}$ is a closed selection relation on X such that $S = (3, +\infty)$ is $\preceq_{\mathbf{s}}$-bounded below, but it has no $\preceq_{\mathbf{s}}$-infimum.

Here is a very simple observation dealing with the infimum of $\preceq_{\mathbf{s}}$-bounded below sets. Its verification is left to the reader.

Proposition 3.2. *Let X be a set, $\preceq_{\mathbf{s}}$ be a selection relation on X, $S \subset X$ be a nonempty set which is $\preceq_{\mathbf{s}}$-bounded below, and let $p \in X$ be a lower $\preceq_{\mathbf{s}}$-bound of S. Then, $p = \inf_{\preceq_{\mathbf{s}}} S$ if and only if $\{p\} = \bigcap \{[p, s]_{\preceq_{\mathbf{s}}} : s \in S\}$.*

Let X be a set, and $\preceq_{\mathbf{s}}$ be a selection relation on X. We shall say that a sequence $\{x_n \in X : n < \omega\}$ is *strictly* $\preceq_{\mathbf{s}}$*-decreasing* (respectively, *strictly* $\preceq_{\mathbf{s}}$*-increasing*) if $x_{n+1} \prec_{\mathbf{s}} x_k$ (respectively, $x_k \prec_{\mathbf{s}} x_{n+1}$) for every $k \leq n$ and $n < \omega$, or in other words if the sequence is linearly ordered by $\preceq_{\mathbf{s}}$ and is strictly decreasing (respectively, strictly increasing) with respect to this order. Finally, we shall say that a sequence $\{x_n : n < \omega\}$ is *strictly* $\preceq_{\mathbf{s}}$*-monotone* if it is either strictly $\preceq_{\mathbf{s}}$-decreasing or strictly $\preceq_{\mathbf{s}}$-increasing.

PROPOSITION 3.3. *Let* $\preceq_{\mathbf{s}}$ *be a selection relation on* X *and* $\{x_n \in X : n < \omega\}$ *be a sequence of distinct points of* X*. Then,* $\{x_n : n < \omega\}$ *has a strictly* $\preceq_{\mathbf{s}}$*-monotone subsequence.*

Proof. We follow an idea of Eric van Douwen [24]. Namely, we are going to apply Ramsey's Lemma [23] (see, also, [20]) that if $\{P_0, P_1\}$ is a partition of $[\omega]^2 = \big\{S \subset \omega : |S| = 2\big\}$, then there is an infinite set $H \subset \omega$ which is homogeneous for this partition, i.e. such that either $[H]^2 \subset P_0$ or $[H]^2 \subset P_1$. To this end, as in the proof of [24: Theorem 2], consider the partition $\{P_0, P_1\}$ of $[\omega]^2$ defined by

$$P_0 = \big\{\{n,m\} \in [\omega]^2 : x_{\min\{n,m\}} = \min\nolimits_{\preceq_{\mathbf{s}}}\{x_n, x_m\}\big\},$$
$$P_1 = \big\{\{n,m\} \in [\omega]^2 : x_{\min\{n,m\}} = \max\nolimits_{\preceq_{\mathbf{s}}}\{x_n, x_m\}\big\}.$$

If H is homogeneous for the partition $\{P_0, P_1\}$, then $\{x_k : k \in H\}$ is a strictly $\preceq_{\mathbf{s}}$-monotone subsequence of $\{x_n : n < \omega\}$. $\square$

In what follows, we will mostly deal with strictly $\preceq_{\mathbf{s}}$-decreasing sequences, but clearly our considerations remain valid also for strictly $\preceq_{\mathbf{s}}$-increasing sequences by considering the reverse relation.

