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APPROXIMATION BY COMPLEX SUMMATION-INTEGRAL TYPE OPERATOR IN COMPACT DISKS

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ABSTRACT. In the present paper we estimate a Voronovskaja type quantitative estimate for a certain type of complex Durrmeyer polynomials, which is different from those studied previously in the literature. Such estimation is in terms of analytic functions in the compact disks. In this way, we present the evidence of overconvergence phenomenon for this type of Durrmeyer polynomials, namely the extensions of approximation properties (with quantitative estimates) from real intervals to compact disks in the complex plane. In the end, we mention certain applications.

1. Inroduction

If $f\colon G \to \mathbb{C}$ is an analytic function in the open set $G \subset \mathbb{C}$, with $\overline{D}_1 \subset G$ (where $D_1 = \{z \in \mathbb{C} : |z| < 1\}$), then S. N. Bernstein proved that the complex Bernstein polynomials converges uniformly to f in $\overline{D}_1$ (see e.g., Lorentz [8: p. 88]). Sorin G Gal has done commendable work in this direction and he estimated upper quantitative estimates for the uniform convergence for the first time. (see e.g. [3: p. 264]). Also exact quantitative estimates for different operators were established in his recent papers see e.g. [2], [4], [6] and [5] etc.

In the recent years and for the real variable case, Abel-Gupta-Mohapatra [1] studied the rate of convergence and established asymptotic expansion of certain Bernstein-Durrmeyer type operators, which are discretely defined at $f(0)$. The aim of the present article is to extend the studies on such operators. Let $R > 1$

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and suppose that $f\colon D_R \to \mathbb{C}$ is analytic in $D_R = \{z \in \mathbb{C} : |z| < R\}$ that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, the complex Bernstein-Durrmeyer type operator defined as

$$M_n(f,z) = (n+1)\sum_{k=1}^{n} p_{n,k}(z) \int_0^1 f(t)p_{n,k-1}(t)\,\mathrm{d}t + f(0)p_{n,0}(z) \tag{1}$$

where $z \in \mathbb{C}$, $n = 1, 2, \dots$ and

$$p_{n,k}(z) := \binom{n}{k} z^k (1-z)^{n-k}.$$

Our results will put in evidence the overconvergence phenomenon for the operators (1). The results established here are the extensions of approximation properties with exact quantitative estimates from the real interval $[0,1]$, to compact disks in the complex plane. Also, the methods used here are different from other complex Bernstein-type operators.

2. Basic results

In the sequel, we shall need the following basic results.

Lemma 1. *For all $e_p = t^p$, $p \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$ we have*

$$M_n(e_{p+1}, z) = \frac{z(1-z)}{n+p+2} M_n'(e_p, z) + \frac{nz+p}{n+p+2} M_n(e_p, z).$$

Proof. For $p = 0$ the relationship is evident from $M_n(e_0, z) = 1$ and $M_n(e_1, z) = \frac{nz}{n+2}$ (see e.g., [1]).

Therefore, let $p \in \mathbb{N}$. Using the equality

$$z(1-z)p_{n,k}'(z) = (k-nz)p_{n,k}(z),$$

we have

$$\begin{aligned} &z(1-z)M_n'(e_p, z) \\ &= (n+1)\sum_{k=1}^{n} z(1-z)p_{n,k}'(z) \int_0^1 p_{n,k-1}(t)t^p\,\mathrm{d}t \\ &= (n+1)\sum_{k=1}^{n} (k-nz)p_{n,k}(z) \int_0^1 p_{n,k-1}(t)t^p\,\mathrm{d}t \end{aligned}$$

$$= (n+1)\sum_{k=1}^{n} p_{n,k}(z)\int_0^1 [\{(k-1)-nt\} + nt + (1-nz)]p_{n,k-1}(t)t^p\,\mathrm{d}t$$

$$= (n+1)\sum_{k=1}^{n} p_{n,k}(z)\int_0^1 t(1-t)p'_{n,k-1}(t)t^p\,\mathrm{d}t + (1-nz)M_n(e_p,z) + nM_n(e_{p+1},z).$$

Integrating by parts the last integral, we get

$$\begin{aligned} z(1-z)M'_n(e_p,z) = & -(p+1)M_n(e_p,z) + (p+2)M_n(e_{p+1},z) \\ & + (1-nz)M_n(e_p,z) + nM_n(e_{p+1},z). \end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2.