Given a selection relation $\preceq_{\mathbf{s}}$ on X, the family

$$\mathscr{S}_{\preceq_{\mathbf{s}}} = \big\{(\leftarrow, x)_{\preceq_{\mathbf{s}}}, (x, \rightarrow)_{\preceq_{\mathbf{s}}} : x \in X\big\}$$

is a subbase for a natural "$\preceq_{\mathbf{s}}$-open" interval topology $\mathscr{T}_{\preceq_{\mathbf{s}}}$ on X, called a *selection topology* [12]. In fact, $\mathscr{T}_{\preceq_{\mathbf{s}}}$ is the usual open interval topology provided $\preceq_{\mathbf{s}}$ is a linear ordering on X. If X is a space with a topology $\mathscr{T}$ and $\preceq_{\mathbf{s}}$ is a closed selection relation, then $\mathscr{T}_{\preceq_{\mathbf{s}}} \subset \mathscr{T}$. The inclusion $\mathscr{T}_{\preceq_{\mathbf{s}}} \subset \mathscr{T}$ is actually equivalent to $(\leftarrow, x)_{\preceq_{\mathbf{s}}}, (x, \rightarrow)_{\preceq_{\mathbf{s}}} \in \mathscr{T}$, $x \in X$, [21] (see, also, [18: Corollary 4.2]). However, there are selection relations $\preceq_{\mathbf{s}}$ which are not closed but all $\preceq_{\mathbf{s}}$-open intervals are open sets [12: Example 3.6] (see, also, [18: Example 4.3]). Motivated by this, a weak selection f for a space $(X, \mathscr{T})$ is called *separately continuous* [18] if $\mathscr{T}_{\preceq_f} \subset \mathscr{T}$. In order to express this property in terms of selection relations, we will say that a selection relation $\preceq_{\mathbf{s}}$ on a space $(X, \mathscr{T})$ is *admissible* if $\mathscr{T}_{\preceq_{\mathbf{s}}} \subset \mathscr{T}$.

THEOREM 3.4. *Let* X *be a countably compact space,* $\preceq_{\mathbf{s}}$ *be an admissible selection relation on* X*, and let* $S = \{x_n : n < \omega\}$ *be a strictly* $\preceq_{\mathbf{s}}$*-decreasing sequence. Then,* $\overline{S} \setminus S$ *is a singleton and* $\overline{S} \setminus S \prec_{\mathbf{s}} x_n$ *for every* $n < \omega$*. In particular, there exists* $\inf_{\preceq_{\mathbf{s}}} S$ *and* $\inf_{\preceq_{\mathbf{s}}} S \in \overline{S} \setminus S$*.*

P r o o f. First of all, let us show that $\overline{S} \setminus S \neq \varnothing$. To this end, suppose that $\overline{S} \setminus S = \varnothing$, i.e. that S is closed in X. Next, for every $n < \omega$, let $S_n = \{x_k : k \geq n\}$ which is also closed in X because $S_n = S \setminus (x_n, \to)_{\preceq_{\mathbf{s}}}$. Thus, we get a decreasing family $\{S_n : n < \omega\}$ of nonempty closed subsets of X. Since X is countably compact, we have that $S_\omega = \bigcap\{S_n : n < \omega\} \neq \varnothing$. However, $S_\omega \subset S$ because $S_n \subset S$ for every $n < \omega$, while $x_n \notin S_\omega$ for every $n < \omega$. A contradiction! Thus, $\overline{S} \setminus S \neq \varnothing$.

Take a point $p \in \overline{S} \setminus S$. Then, $p \neq x_n$ for every $n < \omega$, and we must have that $p \prec_{\mathbf{s}} x_n$ for every $n < \omega$. Indeed, suppose that $x_n \preceq_{\mathbf{s}} p$ for some $n < \omega$. Then, $x_n \prec_{\mathbf{s}} p$ while $x_k \notin (x_n, \to)_{\preceq_{\mathbf{s}}}$ for every $k \geq n$. So, $U = (x_n, \to)_{\preceq_{\mathbf{s}}} \setminus \{x_k : k < n\}$ is a neighbourhood of p such that $U \cap S = \varnothing$, but $p \in \overline{S}$. A contradiction! Thus, $p \prec_{\mathbf{s}} x_n$ for every $n < \omega$. We are also ready to show that $\overline{S} \setminus S$ is a singleton. On the contrary, suppose that $p, q \in \overline{S} \setminus S$ are two distinct points, say $q \prec_{\mathbf{s}} p$. Then, $V = (\leftarrow, p)_{\preceq_{\mathbf{s}}}$ is a neighbourhood of q such that $x_n \notin V$, $n < \omega$, because $p \prec_{\mathbf{s}} x_n$, $n < \omega$. Consequently, $V \cap S = \varnothing$. However, we also have that $V \cap S \neq \varnothing$ because $q \in \overline{S}$. A contradiction! So, $\overline{S} \setminus S$ is a singleton.

We complete the proof by showing that there exists $\inf_{\preceq_{\mathbf{s}}} S$ and $\inf_{\preceq_{\mathbf{s}}} S \in \overline{S} \setminus S$. Let $p \in \overline{S} \setminus S$, and let $y \in X$ be a lower $\preceq_{\mathbf{s}}$-bound for S. Suppose that $p \prec_{\mathbf{s}} y$. Then, $(\leftarrow, y)_{\preceq_{\mathbf{s}}}$ is a neighbourhood of p, hence $(\leftarrow, y)_{\preceq_{\mathbf{s}}} \cap S \neq \varnothing$. However, $y \preceq_{\mathbf{s}} x_n$ for every $n < \omega$. A contradiction! Thus, $y \preceq_{\mathbf{s}} p$ and the proof is completed. □

The following observation is an immediate consequence of Proposition 3.2 and Theorem 3.4.

Corollary 3.5. *Let X be a countably compact space, $\preceq_{\mathbf{s}}$ be an admissible selection relation on X, and let $S = \{x_n : n < \omega\}$ be a strictly $\preceq_{\mathbf{s}}$-decreasing sequence. Then, $p = \inf_{\preceq_{\mathbf{s}}} S$ is a G_δ-point of $[p, \to)_{\preceq_{\mathbf{s}}}$ in the selection topology $\mathscr{T}_{\preceq_{\mathbf{s}}}$.*

In fact, in this case, every strictly $\preceq_{\mathbf{s}}$-decreasing sequence is also convergent.

Corollary 3.6. *Let X be a countably compact space, $\preceq_{\mathbf{s}}$ be an admissible selection relation on X, $S = \{x_n : n < \omega\}$ be a strictly $\preceq_{\mathbf{s}}$-decreasing sequence, $p = \inf_{\preceq_{\mathbf{s}}} S$, and let $T_n = \bigcap\big\{[p, x_k]_{\preceq_{\mathbf{s}}} : k \leq n\big\}$, $n < \omega$. Then, for every open $U \subset X$ containing p, there is an $n < \omega$ such that $T_n \subset U$. In particular, S is convergent to p.*

P r o o f. Let $U \subset X$ be an open set containing p, and contrary to our claim suppose that $F_n = T_n \setminus U \neq \varnothing$ for every $n < \omega$. Then, $\{F_n : n < \omega\}$ is a decreasing sequence of nonempty closed sets in X. Since X is countably compact, there is a point $q \in \bigcap\{F_n : n < \omega\}$. Since $p = \inf_{\preceq_{\mathbf{s}}} S$, by Proposition 3.2, $p = q \notin U$. A contradiction! □

According to Proposition 3.3, this implies the following further consequence.

Corollary 3.7. *Every countably compact space which has a separately continuous weak selection is sequentially compact.*

The fact the every countably compact space with a continuous weak selection is sequentially compact was established by Eric van Douwen, [24: Theorem 2]. In this regard, let us also mention the following result which is credited to a list of authors.

Theorem 3.8. ([1,7,22,24]) *Every pseudocompact space X which has a continuous weak selection is suborderable. In particular, for a Tychonoff space X with $\mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$, the following are equivalent*:

(a) *X is countably compact.*

(b) *X is pseudocompact.*

(c) *X is sequentially compact.*

According to Corollary 3.7, countable compactness and sequential compactness are equivalent in the realm of spaces X which have separately continuous weak selections. On the other hand, there are non-regular countably compact spaces which have continuous weak selections.

Example 3.9. Let X be the space obtained from the ordinal space $\omega_1 + 1$ by making the set of all countable limit ordinals closed in X, see [4: 3.10.B]. Since the topology of X is finer than that of $\omega_1 + 1$, X has a continuous weak selection but is not regular.

Motivated by this, we have the following natural question.

Question 1. Let X be a countably compact space which has a (separately) continuous weak selection. Then, is it true that X is weakly orderable? What about if X is regular?

Let us remark that a countably compact space with a continuous weak selection is not necessarily suborderable which was illustrated by Example 3.9. For some related results, we refer the interested reader to the last Section 5 of the paper.