(i) *For all $n \in \mathbb{N}$ and $p \in \mathbb{N} \cup \{0\}$, we have $M_n(e_p,1) \le 1$.*

(ii) *For all $n, p \in \mathbb{N}$ and $z \in \mathbb{C}$, we have*

$$M_n(e_p,z) = \frac{(n+1)!}{(n+p+1)!}\sum_{k=0}^{\min\{n,p\}} \binom{n}{k}\Delta_1^k F_p(0)z^k,$$

where $F_p(v) = \prod_{j=0}^{p-1}(v+j)$ for all $v \ge 0$,

$$\Delta_1^k F_p(0) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}F_p(k-j)$$

and $\Delta_1^k F_p(0) \ge 0$ for all k and p.

Proof.

(i) For $p = 0$ we have $M_n(e_p,1) = 1$. So let $p \ge 1$. By definition, we get

$$\begin{aligned} M_n(e_p,z) &= (n+1)\sum_{k=1}^{n} p_{n,k}(z)\int_0^1 p_{n,k-1}(t)t^p\,\mathrm{d}t \\ &= (n+1)\sum_{k=1}^{n} p_{n,k}(z)\int_0^1 \binom{n}{k-1}t^{k+p-1}(1-t)^{n-k+1}\,\mathrm{d}t \\ &= (n+1)\sum_{k=1}^{n}\binom{n}{k}z^k(1-z)^{n-k}\frac{n!}{(k-1)!(n-k+1)!}B(k+p,n-k+2) \end{aligned}$$

where B is the Euler's Beta function given by $B(k+p,n-k+2) = \frac{(k+p-1)!(n-k+1)!}{(n+p+1)!}$.

Thus

$$M_n(e_p, 1) = \frac{n(n+1)}{(n+p)(n+p+1)} \leq 1.$$

(ii) We have

$$\begin{aligned}\int_0^1 p_{n,k-1}(t)t^p \,\mathrm{d}t &= \int_0^1 \binom{n}{k-1} t^{k+p-1}(1-t)^{n-k+1} \,\mathrm{d}t \\ &= \frac{n!}{(k-1)!(n-k+1)!} \cdot B(k+p, n-k+2) \\ &= \frac{n!}{(n+p+1)!}[k(k+1)\dots(k+p-1)] = \frac{n!}{(n+p+1)!} F_p(k),\end{aligned}$$

where $F_p(v) = \prod_{j=0}^{p-1}(v+j)$. It is obvious that $F_p(v)$ and its derivatives of any order are ≥ 0 for all $v \geq 0$, which implies that $\Delta_1^k F_p(0) \geq 0$ for all k and p. Therefore

$$\begin{aligned}M_n(e_p, z) &= \frac{(n+1)!}{(n+p+1)!}\left[\sum_{k=0}^{n} p_{n,k}(z)F_p(k) - p_{n,0}(z)F_p(0)\right] \\ &= \frac{(n+1)!}{(n+p+1)!}\sum_{k=0}^{n}\binom{n}{k}\Delta_1^k F_p(0) z^k \\ &= \frac{(n+1)!}{(n+p+1)!}\sum_{k=0}^{\min\{n,p\}}\binom{n}{k}\Delta_1^k F_p(0) z^k,\end{aligned}$$

which proves the lemma. □

Corollary 3. *For all $p, n \in \mathbb{N}\cup\{0\}$ and $|z| \leq r$, $r \geq 1$ we have $|M_n(e_p, z)| \leq r^p$.*

P r o o f. By Lemma 2, it follows that

$$\frac{(n+1)!}{(n+p+1)!}\sum_{k=0}^{\min\{n,p\}}\binom{n}{k}\Delta_1^k F_p(0) \leq 1$$

which implies that

$$|M_n(e_p, z)| \leq \frac{(n+1)!}{(n+p+1)!}\sum_{k=0}^{\min\{n,p\}}\binom{n}{k}\Delta_1^k F_p(0) r^k \leq r^p.$$

□

3. Main results

The first main result one refers to upper estimates.

Theorem 1. *Let* $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *for all* $|z| < R$, $R > 1$ *and take* $1 \leq r < R$. *For all* $|z| \leq r$ *and* $n \in \mathbb{N}$, *we have*

$$|M_n(f,z) - f(z)| \leq \frac{C_r(f)}{n},$$

where $C_r(f) = 2 \sum_{p=2}^{\infty} |c_p| p(p+1) r^p < \infty$.

Proof. First we prove that $M_n(f,z) = \sum_{k=0}^{\infty} c_k M_n(e_k, z)$. Indeed denoting $f_m(z) = \sum_{j=0}^{m} c_j z^j$, $|z| \leq r$ with $m \in \mathbb{N}$, by the linearity of M_n, we have

$$M_n(f_m, z) = \sum_{k=0}^{m} c_k M_n(e_k, z),$$

and it is sufficient to show that for any fixed $n \in \mathbb{N}$ and $|z| \leq r$ with $r \geq 1$, we have $\lim_{m\to\infty} M_n(f_m, z) = M_n(f,z)$. But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$, the norm being defined as $\|f\|_r = \max\{|f(z)| : |z| \leq r\}$ and from the inequality

$$\begin{aligned}
&|M_n(f_m, z) - M_n(f,z)| \\
&\leq |f_m(0) - f(0)| \cdot |(1-z)^n| + (n+1) \sum_{k=1}^{n} |p_{n,k}(z)| \int_0^1 p_{n,k-1}(t) |f_m(t) - f(t)| \, \mathrm{d}t \\
&\leq C_{r,n} ||f_m - f||_r,
\end{aligned}$$

valid for all $|z| \leq r$, where

$$C_{r,n} = (1+r)^n + (n+1) \sum_{k=1}^{n} \binom{n}{k} (1+r)^{n-k} r^k \int_0^1 p_{n,k-1}(t) \, \mathrm{d}t.$$

Therefore we get

$$|M_n(f,z) - f(z)| \leq \sum_{p=0}^{\infty} |c_p| \cdot |M_n(e_p, z) - e_p(z)| = \sum_{p=1}^{\infty} |c_p| \cdot |M_n(e_p, z) - e_p(z)|,$$

as $M_n(e_0, z) = e_0(z) = 1$.

We have two cases:

(i) $1 \leq p \leq n$,

(ii) $p > n$.

Case (i): By Lemma 2, we have

$$M_n(e_p,z)-e_p(z)=z^p\left[\frac{(n+1)!}{(n+p+1)!}\binom{n}{p}\Delta_1^pF_p(0)-1\right]$$
$$+\frac{(n+1)!}{(n+p+1)!}\sum_{k=0}^{p-1}\binom{n}{k}\Delta_1^kF_p(0)z^k$$

and

$$|M_n(e_p,z)-e_p(z)|\le r^p\left[1-\frac{(n+1)!}{(n+p+1)!}\binom{n}{p}\Delta_1^pF_p(0)\right]$$
$$+r^p\left[1-\frac{(n+1)!}{(n+p+1)!}\binom{n}{p}\Delta_1^pF_p(0)\right]$$
$$=2r^p\left[1-\frac{(n+1)!}{(n+p+1)!}\binom{n}{p}\Delta_1^pF_p(0)\right].$$

Hence, we can write

$$\frac{(n+1)!}{(n+p+1)!}\binom{n}{p}\Delta_1^pF_p(0)=\frac{(n+1)!}{(n+p+1)!}\binom{n}{p}p!=\prod_{j=1}^{p}\frac{(n+j-p)}{(n+j+1)}.$$

By using the formula

$$1-\prod_{j=1}^{k}x_j\le\sum_{j=1}^{k}(1-x_j),0\le x_j\le 1,\qquad j=1,2,\dots,k,$$

with $x_j=\frac{(n+j-p)}{(n+j+1)}$ and $k=p$, we obtain

$$1-\prod_{j=1}^{p}\frac{(n+j-p)}{(n+j+1)}\le\sum_{j=1}^{p}(1-\frac{(n+j-p)}{(n+j+1)})=(p+1)\sum_{j=1}^{p}\frac{1}{n+j+1}\le\frac{p(p+1)}{n}.$$

Therefore it follows that

$$|M_n(e_p,z)-e_p(z)|\le\frac{2p(p+1)r^p}{n}.$$

Case (ii): By (i) and for $p>n\ge 1$, we obtain

$$|M_n(e_p,z)-e_p(z)|\le|M_n(e_p,z)|+|e_p(z)|\le 2r^p<2\frac{p(p+1)r^p}{n}.$$

By the cases (i) and (ii), we conclude that for all $p,n\in\mathbb{N}$ one has

$$|M_n(e_p,z)-e_p(z)|\le\frac{2p(p+1)r^p}{n}.$$

Hence, we get

$$|M_n(f,z)-f(z)|\le\frac{2}{n}\sum_{p=1}^{\infty}|c_p|p(p+1)r^p,$$

which proves the theorem. $\square$

We have the following Voronovskaja-type quantitative estimate.