4. Countable compactness and weakly extreme cut points

In this section, we prove the following theorem which is a partial generalisation of Theorem 1.1.

Theorem 4.1. *Let X be a countably compact space with $\mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$, and let $p \in X$ be a non-isolated point. Then, X is zero-dimensional and first countable at p if and only if p is a weakly extreme cut point.*

P r o o f. If X is first countable and zero-dimensional at p, then the statement follows by Proposition 2.2. To show the converse, suppose that $X \setminus \{p\} = U \cup V$ for some disjoint (open) sets $U, V \subset X$ with $\overline{U} \cap \overline{V} = \{p\}$, and let $g \in \mathscr{sel}[\mathscr{F}_2(X)]$ be such that $p \preceq_g x$ for every $x \in X$. Take a point $x_0 \in U$. Then, $(\leftarrow, x_0)_{\preceq_g}$ is a neighbourhood of p and there exists $x_1 \in (\leftarrow, x_0)_{\preceq_g} \cap V$. In an obvious manner, proceeding by induction, we get a strictly $\preceq_g$-decreasing sequence $S = \{x_n : n < \omega\}$ such that $x_{2n} \in U$ and $x_{2n+1} \in V$ for every $n < \omega$. According to Corollary 3.6, S is convergent to a point $q \in X$. Then, $q \in \overline{U}$ because $\{x_{2n} : n < \omega\} \subset U$, and $q \in \overline{V}$ because $\{x_{2n+1} : n < \omega\} \subset V$. Hence, $q = p$. By Corollary 3.5, p is a G_δ-point of $X = [p, \rightarrow)_{\preceq_g}$ with respect to the selection topology $\mathscr{T}_{\preceq_g}$. On the other hand, by Proposition 2.3, X is totally disconnected at p. Hence, by [18: Proposition 5.6], $\{p\}$ is a countable intersection of clopen subsets of X. Since X is countably compact, it finally implies that X is first countable and zero-dimensional at p. The proof is completed. $\square$

Related to Theorem 4.1, let us explicitly mention that countable compactness is substantial to derive that X is first countable at p, see Example 5.3 in the next section. We proceed with some applications of Theorem 4.1 the first of which is based on [5: Theorem 3].

Corollary 4.2. *For infinite cardinals ξ and η, the following are equivalent:*

(a) *∞ is weakly extreme in $L(\xi, \eta)$.*

(b) *Both ξ and η have countable cofinality.*

(c) *The space $L(\xi, \eta)$ is homeomorphic to an ordinal space the last element of which is ∞.*

In particular, ∞ is a cut point of $L(\omega_1, \omega_1)$ but it is not weakly extreme.

P r o o f. Since ξ and η are limit ordinals, ∞ is always a cut point of $L(\xi, \eta)$. Then, (a) $\Longrightarrow$ (b) is an immediate consequence of Theorem 4.1 because, in this case, $L(\xi, \eta)$ will be first countable at ∞. The implication (b) $\Longrightarrow$ (c) follows by [5: Theorem 3], while (c) $\Longrightarrow$ (a) is obvious. $\square$

Another immediate consequence of Theorem 4.1 is the following characterisation of first countable zero-dimensional countably compact spaces.

Corollary 4.3. *Let X be a countably compact space with $\mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$. Then, X is zero-dimensional and first countable if and only if each non-isolated point of X is a weakly extreme cut point.*

Let us remark that each isolated point of X is always weakly extreme. Consequently, one of the conditions in Corollary 4.3 is that each point of X is weakly extreme. This condition implies total disconnectedness of X without any extra hypothesis. To prepare for this, we will use $\mathscr{C}[x]$ to denote the *connected* *component* of the point $x \in X$. Recall that

$$\mathscr{C}[x] = \bigcup\{C \subset X : \ x \in C \ \text{ and } \ C \ \text{ is connected}\}.$$

Proposition 4.4. *Let X be a space with $\mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$. If the set of all weakly extreme points of X is dense in X, then X is totally disconnected.*