Theorem 2. *Let $R > 1$ and suppose that $f\colon D_R \to \mathbb{C}$ is analytic in $D_R = \{z \in \mathbb{C} : |z| < R\}$ that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$. For any fixed $r \in [1, R]$ and for all $n \in \mathbb{N}, |z| \le r$, we have*

$$\left| M_n(f,z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{n} \right| \le \frac{M_r(f)}{n^2},$$

where $M_r(f) = \sum_{k=1}^{\infty} |c_k| k B_{k,r} r^k < \infty$ and

$B_{k,r} = r^2(2k^3+2k) + r(8k^3+k^2+13k+4) + (6k^3+k^2+17k+6) + 4(k-1)^3(1+r)$.

Proof. We denote $e_k(z) = z^k, k = 0,1,2,\dots$ and $\pi_{k,n}(z) = M_n(e_k, z)$. By the proof of Theorem 1, we can write $M_n(f,z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$. Also

$$\begin{aligned}\frac{z(1-z)f''(z) - 2zf'(z)}{n} &= \frac{z(1-z)}{n}\sum_{k=2}^{\infty} c_k k(k-1)z^{k-2} - \frac{2z}{n}\sum_{k=1}^{\infty} c_k k z^{k-1} \\ &= \frac{1}{n}\sum_{k=1}^{\infty} c_k[k(k-1) - k(k+1)z]z^{k-1}.\end{aligned}$$

Thus

$$\begin{aligned}&\left| M_n(f,z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{n} \right| \\ &\le \sum_{k=1}^{\infty} |c_k| \left| \pi_{k,n}(z) - e_k(z) - \frac{(k(k-1) - k(k+1)z)z^{k-1}}{n} \right|,\end{aligned}$$

for all $z \in D_R$, $n \in \mathbb{N}$.

By Lemma 1, for all $n \in \mathbb{N}, z \in \mathbb{C}$ and $k = 0,1,2,\dots$, we have

$$\pi_{k+1,n}(z) = \frac{z(1-z)}{n+k+2}\pi'_{k,n}(z) + \frac{nz+k}{n+k+2}\pi_{k,n}(z).$$

If we denote

$$E_{k,n}(z) = \pi_{k,n}(z) - e_k(z) - \frac{(k(k-1) - k(k+1)z)z^{k-1}}{n},$$

then it is obvious that $E_{k,n}(z)$ is a polynomial of degree less than or equal to k and by simple computation and the use of above recurrence relation, we are led to

$$E_{k,n}(z) = \frac{z(1-z)}{n+k+1}E'_{k-1,n}(z) + \frac{nz+k-1}{n+k+1}E_{k-1,n}(z) + X_{k,n}(z),$$

where

$$X_{k,n}(z)=\frac{z^{k-2}}{n(n+k+1)}\big[z^2(2k^3+2k)+z(-4k^3+9k^2-9k+4)$$
$$+(2k^3-9k^2+13k-6)\big],$$

for all $k\geq 1$, $n\in\mathbb{N}$ and $|z|\geq r$.

Using the estimate in the proof of Theorem 1, we have

$$|\pi_{k,n}(z)-e_k(z)|\leq\frac{2k(k+1)r^k}{n},$$

for all $k,n\in\mathbb{N},|z|\leq r$, with $1\leq r$.

For all $k,n\in\mathbb{N},k\geq 1$ and $|z|\leq r$, it follows

$$|E_{k,n}(z)|\leq\frac{r(1+r)}{n+k+1}|E'_{k-1,n}(z)|+\frac{nr+k-1}{n+k+1}|E_{k-1,n}(z)|+|X_{k,n}(z)|.$$

Since $\frac{r(1+r)}{n+k+1}\leq\frac{r(1+r)}{n}$ and $\frac{nr+k-1}{n+k+1}\leq r$, it follows

$$|E_{k,n}(z)|\leq\frac{r(1+r)}{n}|E'_{k-1,n}(z)|+r|E_{k-1,n}(z)|+|X_{k,n}(z)|.$$