Proof. Let $p, q \in X$ be distinct points, say $p \prec_g q$ for some $g \in \mathscr{sel}[\mathscr{F}_2(X)]$. According to [13: Theorem 4.1] (see, also, [18: Theorem 6.1]) it suffices to show that the connected component $\mathscr{C}[p]$ of p in X is not equal to the connected component $\mathscr{C}[q]$ of q. To the contrary, assume that $\mathscr{C}[p] = \mathscr{C}[q]$. Then, by [10: Lemma 2.5], $\varnothing \neq (p, q)_{\preceq_g} \subset \mathscr{C}[p]$. Hence, there exists a weakly extreme point $z \in (p, q)_{\preceq_g}$. That is, there exists $f \in \mathscr{sel}[\mathscr{F}_2(X)]$ such that $z \preceq_f x$ for every $x \in X$. Since $\mathscr{C}[p]$ is connected, by a result of Eilenberg [3], $\mathscr{C}[q]$ has exactly two continuous weak selections, namely $f \restriction \mathscr{F}_2(\mathscr{C}[q])$ and the one generated by the selection relation reverse to $\preceq_f$. Hence, either $z \prec_g x$ for every $x \in \{p, q\}$, or $x \prec_g z$ for every $x \in \{p, q\}$. However, $p \prec_g z \prec_g q$. A contradiction! □

5. Countable compactness and first countability

Let us recall that, for a non-isolated point p of X, $\mathsf{sa}(p, X)$ denotes the least cardinal γ such that there exists $A \subset X \setminus \{p\}$ with $|A| \leq \gamma$ and $p \in \overline{A}$, see [6,15]. Whenever p is isolated in X, set $\mathsf{sa}(p, X) = 0$. The cardinal number $\mathsf{sa}(p, X)$ has the meaning of an *approaching number* of X in p, and might be compared with the *tightness* $\mathsf{t}(p, X)$ of X in p, see [6,15].

Let $p \in X$ and $g \in \mathscr{sel}[\mathscr{F}_2(X)]$. Following [6], we consider the *left approach* to p with respect to g defined by $\lambda_g(p, X) = \mathsf{sa}\big(p, (\leftarrow, p]_{\preceq_g}\big)$, and, respectively, the *right approach* $\rho_g(p, X) = \mathsf{sa}\big(p, [p, \rightarrow)_{\preceq_g}\big)$. Finally, let

$$\mu_g(p, X) = \max\big\{\lambda_g(p, X), \rho_g(p, X)\big\}.$$

According to [6: Theorem 4.1], p is a G_δ-point of X provided $\mu_g(p, X) \leq \omega$ for some $g \in \mathscr{sel}[\mathscr{F}_2(X)]$. Consequently, in this case, if X is a regular countably compact space, then it must be first countable at p. In this section, we first provide the following refinement of [6: Theorem 4.1] for the case of countably compact spaces.

Lemma 5.1. *For a countably compact space X with $\mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$, and $p \in X$, the following are equivalent*:

(a) *X is first countable at p.*

(b) *$\mu_g(p, X) \leq \omega$ for every $g \in \mathscr{sel}[\mathscr{F}_2(X)]$.*

(c) *$\mu_g(p, X) \leq \omega$ for some $g \in \mathscr{sel}[\mathscr{F}_2(X)]$.*

To prepare for the proof of Lemma 5.1, we need the following proposition.

Proposition 5.2. *Let X be a countably compact space, $\preceq_{\mathbf{s}}$ be an admissible selection relation on X, and let $p \in X$ be such that $p \in \overline{A}$ for some countable set $A \subset (p, \to)_{\preceq_{\mathbf{s}}}$. Then, $[p, \to)_{\preceq_{\mathbf{s}}}$ has a countable local base at p of $\mathscr{T}_{\preceq_{\mathbf{s}}}$-open sets.*

Proof. We proceed in a way very similar to Corollary 3.6. Namely, let $\mathscr{A}$ be the set of all nonempty finite subsets of A. For every $\alpha \in \mathscr{A}$, let

$$U_\alpha = \bigcap \bigl\{ [p, x)_{\preceq_{\mathbf{s}}} :\ x \in \alpha \bigr\} \quad \text{and} \quad F_\alpha = \bigcap \bigl\{ [p, x]_{\preceq_{\mathbf{s}}} :\ x \in \alpha \bigr\}.$$

Then, $p \in U_\alpha \subset \overline{U_\alpha} \subset F_\alpha$, $\alpha \in \mathscr{A}$, and $\bigcap \bigl\{ F_\alpha :\ \alpha \in \mathscr{A} \bigr\} = \{p\}$. Since X is countably compact and A is countable, exactly in the same way as in Corollary 3.6, every open subset of X containing p will contain also an F_α for some $\alpha \in \mathscr{A}$. Since $\preceq_{\mathbf{s}}$ is an admissible selection relation on X, $\bigl\{ U_\alpha :\ \alpha \in \mathscr{A} \bigr\}$ will be a local base at p in $[p, \to)_{\preceq_{\mathbf{s}}}$. ☐

Proof of Lemma 5.1. The implications (a) $\implies$ (b) $\implies$ (c) are obvious. To show that (c) $\implies$ (a), suppose that $\mu_g(p, X) \le \omega$ for some $g \in \mathscr{sel}[\mathscr{F}_2(X)]$. If p is not isolated in $[p, \to)_{\preceq_g}$, then, by Proposition 5.2, $[p, \to)_{\preceq_g}$ will be first countable at p because $\mathsf{sa}\bigl(p, [p, \to)_{\preceq_g}\bigr) \le \omega$. Using the reverse relation of $\preceq_g$, the same is true for $(\leftarrow, p]_{\preceq_g}$. Hence, X is first countable at p. ☐

Lemma 5.1 is not true for arbitrary spaces and, in general, $\mu_g(p, X)$ is not the same for all $g \in \mathscr{sel}[\mathscr{F}_2(X)]$. Here is an example.

Example 5.3. There exists a space X with $\mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$, and a point $p \in X$ such that p is a weakly extreme cut point of X, but there are weak selections $f, g \in \mathscr{sel}[\mathscr{F}_2(X)]$ with $\mu_f(p, X) \le \omega < \mu_g(p, X)$.

Proof. Let X be the space obtained from the ordinal space $\omega_\omega + 1$ be promoting all ordinals ω_n, $n < \omega$, to be isolated, and by changing the local base at $p = \omega_\omega \in X$ to be of all sets $U \subset X$ for which $p \in U$ and there exists $m < \omega$ and ordinals $\alpha_n < \omega_n < \alpha_{n+1}$, $n \ge m$, such that $(\alpha_n, \omega_n] \subset U$ for every $n \ge m$. The resulting topology on $X = \omega_\omega + 1$ is finer than the original one, so X has a continuous weak selection f such that $x \preceq_f p$ for every $x \in X$. In fact, f is defined by $f(S) = \min S$, $S \in \mathscr{F}_2(X)$, where the minimum is taken with respect to the usual linear order $\le$ on $X = \omega_\omega + 1$. Then $X = (\leftarrow, p]_{\preceq_f}$ and

$$A = \{\omega_n :\ n < \omega\} \subset (\leftarrow, p)_{\preceq_f} = X \setminus \{p\}$$

is such that $p \in \overline{A}$. Hence, $\mu_f(p, X) \le \omega$. On the other hand, considering A in the reverse order and making each element of A bigger than any element of $X \setminus A$, we get another linear order $\preceq$ on X. Then, we may define another continuous weak selection g for X by $g(S) = \min_{\preceq} S$, $S \in \mathscr{F}_2(X)$. Now $\preceq_g = \preceq$, and we have that $X \setminus A = (\leftarrow, p]_{\preceq}$ and $\overline{A} = [p, \to)_{\preceq}$. This demonstrates that p

is a cut point of X because $U = (\leftarrow, p)_{\preceq}$ and $V = (p, \rightarrow)_{\preceq} = A$ are disjoint open subsets of X such that $X \setminus \{p\} = U \cup V$ and $\overline{U} \cap \overline{V} = \{p\}$. Finally, let us show that $\mu_g(p, X) > \omega$. So, let $B \subset (\leftarrow, p)_{\preceq}$ be a countable set. Since each ordinal ω_{n+1}, $n < \omega$, is uncountable and regular, for every $n < \omega$ there exists an α_{n+1}, with $\omega_n < \alpha_{n+1} < \omega_{n+1}$, such that $\beta < \alpha_{n+1}$ for every $\beta \in B \cap \omega_{n+1}$. Then, $W = \{p\} \cup \bigcup \big\{(\alpha_{n+1}, \omega_{n+1}]_{\preceq} :\ n < \omega\big\}$ is a neighbourhood of p in X such that $W \cap B = \varnothing$. Consequently, $\mathsf{sa}\big(p, (\leftarrow, p]_{\preceq}\big) > \omega$. The proof is completed. $\square$