Now we shall find the estimation of $|E'_{k-1,n}(z)|$ for $k\geq 1$. Taking into account the fact that $E_{k-1,n}(z)$ is a polynomial of degree $\leq k-1$, we have

$$\begin{aligned}|E'_{k-1,n}(z)|&\leq\frac{k-1}{r}\|E_{k-1,n}\|_r\\&\leq\frac{k-1}{r}\left[\|\pi_{k-1,n}-e_{k-1}\|_r+\left\|\frac{(k-1)e_{k-2}[(k-2)-ke_1]}{n}\right\|_r\right]\\&\leq\frac{k-1}{r}\left[\frac{2(k-1)kr^{k-1}}{n}+\frac{r^{k-2}(k-1)k(1+r)}{n}\right]\\&\leq\frac{(k-1)^2k}{n}\left[2r^{k-1}+\frac{1+r}{r}r^{k-1}\right]\leq\frac{4(k-1)^2kr^{k-1}}{n}.\end{aligned}$$

Thus

$$\frac{r(1+r)}{n}|E'_{k-1,n}(z)|\leq\frac{4(k-1)^2k(1+r)r^k}{n^2}$$

and

$$|E_{k,n}(z)|\leq\frac{4(k-1)^2k(1+r)r^k}{n^2}+r|E_{k-1,n}(z)|+|X_{k,n}(z)|,$$

where

$$|X_{k,n}(z)|\leq\frac{r^{k-2}}{n^2}\left[r^2(2k^3+2k)+r(4k^3+9k^2+9k+4)\right.$$
$$\left.+(2k^3+9k^2+13k+6)\right]\leq\frac{r^k}{n^2}A_{k,r},$$

for all $|z| \le r$, $k \ge 1$, $n \in \mathbb{N}$, where

$$A_{k,r} = r^2(2k^3 + 2k) + r(4k^3 + 9k^2 + 9k + 4) + (2k^3 + 9k^2 + 13k + 6).$$

Thus for all $|z| \le r$, $k \ge 1$, $n \in \mathbb{N}$

$$|E_{k,n}(z)| \le r|E_{k-1,n}(z)| + \frac{r^k}{n^2} B_{k,r},$$

where $B_{k,r}$ is a polynomial of degree 3 in k defined as

$$B_{k,r} = A_{k,r} + 4(k-1)^2 k(1+r).$$

But $E_{0,n}(z) = 0$, for any $z \in C$ and therefore by writing last inequality for $k = 1, 2, \dots$ we easily obtain step by step the following

$$|E_{k,n}(z)| \le \frac{r^k}{n^2} \sum_{j=1}^{k} B_{j,r} \le \frac{kr^k}{n^2} B_{k,r}.$$

We conclude that

$$\left| M_n(f,z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{n} \right|$$
$$\le \sum_{k=1}^{\infty} |c_k||E_{k,n}| \le \frac{1}{n^2} \sum_{k=1}^{\infty} |c_k| k B_{k,r} r^k.$$

As $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3) z^{k-4}$ and the series is absolutely convergent in $|z| \le r$, it easily follows that $\sum_{k=4}^{\infty} |c_k| k(k-1)(k-2)(k-3) r^{k-4} < \infty$, which implies that $\sum_{k=1}^{\infty} |c_k| k B_{k,r} r^k < \infty$. This completes the proof of theorem. □

Finally, we will obtain the exact order in approximation by this type of complex Bernstein-Durrmeyer polynomials and by their derivatives. In this sense, we present the following result.

Theorem 3. *Let $R > 1$ and suppose that $f\colon D_r \to \mathbb{C}$ is analytic in D_R, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$. If f is a polynomial of degree > 0, then for any $r \in [1, R)$, we have*

$$\|M_n(f,\cdot) - f\|_r \ge \frac{C_r(f)}{n}, \qquad n \in \mathbb{N}$$

where $C_r(f)$ depends only on f and r.

Proof. For all $f \in D_r$ and $n \in \mathbb{N}$, we have

$$M_n(f,z) - f(z) = \frac{1}{n}\Bigg[z(1-z)f''(z) - 2zf'(z) + \frac{1}{n}\left\{n^2\Big(M_n(f,z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{n}\Big)\right\}\Bigg].$$

Also, we have

$$\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r.$$

It follows

$$\|M_n(f,\cdot) - f\|_r \geq \frac{1}{n}\Bigg[\|e_1(1-e_1)f'' - 2e_1f'\|_r - \frac{1}{n}\left\{n^2\left\|M_n(f,\cdot) - f - \frac{e_1(1-e_1)f'' - 2e_1f'}{n}\right\|_r\right\}\Bigg].$$

Taking into account that by hypothesis f is not a polynomial of degree 0 in D_R, we get $\|e_1(1-e_1)f'' - 2e_1f'\|_r > 0$.