In conclusion, let us also mention the following consequence of Theorem 3.8 and Proposition 5.2. We let,

$$\mu_g(X) = \min\big\{\gamma :\ \mu_g(p, X) \leq \gamma \text{ for every } p \in X\big\}.$$

Corollary 5.4. *Let X be a countably compact space such that $\mu_g(X) \leq \omega$ for some $g \in \mathscr{sel}[\mathscr{F}_2(X)]$. Then, X is a Tychonoff space and, in particular, is suborderable.*

Proof. Take a point $p \in X$. If p is non-isolated in $[p, \rightarrow)_{\preceq_g}$, by Proposition 5.2, $[p, \rightarrow)_{\preceq_g}$ has a countable base at p consisting of $\mathscr{T}_{\preceq_g}$-open sets of $[p, \rightarrow)_{\preceq_g}$. According to a recent result of [19], $(X, \mathscr{T}_{\preceq_g})$ is a Tychonoff space. Hence, $[p, \rightarrow)_{\preceq_g}$ is itself a Tychonoff space at p. Exactly in the same way, using the reverse relation of $\preceq_g$, the interval $(\leftarrow, p]_{\preceq_g}$ is also a Tychonoff space at p. Consequently, so is X. The latter part of this statement now follows by Theorem 3.8. The proof is completed. $\square$

6. Countable compactness and zero-dimensionality

Another extreme-like property was introduced in [13], and studied also in [9]. The following theorem summarises [13: Theorem 1.5] and [9: Theorem 2.1].

Theorem 6.1. ([9, 13]) *Let X be a space, with $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$. Then, the set $\big\{f(X) :\ f \in \mathscr{sel}[\mathscr{F}(X)]\big\}$ is dense in X if and only if X has a clopen π-base. Moreover, X is totally disconnected whenever the set $\big\{f(X) :\ f \in \mathscr{sel}[\mathscr{F}(X)]\big\}$ is dense in X.*

Here, a family $\mathscr{P}$ of open subsets of X is a *π-base* (sometimes, called also a *pseudobase*) for X if every nonempty open subset of X contains some nonempty member of $\mathscr{P}$.

Let us emphasise that there is a space X which is not zero-dimensional (actually, not regular), but the set $\big\{f(X) :\ f \in \mathscr{sel}[\mathscr{F}(X)]\big\}$ is dense in X, see [17: Example 4.4]. Here, we prove the following theorem.

Theorem 6.2. *Let X be a regular countably compact space with $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$. Then, the set $\big\{f(X) :\ f \in \mathscr{sel}[\mathscr{F}(X)]\big\}$ is dense in X if and only if X is totally disconnected.*

To prepare for the proof of Theorem 6.2, let us recall that a point $p \in X$ is 0-*approachable* if it is an isolated point of X, and that it is ω-*approachable* if there exists an open subset $U \subset X \setminus \{p\}$ such that $\overline{U} = U \cup \{p\}$ and p has a countable clopen base in $\overline{U}$, see [9]. We say that $p \in X$ is *countably-approachable* if it is either 0-approachable or ω-approachable. According to [9: Lemma 4.2], if $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$, then for every countably-approachable point $p \in X$ there exists an $f \in \mathscr{sel}[\mathscr{F}(X)]$ such that $f(X) = p$. In fact, it was proved in [9] that, for a space X with $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$, the set $\{f(X) : f \in \mathscr{sel}[\mathscr{F}(X)]\}$ is dense in X if and only if the set of all countably-approachable points of X is dense in X.

Proposition 6.3. *Let X be a countably compact space which is totally disconnected in a point $p \in X$ and has a continuous weak selection. Then, the following are equivalent*:

(a) $\mathsf{sa}(p, X) \leq \omega$.