Indeed, supposing the contrary it follows that $z(1-z)f''(z) - 2zf'(z) = 0$ for all $|z| \leq r$, that is $(1-z)f''(z) - 2f'(z) = 0$ for all $|z| \leq r$ with $z \neq 0$. The last equality is equivalent to $[(1-z)f'(z)]' - f'(z) = 0$, for all $|z| \leq r$ with $z \neq 0$. Therefore we get $(1-z)f'(z) - f(z) = C$, with C a constant, that is $f(z) = \frac{Cz}{1-z}$, for all $|z| \leq r$ with $z \neq 0$.

But since f is analytic in $\overline{D}_r$ and $r \geq 1$, we necessarily have $C = 0$ (contrariwise, we would get that $f(z)$ is not differentiable at $z = 1$, which is impossible), a contradiction with the hypothesis.

Now by Theorem 2, we have

$$n^2\left\|M_n(f,\cdot) - f - \frac{e_1(1-e_1)f'' - 2e_1f''}{n}\right\|_r \leq M_r(f).$$

Therefore there exists an index n_0 depending only on f and r, such that for all $n \geq n_0$, we have

$$\|e_1(1-e_1)f'' - 2e_1f'\|_r - \frac{1}{n}\left\{n^2\left\|M_n(f,\cdot) - f - \frac{e_1(1-e_1)f'' - 2e_1f'}{n}\right\|_r\right\} \geq \frac{1}{2}\|e_1(1-e_1)f'' - 2e_1f'\|_r,$$

which immediately implies

$$\|M_n(f,\cdot) - f\|_r \geq \frac{1}{2n}\|e_1(1-e_1)f'' - 2e_1f'\|_r, \qquad \text{for all} \quad n \geq n_0.$$

For $n \in \{1, 2, \dots, n_0 - 1\}$ we obviously have $\|M_n(f,\cdot) - f\|_r \geq \frac{M_{r,n}(f)}{n}$ with $M_{r,n}(f) = n\|M_n(f,\cdot) - f\|_r > 0$. Indeed, since $M_n(f,z)$ is a polynomial of degree $\leq n$, from the equality $M_n(f,z) = f(z)$ for all $|z| \leq r$, it necessarily follows that $f(z)$ is a polynomial of degree $\leq n$. Let $f(z) = \sum_{k=0}^{n} a_k z^k$. We get

$$M_n(f,z) = \sum_{k=0}^{n} a_k M_n(e_k,z) = \sum_{k=0}^{n} a_k z^k.$$

But by Lemma 1 (or by Lemma 2, (ii)), it is clear that the coefficient of e_k in $M_n(e_k,z)$ is equal to 1 only for $k = 0$. This necessarily implies that f is a constant function, contradicting the hypothesis i.e. f is a polynomial of degree > 0. Therefore finally we obtain $\|M_n(f,\cdot) - f\|_r \geq \frac{C_r(f)}{n}$ for all n, where

$$C_r(f) = \min\Big\{M_{r,1}(f), M_{r,2}(f), \ldots, M_{r,n_0-1}(f), \frac{1}{2}\|e_1(1-e_1)f'' - 2e_1 f'\|_r\Big\},$$

which completes the proof. □

As a consequence of Theorem 1 and Theorem 3, we have the following:

Corollary 4. *Let $R > 1$ and suppose that $f\colon D_R \to \mathbb{C}$ is analytic in D_R. If f is not a polynomial of degree zero, then for any $r \in [1, R)$, we have*

$$\|M_n(f,\cdot) - f\|_r \sim \frac{1}{n}, \qquad n \in \mathbb{N},$$

where the constants in the equivalence depend only on f and r.

4. Applications

As a first application of the approximation properties of these Durrmeyer-type polynomials, we can mention some shape preserving properties. Thus, reasoning exactly as in the case of complex Bernstein polynomials in [7], one can prove that beginning with an index, the Durrmeyer-type polynomials in the present paper approximate the analytic functions, preserving in addition,the classical properties of univalence, star likeness, convexity and spiral likeness in geometric function theory. Also, as a potential application, we can mention the possibility to represent some C_0-semigroups generated by a complex one-dimensional second-order differential equation acting on the space of analytic functions in an open disk, as a limit of iterates of these complex polynomials, exactly as it was done in the classical well-known case of positive linear operators acting on spaces of continuous functions of real variable.

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