(b) *p is countably-approachable.*

Proof. The statement follows immediately by the definitions if p is an isolated point of X. Suppose that p is non-isolated. Then, (b) $\implies$ (a) is true without any extra hypotheses on X. So, let $\mathsf{sa}(p, X) \leq \omega$, and let $g \in \mathscr{sel}[\mathscr{F}_2(X)]$. Since p is non-isolated and $\mathsf{sa}(p, X) \leq \omega$, there exists a countable set $A \subset X$ such that $p \in \overline{A}$ and $A \subset (\leftarrow, p)_{\preceq_g}$ or $A \subset (p, \rightarrow)_{\preceq_g}$, say $A \subset (p, \rightarrow)_{\preceq_g}$. Then, by Proposition 5.2, $[p, \rightarrow)_{\preceq_g}$ has a countable local base at p of $\mathscr{T}_{\preceq_g}$-open sets. However, by hypothesis, $\{p\}$ is an intersection of clopen subsets of X. Hence, by [18: Proposition 5.6], p is a countable intersection of clopen subsets of $[p, \rightarrow)_{\preceq_g}$. Since X is also countably compact, this finally implies that $[p, \rightarrow)_{\preceq_g}$ is zero-dimensional at p. □

Proof of Theorem 6.2. Suppose that X is totally disconnected, and $U \subset X$ is a nonempty open set. If U is finite, then it consists of isolated points and, in particular, will be clopen. Hence, there exists $g \in \mathscr{sel}[\mathscr{F}(X)]$ with $g(X) \in U$, see, e.g., [13: Lemma 2.1]. Actually, in this case, any point of U will be selection maximal. Suppose that U is infinite and $\preceq_{\mathbf{s}}$ is a closed selection relation on X. Then, U contains a sequence of distinct points and, by Proposition 3.3 and Corollary 3.6, U contains a strictly $\preceq_{\mathbf{s}}$-monotone sequence $S = \{x_n : n < \omega\}$ which is convergent in X to a point $p \in \overline{U}$. In particular, we now have that $\mathsf{sa}(p, X) \leq \omega$ and, by Proposition 6.3, p must be countably-approachable. Finally, by [9: Lemma 4.2], there exists an $f \in \mathscr{sel}[\mathscr{F}(X)]$ such that $f(X) = p \in \overline{U}$. Since X is regular, this implies that the set $\{f(X) : f \in \mathscr{sel}[\mathscr{F}_2(X)]\}$ is dense in X. According to Theorem 6.1, the proof is completed. □

Let us explicitly mention the following consequence of Theorems 3.8 and 6.2 which sheds some light on [17: Question 3].

COROLLARY 6.4. *If X is a countably compact Tychonoff space with $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$, then the set $\{f(X) : f \in \mathscr{sel}[\mathscr{F}(X)]\}$ is dense in X iff X is zero-dimensional.*

Proof. If X is a countably compact Tychonoff space with $\mathscr{sel}[\mathscr{F}(X)] \neq \varnothing$, then, by Theorem 3.8, it is suborderable. However, in the presence of suborderability, total disconnectedness is equivalent to zero-dimensionality. Hence, the statement is an immediate consequence of Theorem 6.1 and [13: Theorem 1.3]. □

According to Lemma 5.1 and Corollary 5.4, every countably compact space X, with $\mu_g(X) \leq \omega$ for some $g \in \mathscr{sel}[\mathscr{F}_2(X)] \neq \varnothing$, is a first countable Tychonoff space. By Theorem 1.1 and Corollary 6.4, this implies the following further consequence.

COROLLARY 6.5. *Let X be a countably compact space such that $\mu_g(X) \leq \omega$ for some $g \in \mathscr{sel}[\mathscr{F}_2(X)]$. Then, the following are equivalent*:

(a) *X is selection pointwise-maximal.*

(b) *$X = \{f(X) : f \in \mathscr{sel}[\mathscr{F}(X)]\}$.*

(c) *the set $\{f(X) : f \in \mathscr{sel}[\mathscr{F}(X)]\}$ is dense in X.*

(d) *X is zero-dimensional.*

Let us remark that, by [9: Corollary 4.5], (b) and (c) of Corollary 6.5 are equivalent for every first countable regular space.

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Department of Mathematics
Faculty of Science
University of Malta
Msida MSD 2080
MALTA
E-mail: david.buhagiar@um.edu.mt
valentin.gutev@um.edu.mt